MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **86** (2021) 549-558

ISSN 0340 - 6253

A Lower Bound for Graph Energy in Terms of Minimum and Maximum Degrees

Saieed Akbari[1], Maryam Ghahremani[2], Mohammad Ali Hosseinzadeh[3], Somayeh Khalashi Ghezelahmad[2], Hamid Rasouli[2], Abolfazl Tehranian[2]

[1]*Department of Mathematical Sciences, Sharif University of Technology, Tehran, Iran*
s_akbari@sharif.edu

[2]*Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran*
maryam.ghahremani@srbiau.ac.ir, s.ghezelahmad@srbiau.ac.ir, hrasouli@srbiau.ac.ir, tehranian@srbiau.ac.ir

[3]*Faculty of Engineering Modern Technologies, Amol University of Special Modern Technologies, Amol, Iran*
ma.hoseinzade@gmail.com

(Received September 4, 2020)

Abstract

The energy of a graph G, denoted by $\mathcal{E}(G)$, is defined as the sum of absolute values of all eigenvalues of G. In (*MATCH Commun. Math. Comput. Chem.* **83** (2020) 631–633) it was conjectured that for every graph G with maximum degree $\Delta(G)$ and minimum degree $\delta(G)$ whose adjacency matrix is non-singular, $\mathcal{E}(G) \geq \Delta(G) + \delta(G)$ and the equality holds if and only if G is a complete graph. Here, we prove the validity of this conjecture for planar graphs, triangle-free graphs and quadrangle-free graphs.

1 Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. By *order* of G, we mean the number of vertices of G. The minimum and maximum degrees of G are denoted by $\delta(G)$ and $\Delta(G)$, or simply by δ and Δ, respectively. For any $v \in V(G)$, the *open* *neighborhood* and the *closed neighborhood* of v in G are $N(v) = \{u \in V(G) : uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$, respectively. Also the degree of $v \in V(G)$ is $d_G(v) = |N(v)|$ or simply $d(v)$. Let $S \subseteq V(G)$. By $\langle S \rangle$, we mean the subgraph of G induced by S. The path and the cycle of order n are denoted by P_n and C_n, respectively. A complete graph of order n is denoted by K_n and a complete bipartite graph with part sizes m and n is denoted by $K_{m,n}$. A *wheel graph* is a graph formed by joining a single vertex to all vertices of a cycle. We use W_n to denote the wheel graph of order n. A graph is *triangle-free* and *quadrangle-free* if it has no subgraph isomorphic to C_3 and C_4, respectively. A $\{1,2\}$-*factor* is a spanning subgraph of G which is a disjoint union of a matching and a 2-regular subgraph of G. A *subdivision* of an edge uv in a graph is the operation of replacing uv with a path u, w, v through a new vertex w. A subdivision of a graph G, is a graph obtained from G by successive edge subdivisions. Equivalently, it is a graph obtained from G by replacing edges with pairwise internally disjoint paths. A graph which can be drawn in the plane in such a way that edges meet only at points corresponding to their common ends is called a *planar graph*, and such a drawing is called a *planar embedding* of the graph. A graph is *outerplanar* if it has a planar embedding in which all vertices lie on the boundary of its outer face.

