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Energy of Graphs Containing Disjoint Cycles

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Abstract

Let G be a graph. The energy $\mathcal{E}(G)$ is the sum of the absolute values of the eigenvalues of the adjacency matrix of G. In [Energy, matching number and odd cycles of graphs, Linear Algebra Appl. 577 (2019) 159–167] it has been proved that for a graph G whose cycles are odd and vertex disjoint, if from each cycle of G, we remove an arbitrary edge to obtain a tree T, then $\mathcal{E}(G) \geq \mathcal{E}(T)$. There is a gap in the proof. In this paper, we correct the proof and generalize this result by showing that if G is a graph all of whose cycles are vertex disjoint and the length of each cycle is not 0, modulo 4, then for any spanning tree of G, $\mathcal{E}(G) \geq \mathcal{E}(T)$. Finally we give an upper bound on $\mathcal{E}(G)$ of a graph G all of whose cycles are vertex disjoint.

1 Introduction

For a graph G, denote the set of vertices and the set of edges of G by $V(G)$ and $E(G)$, respectively. The number of vertices of G is called the *order* of G. The *adjacency matrix* of a graph G, $A(G) = [a_{ij}]$, is an $n \times n$ matrix, where $a_{ij} = 1$ if $v_iv_j \in E(G)$, and $a_{ij} = 0$,

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otherwise. The eigenvalues of G will be referred to the *eigenvalues* of $A(G)$. The *cycle* of order n is denoted by C_n. A *matching* M in G is a set of pairwise non-adjacent edges, that is, no two edges in M share a common vertex. A matching is said to be *maximum* if it has the largest number of edges among all matchings of G. The number of edges of a maximum matching of G is called the *matching number* of G, denoted by ν. A k*-matching* is a matching of size k. Denote by $m(G,k)$ the number of k-matchings of the graph G.

Let $x^n + a_1x^{n-1} + \cdots + a_n$ be the characteristic polynomial of G. We recall the Sachs theorem [3] for the coefficients of the characteristic polynomial of a graph, that is

$$a_i = a_i(G) = \sum_{S\in\mathcal{L}_i} (-1)^{k(S)} 2^{c(S)}, \tag{1}$$

where $\mathcal{L}_i$ denotes the set of Sachs subgraphs of G of order i, that is, the subgraphs in which every component is either a K_2 or a cycle, $k(S)$ is the number of components of S and $c(S)$ is the number of cycles contained in S.

The energy of a graph G of order n, $\mathcal{E}(G)$, is the sum of absolute values of the eigenvalues of its adjacency matrix. This spectrum-based graph invariant has been much studied in both chemical and mathematical literature. For details of the mathematical theory of this, nowadays very popular, graph-spectral invariant see the book [9], the recent papers [1, 2, 4–7] and the references cited therein.

Moreover, $\mathcal{E}(G)$ can be expressed as the Coulson integral formula [8]

$$\mathcal{E}(G) = \frac{1}{\pi}\int_0^{+\infty} \frac{1}{x^2} \ln\left[\left(\sum_{j=0}^{\lceil \frac{n}{2}\rceil} b_{2j}x^{2j}\right)^2 + \left(\sum_{j=0}^{\lceil \frac{n}{2}\rceil} b_{2j+1}x^{2j+1}\right)^2\right] dx.$$

where $b_i(G) = |a_i(G)|$, $i = 0, 1, \ldots, n$.

The following result was proved in [2].

Lemma 1. [2, Lemma 8] *Let G be a graph whose cycles have odd lengths and are vertex disjoint. If from each cycle of G, we remove one edge to obtain a tree T, then $\mathcal{E}(G) \geq \mathcal{E}(T)$.*

It seems that the proof given in [2] has a gap and for the determining of the coefficients of the characteristic polynomial the author does not consider Sachs subgraphs with an even number of odd cycles as a Sachs subgraph. In this paper we generalize this result and present a correct proof for Lemma 1. Before proving our main result we need some results.

Lemma 2. [1, Corollary 3] *Adding any number of edges to each part of a bipartite graph, does not decrease its energy.*

Proof of Lemma 1. If we remove an arbitrary edge from each cycle, then we obtain a bipartite graph (indeed a tree) such that every removed edge has two vertices in one of the parts. So by Lemma 2, we are done. ■

Lemma 3. *Let G be a graph all of whose cycles are vertex disjoint and the length of every cycle is 2 (mod 4). If from each cycle of G, we remove one edge to obtain a tree T, then $\mathcal{E}(G) \geq \mathcal{E}(T)$.*

