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Energy, Randić Index and Maximum Degree of Graphs

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Abstract

The energy $\mathcal{E}(G)$ of a graph G is the sum of the absolute values of all the eigenvalues of its adjacency matrix $A(G)$. For a graph G with Randić index $R(G)$ and maximum degree $\Delta(G)$, we prove that $\mathcal{E}(G) \leqslant 2\sqrt{\Delta(G)}R(G)$ and the equality holds if and only if G is the union of the path P_2.

1 Introduction

Graphs considered in this paper are simple, undirected and without isolated vertices. For a graph G, let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. The *degree* $\deg(v)$ of a vertex v in G is the number of vertices adjacent to v, and the *maximum degree* of G is denoted by $\Delta(G)$. For any two vertex-disjoint graphs G and H, the *union* $G \cup H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

The *graph energy* $\mathcal{E}(G)$ of G, proposed by Gutman [8], is the sum of the absolute values of all the eigenvalues of the adjacency matrix $A(G)$, i.e. $\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A(G)$.

The theory of graph energy is well developed nowadays. In the last two decades, many researchers devoted to determine the lower and upper bounds of graph energy in terms of various graph parameters, such as matching number [2, 17–19], vertex cover number [20]

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and so on [1, 6, 7, 9–11, 13–16]. For more detailed results on graph energy, we refer the reader to the book [12].

For a graph G with Randić index $R(G)$, G. Arizmendi and O. Arizmendi [5] recently proved that $\mathcal{E}(G) \geqslant 2R(G)$ and the equality holds if and only if G is the union of complete bipartite graphs. Along this line, we prove that $\mathcal{E}(G) \leqslant 2\sqrt{\Delta(G)}R(G)$ and the equality holds if and only if G is the union of the path P_2.

2 Main Results

For a graph G, the *vertex energy* $\mathcal{E}(v)$ of the vertex v was proposed in [4]. The following lemma gives an upper bound for vertex vertex in terms of the vertex degree.

Lemma 1. *[3] For a graph G and a vertex $v_i \in V(G)$, $\mathcal{E}(v_i) \leqslant \sqrt{\deg(v_i)}$ and the equality holds if and only if the connected component containing v_i is isomorphic to S_n and v_i is its center.*

We now give the upper bound of graph energy in terms of Randić index.

Theorem 2. *Let G be a graph with Randić index $R(G)$ and maximum degree $\Delta(G)$. Then $\mathcal{E}(G) \leqslant 2\sqrt{\Delta(G)}R(G)$ and the equality holds if and only if G is the union of the path P_2.*

Proof. By the proof of Theorem 6 in [5], the energy can be calculated as

$$\mathcal{E}(G) = \sum_{e \in E(G)} \mathcal{E}(e) = \sum_{uw \in E(G)} \left(\frac{\mathcal{E}(v)}{\deg(v)} + \frac{\mathcal{E}(w)}{\deg(w)} \right).$$

By Lemma 1, we have

$$\begin{aligned}
\mathcal{E}(G) &= \sum_{uw \in E(G)} \left(\frac{\mathcal{E}(v)}{\deg(v)} + \frac{\mathcal{E}(w)}{\deg(w)} \right) \leqslant \sum_{uw \in E(G)} \left(\frac{\sqrt{\deg(v)}}{\deg(v)} + \frac{\sqrt{\deg(w)}}{\deg(w)} \right) \\
&= \sum_{uw \in E(G)} \left(\frac{1}{\sqrt{\deg(v)}} + \frac{1}{\sqrt{\deg(w)}} \right) = \sum_{uw \in E(G)} \frac{\sqrt{\deg(v)} + \sqrt{\deg(w)}}{\sqrt{\deg(v)\deg(w)}} \\
&\leqslant 2\sqrt{\Delta(G)} \sum_{uw \in E(G)} \frac{1}{\sqrt{\deg(v)\deg(w)}} = 2\sqrt{\Delta(G)}R(G).
\end{aligned}$$

The equality holds if and only for any vertex $v \in V(G)$, $\mathcal{E}(v) = \sqrt{\deg(v)} = \sqrt{\Delta(G)}$. By Lemma 1, we can obtain that G is the union of the path P_2. $\blacksquare$

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