Let G be a graph and $V(G) = \{v_1, \dots, v_n\}$. The *adjacency matrix* of G, $A(G) = [a_{ij}]$, is an $n \times n$ matrix, where $a_{ij} = 1$ if $v_i v_j \in E(G)$, and $a_{ij} = 0$, otherwise. Thus $A(G)$ is a symmetric matrix and all eigenvalues of $A(G)$ are real. By eigenvalues of a graph G, we mean the eigenvalues of $A(G)$. The largest eigenvalue of G is called the *spectral radius* of G. For a graph G, let $\det A(G) \neq 0$. Then there exists $\sigma \in S_n$ such that $a_{1\sigma(1)} = \cdots = a_{n\sigma(n)} = 1$. This transversal is corresponding to a $\{1,2\}$-factor in G. The *energy* of a graph G, $\mathcal{E}(G)$, is defined as the sum of absolute values of eigenvalues of G. The concept of graph energy was first introduced by Gutman in 1978, see [7]. For more properties of the energy of graphs the reader is referred to [8]. Some lower bounds for the energy of graphs have been obtained by several authors. For quadrangle-free graphs, Zhou [11] studied the problem of bounding the graph energy in terms of the minimum degree together with other parameters. In [9], it was proved that for a connected graph G, $\mathcal{E}(G) \geq 2\delta(G)$ and the equality holds if and only if G is a complete multipartite graph with the equal size of parts. In [4], this lower bound was improved by showing that if G is a connected graph with average degree $\overline{d}$, then $\mathcal{E}(G) \geq 2\overline{d}$ and the equality holds if and only if G is a complete multipartite graph with the equal size of parts. Also in [4] the authors proposed the following conjecture.

Conjecture. *For every graph G whose adjacency matrix is non-singular, $\mathcal{E}(G) \geq \Delta(G) + \delta(G)$ and the equality holds if and only if G is a complete graph.*

In this paper, we attempt to establish the validity of the conjecture for three classes of graphs, triangle-free, quadrangle-free and planar graphs. The following lemmas are needed in the sequel.

Lemma 1. *[2] Let G be a graph of order n. If G has a $\{1,2\}$-factor, then $\mathcal{E}(G) \geq n$. In particular, if $A(G)$ is non-singular, then $\mathcal{E}(G) \geq n$.*

Lemma 2. *[3] Let G be a graph and $H_1, \ldots, H_k$ be its k vertex-disjoint induced subgraphs. Then $\mathcal{E}(G) \geq \sum_{i=1}^{k} \mathcal{E}(H_i)$.*

Lemma 3. *[2] If n is an odd positive integer, then $\mathcal{E}(C_n) \geq n + 1$.*

Lemma 4. *[1] If $n \geq 9$, then $\mathcal{E}(C_n) \geq n + 2$.*

2 The validity of conjecture for triangle–free and quadrangle–free graphs

In this section, it is shown that the conjecture holds for two classes of graphs, triangle-free and quadrangle-free graphs. First we prove the conjecture for triangle-free graphs.

Theorem 5. *Let G be a triangle-free graph which has a $\{1,2\}$-factor. Then for any two adjacent vertices u and v, $\mathcal{E}(G) \geq d(u) + d(v)$.*

Proof. Let u and v be two adjacent vertices of G. Since G is triangle-free, $N(u) \cap N(v) = \varnothing$. This implies that $d(u) + d(v) \leq n$, where $n = |V(G)|$. Now, since G has a $\{1,2\}$-factor, by Lemma 1, $\mathcal{E}(G) \geq n \geq d(u) + d(v)$. ■

Corollary 1. *The conjecture holds for triangle-free graphs. In particular, every bipartite graph satisfies the conjecture.*

Theorem 6. *Let G be a quadrangle-free graph which has a $\{1,2\}$-factor. Then $\mathcal{E}(G) \geq \Delta(G) + \delta(G)$.*

Proof. The result holds for K_2. So, let G be a graph of order $n \geq 3$ and u be a vertex of G with $d(u) = \Delta$. First suppose that $d(u) < n - 1$. Consider a vertex v not adjacent to

u. Since G is quadrangle-free, $|N(u) \cap N(v)| \leq 1$. Thus $\Delta + \delta \leq d(u) + d(v) \leq n-1$. Now, applying Lemma 1 yields the result. Next, assume that $d(u) = n-1$. Since G is quadrangle-free, the degree of each vertex of $N(u)$ is at most 2. If there exists a vertex w with degree 1, then using Lemma 1, we obtain $\mathcal{E}(G) \geq n \geq \Delta(G) + \delta(G)$. Otherwise, for each $w \in N(u)$, $d(w) = 2$. Therefore, G is a union of some edge-disjoint triangles having a vertex in common. Hence, G has a $\{1,2\}$-factor, say F, consisting of a triangle and some P_2-components. By considering the components of F as vertex-disjoint induced subgraphs and applying Lemmas 2 and 3, we have $\mathcal{E}(G) \geq n+1 \geq \Delta + \delta$. ■

Corollary 2. *The conjecture holds for quadrangle-free graphs.*

Now, we prove the validity of the conjecture for the class of graphs whose maximum eigenvalues are integer.

