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On the Graovac–Ghorbani Index for Bicyclic Graphs with No Pendent Vertices

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Abstract

Let $G = (V, E)$ be a simple, undirected and connected graph on n vertices. The Graovac–Ghorbani index of a graph G is defined as

$$ABC_{GG}(G) = \sum_{uv \in E(G)} \sqrt{\frac{n_u + n_v - 2}{n_u n_v}},$$

where n_u is the number of vertices closer to vertex u than vertex v of the edge $uv \in E(G)$ and n_v is defined analogously. Bicyclic graphs with no pendent vertices are composed by three infinite families of graphs. In this paper, we give a lower bound for all graphs in one of these families, and prove that this bound is sharp by presenting its extremal graphs. Additionally, we conjecture a sharp lower and upper bounds to the ABC_{GG} index for all bicyclic graphs.

1 Introduction

Let $G = (V, E)$ be a simple undirected and connected graph such that $n = |V|$ and $m = |E|$. The degree of a vertex $v \in V$, denoted by d_v, is the number of edges incidents to v. The Graovac-Ghorbani index, [7], is defined as

$$ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{n_u + n_v - 2}{n_u n_v}}, \quad (1)$$

where n_u is the number of vertices closer to vertex u than vertex v of the edge $uv \in E(G)$ and n_v is defined analogously. Note that equidistant vertices from u and v are not taken into account to compute n_u and n_v in Equation (1). The problem of finding graphs with maximum or minimum Graovac-Ghorbani index turns to be a difficult problem for general graphs. Some papers have been published in order to find extremal graphs to the Graovac-Ghorbani index of trees [9], unicyclic [3] and bipartite graphs [4]. Some interesting results on this topic can be found at [1,2,5,6,11]. In 2016, Das in [1] posed the following question: "Which graph has minimal ABC_{GG} index among all bicyclic graphs?" Motivated by this question we considered the ABC_{GG} index for bicyclic graphs with no pendent vertices.

In this paper, we explicitly give the ABC_{GG} index for some bicyclic graphs of no pendent vertices and present a sharp lower bound. Also, we conjecture a lower bound to the ABC_{GG} for all bicyclic graphs with no pendent vertices.

2 Preliminaries

A connected graph G of order n is called a bicyclic graph if G has $n+1$ edges. Bicyclic graphs with no vertex of degree one are bicyclic graphs with no pendent vertices. Let $\mathcal{B}_n$ be the set of all bicyclic graphs of order n with no pendent vertices. According to [8], there are three types of bicyclic graphs containing no pendent vertices, which we denote here by $B_1(n)$, $B_2(n)$ and $B_3(n)$. We use integers p, $q \geq 3$ to denote the size of the cycles, and $l \geq 1$ to denote the length of a path (i.e., the number of edges of a path). Let $B_1(p,q)$ be the set of bicyclic graphs obtained from two vertex-disjoint cycles C_p and C_q by identifying a vertex u of C_p and a vertex v of C_q such that $n = p+q-1$. Observe that all graphs in $B_1(p,q)$ have the same number of vertices but are not isomomorphic since the size of the cycles are not the same. Let $B_2(p,l,q)$ be the bicyclic graph obtained from two vertex-disjoint cycles C_p e C_q, by joining vertices v_1 of C_p and u_l of C_q by a new path $v_1u_1, \ldots, u_{l-1}u_l$ with length l, such that $n = p+q+l-1$. Let $B_3(p,l,q)$ be the bicyclic graph obtained from a cycle C_{p+q-2l} with vertex set given by $v_1v_2v_3, \ldots, v_{p+q-2l-1}v_{p+q-2l}v_1$ by joining vertices v_1 and v_{p-l-2} by a new path $v_1u_1u_2, \ldots, u_{l-2}u_{l-1}u_lv_{p-l-2}$ with length

l, where $n = p + q - l - 1$. Thus,

$$B_1(n) = \bigcup_{p,q\geq 3} B_1(p,q),\ B_2(n) = \bigcup_{p,q\geq 3, l\geq 1} B_2(p,l,q) \text{ and } B_3(n) = \bigcup_{p,q\geq 3, l\geq 1} B_3(p,l,q).$$

Now, it is clear that $\mathcal{B}_n = B_1(n) \cup B_2(n) \cup B_3(n)$. In Figure 1 the general form of the graphs in families $B_1(n), B_2(n)$ and $B_3(n)$ is displayed.