Proof. Suppose that $C_{4l_1+2}, \ldots, C_{4l_s+2}$ are all cycles of G, and $r_i = 2l_i + 1$, $1 \leq i \leq s$. Now, we obtain the sign of every term contributing to a_{2k} according to (1). The number of Sachs subgraphs of order $2k$ each of whose component is K_2 is $m(G,k)$. So one term of a_{2k} is $(-1)^k m(G,k)$. Now, suppose that we have a Sachs subgraph of order $2k$ containing j_i cycles of order $4l_i + 2$, for $i = 1, \ldots, s$. Then the number of connected components of this Sachs subgraph is:

$$\frac{2k - \sum_{i=1}^{s} 2j_i r_i}{2} + \sum_{i=1}^{s} j_i = k - \sum_{i=1}^{s} j_i r_i + \sum_{i=1}^{s} j_i \ ,$$

since r_i is odd, the number of connected components is k modulo 2. Since by [9] $b_{2k} = (-1)^k a_{2k}$, thus every term in b_{2k} is positive. This yields that $b_{2k} \geq m(G,k)$. Now, if T is a spanning tree of G, we have $b_{2k} \geq m(G,k) \geq m(T,k) = b'_{2k}$, where b'_{2k} is the coefficient of x^{n-2k} in the characteristic polynomial of T. On the other hand, $a'_{2k+1} = 0$. Note that since G is bipartite, $b_{2k+1} = 0$, for $k = 0, 1, \ldots, \lfloor \frac{n-1}{2} \rfloor$. Thus we have,

$$\left(\sum_{k \geq 0} b_{2k} x^{2k} \right)^2 \geq \left(\sum_{k \geq 0} b'_{2k} x^{2k} \right)^2, \quad \sum_{k \geq 0} b_{2k+1} x^{2k+1} = \sum_{k \geq 0} b'_{2k+1} x^{2k+1} = 0.$$

Thus, $\mathcal{E}(G) \geq \mathcal{E}(T)$. ■

Theorem 4. *Let G be a graph all of whose cycles are vertex disjoint and the length of each cycle is not 0, modulo 4. Suppose that T is an arbitrary spanning tree of G, then $\mathcal{E}(G) \geq \mathcal{E}(T)$.*

Proof. Suppose that G has t cycles $C_1, \ldots, C_t$ of odd lengths and s cycles $C'_1, \ldots, C'_s$ of even lengths. Let e_i, $1 \leq i \leq t$, be an arbitrary edge of cycle C_i, and e'_j, $1 \leq j \leq s$, be an arbitrary edge of cycle C'_j. Let $H = G \setminus \{e_1, \ldots, e_t\}$. Then H is a bipartite graph each of whose cycle has length 2 module 4. If $T = H \setminus \{e'_1, \ldots, e'_s\}$, then by Lemma 3, $\mathcal{E}(H) \geq \mathcal{E}(T)$. Since, H is a bipartite graph, then by Lamma 2, $\mathcal{E}(G) \geq \mathcal{E}(H)$ and this completes the proof. ∎

Conjecture 5. *Let G be a C_4-free graph whose cycles are vertex disjoint. If from each cycle of G, we remove an arbitrary edge to obtain a tree T, then $\mathcal{E}(G) \geq \mathcal{E}(T)$.*

Remark 6. By means of a computer-aided search [10], for connected C_4-free unicyclic graphs of order up to 14, we see that Conjecture 5 is correct.

In [2], Ashraf obtained a lower bound on $\mathcal{E}(G)$ in terms of matching number ν. Moreover, for graphs with vertex-disjoint cycles, she proved that $\mathcal{E}(G) \geq 2\nu + k$, where k denotes the number of odd cycles of G with length at least 5. We now give an upper bound on $\mathcal{E}(G)$ of a graph G all of whose cycles are vertex disjoint.

Theorem 7. *Let G be a connected graph with k cycles all of whose cycles are vertex disjoint. If $|E(G)| = m$, then*

$$\mathcal{E}(G) \leq 2\sqrt{\left(\nu + \frac{k}{2}\right) m},$$

where ν is the matching number of graph G.

Proof. Since $\mathcal{E}(G) = \sum_{\lambda_i \neq 0} |\lambda_i|$, by Cauchy-Schwarz inequality, we have,

$$\mathcal{E}(G) \leq \sqrt{\underbrace{1 + \cdots + 1}_{r}}\sqrt{\sum_{\lambda_i \neq 0} |\lambda_i|^2},$$

where $r = rank(A)$. Thus $\mathcal{E}(G) \leq \sqrt{2rm}$. By Theorem 1.1 of [11], $\frac{r-k}{2} \leq \nu$ and this implies that $\mathcal{E}(G) \leq 2\sqrt{(\nu + \frac{k}{2})m}$. ∎

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