Theorem 7. *The conjecture holds for a graph whose spectral radius is integer.*

Proof. Let G be a graph of order n with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Note that since $A(G)$ is non-singular, for $i = 1, \ldots, n$, $\lambda_i \neq 0$. Since for every real number $x > 0$, $x - \ln x \geq 1$, we have

$$\mathcal{E}(G) = \lambda_1 + \sum_{i=2}^{n} |\lambda_i| \geq \lambda_1 + (n-1) + \sum_{i=2}^{n} \ln|\lambda_i| = \lambda_1 + (n-1) + \ln \prod_{i=2}^{n} |\lambda_i|.$$

By [5, Theorem 3.8], we know that $\lambda_1 \geq \delta$. Now, since $A(G)$ is non-singular and λ_1 is integer, $\prod_{i=2}^{n} |\lambda_i| = \frac{|\det A(G)|}{\lambda_1}$ is a non-zero rational number which is an algebraic integer. Hence, $\ln \prod_{i=2}^{n} |\lambda_i| \geq 0$. This implies that $\mathcal{E}(G) \geq \delta + \Delta$. Also the equality holds if and only if $\Delta = n-1$, $\delta = \lambda_1$ and $\prod_{i=2}^{n} |\lambda_i| = 1$. In the equality case, since $\delta = \lambda_1$, by [5, Theorem 3.8], we find that the graph is regular and since $\Delta = n-1$, the graph is complete. ■

3 The validity of conjecture for planar graphs

In this section, we check the validity of conjecture for planar graphs. In order to prove the results of this section, we need the following lemmas.

Lemma 8. *[6] A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.*

Lemma 9. *[10, p. 275] A graph is outerplanar if and only if it has neither a subdivision of K_4 nor a subdivision of $K_{2,3}$ as a subgraph.*

Lemma 10. *[10, Proposition 6.1.20] Every outerplanar graph has a vertex of degree at most* 2.

Now, we are ready to prove the next theorem.

Theorem 11. *There is no planar graph G of order n with $\delta(G) \geq 4$ and $\Delta(G) = n-1$.*

Proof. By the contrary, suppose that G is a planar graph of order n such that $\Delta(G) = n-1$ and $\delta(G) \geq 4$. Let $d(u) = n-1$. By Lemma 8, G does not contain a subdivision of K_5 or $K_{3,3}$, and so $G-u$ has neither a subdivision of K_4 nor a subdivision of $K_{2,3}$ as a subgraph. Thus by Lemma 9, $G-u$ is outerplanar and so by Lemma 10, $\delta(G-u) \leq 2$. This implies that $\delta(G) \leq 3$, a contradiction. ■

Theorem 12. *Let G be a planar graph of order n which has a $\{1,2\}$-factor. Then $\mathcal{E}(G) \geq \Delta(G) + \delta(G)$, where $\delta \leq i+1$ and $\Delta = n-i$, for $i = 1,2,3,4$.*

Proof. Since G is planar, by [10, p. 243], $\delta \leq 5$. First note that if $\delta \leq i$, for $i = 1,2,3,4$, then $\delta + \Delta \leq n \leq \mathcal{E}(G)$ and we are done. Hence we may assume that $\delta = i+1$. Let F be a $\{1,2\}$-factor of G which consists of cycles $C^{(1)}, \dots, C^{(l)}$ and t copies of P_2. We may consider two cases:

Case 1. If n is odd, then F contains at least one odd cycle. One may assume that every odd cycle in F is an induced odd cycle, because if we have an odd cycle with a chord, then there is a chord which partitions the vertices of odd cycle into an induced odd cycle and some matchings. Now, by Lemma 2 we obtain,

$$\mathcal{E}(G) \geq \sum_{i=1}^{l} \mathcal{E}(C^{(i)}) + t\mathcal{E}(P_2) = \sum_{i=1}^{l} \mathcal{E}(C^{(i)}) + 2t.$$

Note that by Lemma 1, if $C^{(j)}$ is an even cycle, for some j, then $\mathcal{E}(C^{(j)}) \geq |V(C^{(j)})|$. Now, since n is odd, at least one of the components of F is an odd cycle and so Lemma 3 implies that

$$\mathcal{E}(G) \geq n+1 = (n-i) + (i+1) \geq \Delta + \delta.$$

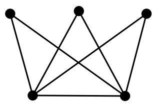
Case 2. If n is even and F contains an odd cycle, then by a similar argument as previous case, we find that $\mathcal{E}(G) \geq \Delta + \delta$. Also, one can assume that F has no even cycle, because the vertex set of every even cycle can be partitioned into disjoint copies of P_2. Thus we may assume that F is a perfect matching. Let $d(u) = \Delta = n-i$ and uw be a P_2-component of F. We have $d(w) \geq \delta = i+1$. Let $S = V(G) \setminus N[u] = \{v_1, \dots, v_{i-1}\}$ and

k be the number of P_2-components of F such that w is adjacent to both vertices of each of these P_2-components. Then the number of P_2-components of F, such that w is adjacent to exactly one of its vertices is at least $i - 2k$. We claim that there are four vertices of G such that the induced subgraph on these vertices is K_4 or $K_4 \setminus e$, for some edge e. If there exists one of k, P_2-components of F, say ab, such that $|\{a, b\} \cap S| \leq 1$, then $\langle u, w, a, b\rangle$ is either K_4 or $K_4 \setminus e$. Thus we may assume that for every P_2-component xy in which w is adjacent to both vertices, $\{x, y\} \subseteq S$. If there exists one of $i - 2k$, P_2-components, say ab, such that $\{a, b\} \cap S = \varnothing$, then $\langle u, w, a, b\rangle$ is $K_4 \setminus e$. Otherwise, $|S| \geq 2k + i - 2k = i$, a contradiction and the claim is proved. Now, since $\mathcal{E}(K_4) = 6$ and $\mathcal{E}(K_4 \setminus e) \geq 5$, we find that

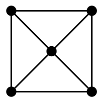
$$\mathcal{E}(G) \geq n + 1 = (n - i) + (i + 1) \geq \Delta + \delta,$$

and the proof is complete. ■

Remark 13. A computer search shows that if G is a graph of order 5 such that $\delta \geq 2$, $\Delta = 4$ and $\mathcal{E}(G) < 7$, then G is one of the following graphs:

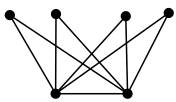


(a)

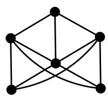


(b)

Remark 14. By a computer search, we noted that, if G is a graph of order 6 such that $\delta \geq 2$, $\Delta = 5$ and $\mathcal{E}(G) < 8$, then G is one of the following graphs. Note that the Graph (b) is not planar.

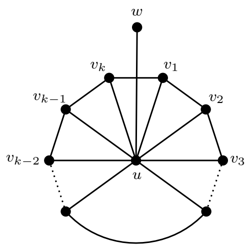


(a)



(b)

Remark 15. Let G_k, $3 \leq k \leq 7$, be the following graph. Among all graphs which are obtained by joining the vertex w to an arbitrary number of vertices v_i, $i = 1, \ldots, k$, the graph G_k has the minimum energy. Moreover, $\mathcal{E}(G_3) = 7.028$, $\mathcal{E}(G_4) = 7.362$, $\mathcal{E}(G_5) = 10.158$, $\mathcal{E}(G_6) = 12$ and $\mathcal{E}(G_7) = 13.290$. These are obtained by a computer search.



The graph G_k.

Remark 16. Using a computer search, one can find that for each planar graph G of order at most 9 which contains a $\{1,2\}$-factor, except W_5, $\mathcal{E}(G) \geq \Delta(G) + \delta(G)$.

Theorem 17. *Let G be a planar graph of order n with $\delta = 3$ and $\Delta = n-1$. If G has a $\{1,2\}$-factor, then $\mathcal{E}(G) \geq \Delta(G) + \delta(G)$.*

Proof. By Remark 16, we may suppose that $n \geq 10$. Let F be a $\{1,2\}$-factor of G. We consider two cases:

Case 1. Assume that n is even. If F has at least two induced odd cycles, then by Lemmas 2 and 3, $\mathcal{E}(G) \geq n+2 = \Delta+\delta$ and we are done. Note that since n is even, if one of the components of F is an induced odd cycle, then F contains at least two induced odd cycles. Hence, we may assume that F is a perfect matching. Let $d(u) = n-1$ and uw be a P_2-component of F. Since $d(w) \geq 3$, there exist two vertices v and z such that $v \in N(w)$ and vz is a P_2-component of F. Let $H = \langle u, v, w, z \rangle$. If $H = K_4$, then $\mathcal{E}(K_4) = 6$ and we are done. Otherwise, let xy be a P_2-component of F which is different from uw and vz. Let $K = \langle x, y, u, v, w, z \rangle$. We have $d_K(u) = 5$ and $\delta(K) \geq 2$. If $\mathcal{E}(K) \geq 8$, then there is nothing to prove. If $\mathcal{E}(K) < 8$, then by Remark 14, G is one of the Graphs (a) or (b). The Graph (a) has no perfect matching and the Graph (b) is not planar, a contradiction.

Case 2. Suppose that n is odd. Clearly, one of the components of F is an induced odd cycle. If F contains at least two induced odd cycles, then Lemmas 1, 2 and 3 give the result. Let $d(u) = n-1$. First, assume that $u \in V(C)$, where C is an odd cycle which is a component of F. Since u is adjacent to all vertices of C, u lies on a triangle. Hence, one can assume that F is a disjoint union of one C_3 and $\frac{n-3}{2}$, P_2-components. Let $V(C_3) = \{u, v, w\}$ and ab be a P_2-component of F. Let $K = \langle u, v, w, a, b \rangle$. We have $\Delta(K) = 4$ and $\delta(K) \geq 2$. If $\mathcal{E}(K) \geq 7$, then we are done. Otherwise, K is one the Graphs (a) and (b) in Remark 13. It is easy to see that this happens only if a is adjacent to exactly one of the vertices v and w, and b is adjacent to the other one. Now, consider $\langle u, v, w, a, b, c, d, r, s\rangle$, where cd and rs are P_2-components of F (since $n \geq 11$, cd and rs exist). Note that the situation of cd and rs are the same as ab, that is c is adjacent to one of the v and w, and d is adjacent to another one and this holds for r and s, too. This graph contains a subdivision of $K_{3,3}$ with parts $\{u, v, w\}$ and $\{a, c, r\}$, which contradicts Lemma 8. Next, suppose that $u \in V(P_2)$, where $P_2 = uw$ is a P_2-component of F. If one of the components of F is an induced odd cycle of order at least 9, then Lemma 4 yields the result. Hence, assume that every induced odd cycle component of F has order at most 7. Now, consider the subgraph induced by the P_2-component uw and an induced odd cycle of order at most 7. Using Remark 15, we obtain that $\mathcal{E}(G) \geq n + 2 = \Delta + \delta$. This completes the proof. ■

Theorem 18. *Let G be a planar graph of order n with $\delta \geq 4$ and $\Delta = n - 2$. If G has a $\{1, 2\}$-factor, then $\mathcal{E}(G) \geq \Delta(G) + \delta(G)$.*

Proof. By Remark 16, we can assume that $n \geq 10$. Let F be a $\{1, 2\}$-factor of G. We consider two cases:

Case 1. Assume that n is even. If F has at least two induced odd cycles, then by Lemmas 2 and 3, $\mathcal{E}(G) \geq n + 2 = \Delta + \delta$ and we are done. Note that since n is even, if one of the components of F is an induced odd cycle, then F contains at least two induced odd cycles. Hence, we may assume that F is a perfect matching. Let $d(u) = n - 2$ and uw be a P_2-component of F. Since $d(w) \geq 4$ and $n \geq 10$, there exist vertices $x \in N(w) \cap N(u)$ and $a, b, y \in N(u) \setminus \{w, x\}$ such that xy and ab are two P_2-components of F. Let $K = \langle u, w, x, y, a, b\rangle$. We have $\Delta(K) = 5$ and $\delta(K) \geq 2$. If K contains the subgraph K_4, then as $\mathcal{E}(K_4) = 6$, by Lemma 2 we are done. Otherwise, since the Graph (a) in Remark 14 has four independent vertices of degree 2 and the Graph (b) in Remark 14 is not planar, one can see that K is different from the Graphs (a) and (b). Thus $\mathcal{E}(K) \geq 8$. Now, by considering the vertex-disjoint induced subgraphs K and the components of $F \setminus V(K)$ and using Lemma 2, we obtain the result.

Case 2. Suppose that n is odd. Clearly, one of the components of F is an induced odd cycle, say C. If F contains at least two induced odd cycles, then Lemmas 1, 2 and 3 give the result. Let $d(u) = n - 2$.

First, suppose that $u \in V(C)$. Since u is adjacent to all other vertices of C, except at most one vertex, u lies on a triangle. Hence, one can assume that F is a disjoint union of one C_3 and some P_2-components. Let $V(C_3) = \{u, v, w\}$. Since $d(u) = n - 2$, there exists a P_2-component of F, say ab, such that $a, b \in N(u)$. Let $H = \langle u, v, w, a, b\rangle$. Note that $\Delta(H) = 4$ and $\delta(H) \geq 2$. If $\mathcal{E}(H) \geq 7$, then by Lemma 2, the result is obtained. Otherwise, H is one the Graphs (a) or (b) in Remark 13. It is easy to see that this happens only if a is adjacent to exactly one of the vertices v or w, and b is adjacent to the other one. Now, consider $\langle u, v, w, a, b, c, d, r, s\rangle$, where $c, d, r, s \in N(u) \setminus \{v, w, a, b\}$ and cd, rs are P_2-components of F (since $n \geq 11$, cd and rs exist). Note that the situation of cd and rs are the same as ab, that is c is adjacent to one of the v and w, and d is adjacent to another one and this holds for r and s, too. Thus this graph contains a subdivision of $K_{3,3}$ with parts $\{u, v, w\}$ and $\{a, c, r\}$, which contradicts Lemma 8.

Next, suppose that $u \in V(P_2)$, where $P_2 = uw$ is a P_2-component of F. If $|V(C)| \geq 9$, then Lemmas 2 and 4 give the result. So let $|V(C)| \leq 7$. If u or w is adjacent to all vertices of C, then consider $\langle u, w, V(C)\rangle$. Now, by Lemma 2 and Remark 15, the assertion holds. Suppose that none of u and w is adjacent to all vertices of C. Since $d(u) = n - 2$, u is adjacent to all vertices of P_2-components of F except u. First let $C = C_3$. Since w is not adjacent to at least one of the vertices of C_3 and $d(w) \geq 4$, there exists a P_2-component of F, say fh, such that $K = \langle u, w, f, h\rangle$ is either K_4 or $K_4 \setminus e$, for some edge e. Note that $\mathcal{E}(C_3) = 4$ and $\mathcal{E}(K) \geq 5$. Since the order of $C_3 \cup K$ is 7 and $\mathcal{E}(C_3 \cup K) \geq 9$, Lemma 2 yields the result.

In the sequel suppose that $|V(C)| \in \{5, 7\}$. If $N(w) \not\subseteq V(C) \cup \{u\}$, then since $\mathcal{E}(C) \geq |V(C)| + 1$, a similar proof as we did for C_3 works. Thus assume that $N(w) \subseteq V(C) \cup \{u\}$. We know that u is adjacent to all vertices of C except one vertex. Let $V(C) = \{v_1, \ldots, v_{2k+1}\}$ and without loss of generality, assume that u is not adjacent to v_{2k+1}. Since $d(w) \geq 4$, w is adjacent to v_t, for some $t \neq 1, 2$. Now, the triangle u, v_{t-2}, v_{t-1} and the edges $wv_t, v_{t+1}v_{t+2}, \ldots, v_{t-4}v_{t-3}$, where indices of vertices are considered modulo $2k + 1$, union all P_2-components of F except uw, form a $\{1, 2\}$-factor for G, in which u lies on C_3 and the proof is complete. ■

Theorem 19. *If $G \neq W_5$ is a connected planar graph containing a $\{1, 2\}$-factor, then $\mathcal{E}(G) \geq \Delta(G) + \delta(G)$.*

Proof. By Remark 16, the result holds for graphs of order at most 9. Let G be a planar graph of order $n \geq 10$. If $n \geq \Delta + \delta$, then Lemma 1 gives the result. Since G is planar, $\delta \leq 5$. Hence, if $\Delta \leq n - 5$, then $n \geq \Delta + \delta$ and we are done. Thus assume that $\Delta + \delta = n + i$, $i = 1, 2, 3, 4$. If $\Delta = n - 1$, then the result is obtained by Theorems 11, 12 and 17. If $\Delta = n - 2$, then Theorems 12 and 18 yield the result. If $\Delta = n - 3$, then

Theorem 12 implies the result for the case $\delta \leq 4$. Note that by [10, Theorem 6.1.23], each planar graph of order n has at most $3n - 6$ edges. If $\Delta = n - 3$ and $\delta = 5$, then the number of edges of G is at least $\frac{1}{2}(n - 3 + 5(n - 1)) = 3n - 4$ which is impossible. Also if $\Delta = n - 4$, then the result is a consequence of Theorem 12. ■

Since the graph W_5 has singular adjacency matrix and each complete graph of order at least 5 is not planar, as a consequence of Theorem 19, we give the following corollary.

Corollary 3. *The conjecture holds for planar graphs.*

Acknowledgments: This research of the first author was supported by grant number (G981202) from the Sharif University of Technology. The authors would like to thank Fatemeh Koorepazan Moftakhar for her preparing computational program search.

References

[1] A. Aashtab, S. Akbari, E. Ghasemian, A. H. Ghodrati, M. A. Hosseinzadeh, F. Koorepazan Moftakhar, On the minimum energy of regular graphs, *Lin. Algebra Appl.* **581** (2019) 51–71.

[2] S. Akbari, M. Ghahremani, I. Gutman, F. Koorepazan Moftakhar, Orderenergetic graphs, *MATCH Commun. Math. Comput. Chem.* **84** (2020) 325–334.

[3] S. Akbari, E. Ghorbani, M. R. Oboudi, Edge addition, singular values, and energy of graphs and matrices, *Lin. Algebra Appl.* **430** (2009) 2192–2199.

[4] S. Akbari, M. A. Hosseinzadeh, A short proof for graph energy is at least twice of minimum degree, *MATCH Commun. Math. Comput. Chem.* **83** (2020) 631–633.

[5] D. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs – Theory and Application*, Academic Press, New York, 1980.

[6] K. Kuratowski, Sur Ie probleme des courbes gauches en topologie, *Fund. Math.* **15** (1930) 271–283.

[7] I. Gutman, The energy of a graph, *Ber. Math. Statist. Sekt. Forschungsz. Graz.* **103** (1978) 1–22.

[8] X. Li, Y. Shi, I. Gutman, *Graph Energy*, Springer, New York, 2012.

[9] X. Ma, A low bound on graph energy in terms of minimum degree, *MATCH Commun. Math. Comput. Chem.* **81** (2019) 393–404.

[10] D. B. West, *Introduction to Graph Theory*, Prentice Hall, Upper Saddle River, 2001.

[11] B. Zhou, Lower bounds for the energy of quadrangle–free graphs, *MATCH Commun. Math. Comput. Chem.* **55** (2006) 91–94.