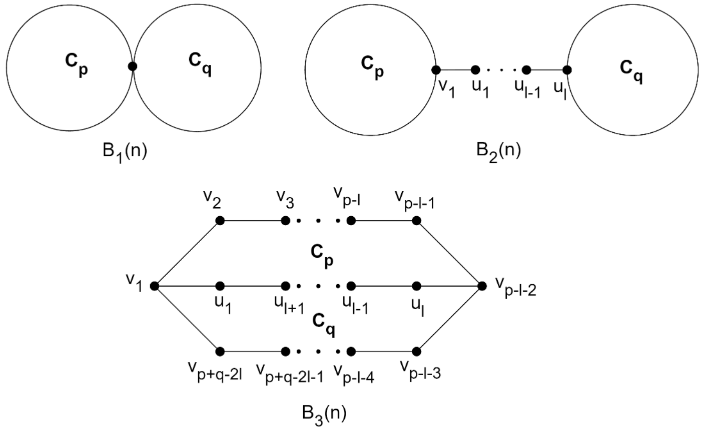

Figure 1. Families $B_1(n)$, $B_2(n)$ and $B_3(n)$ of bicyclic graphs with no pendent vertices

3 ABC_{GG} index for all graphs $G \in B_1(n)$

In this section, we give an explicit formula to the ABC_{GG} index of any graph in $B_1(n)$. In order to prove it, we consider the following cases:

- If n is odd there are two possibilities: either C_p and C_q are both odd cycles or C_p and C_q are both even cycles.
- If n is even, C_p is an odd cycle and C_q is an even cycle.

Throughout the proofs of next lemmas, we define

$$f(u,v) = \sqrt{\frac{n_u + n_v - 2}{n_u n_v}},$$

for any edge $uv \in E(G)$, and we write $G[H]$ for the subgraph induced in G by the vertex set of graph H. In Lemmas 3.1, 3.2 and 3.3 we present the ABC_{GG} for all graphs in $B_1(n)$. Note that for a fixed n some non-isomorphic graphs can be obtained by varying p and q such that $n = p + q - 1$.

Lemma 3.1. *Let $G \in B_1(n)$ be a graph on $n = p + q - 1$ vertices such that C_p and C_q are odd cycles. Then*

$$\begin{aligned} ABC_{GG}(G) &= 2\,(p-1)\sqrt{\frac{p+q-4}{(p-1)(p+2q-3)}} + \frac{2\,\sqrt{p-3}}{p-1} + \\ &+ 2\,(q-1)\sqrt{\frac{p+q-4}{(2\,p+q-3)(q-1)}} + \frac{2\,\sqrt{q-3}}{q-1}. \qquad (2) \end{aligned}$$

Proof. Let G be the graph labeled as Figure 2. Let $H_1 = G[C_p]$ be the graph induced by vertices $\{v_1, v_2, ..., v_p\}$ and consider the edge $(v_1, v_p) \in E(H_1)$. Note that $n_{v_1} = \frac{p+2q-3}{2}$ and $n_{v_p} = \frac{p-1}{2}$. Taking the advantage of the symmetry, we can observe that this same situation occurs $p-1$ times which can be written as $n_{v_i} = \frac{p+2q-3}{2}$ and $n_{v_{i+1}} = \frac{p-1}{2}$ for $i = \left\{1, \ldots, \frac{p-1}{2}\right\}$. For $i \in \left\{\frac{p+3}{2}, \ldots, p-1\right\}$ $n_{v_i} = \frac{p-1}{2}$ and $n_{v_{i+1}} = \frac{p+2q-3}{2}$. The remaining edge $(v_{\frac{p+1}{2}}, v_{\frac{p+3}{2}})$ has $n_{v_{\frac{p+1}{2}}} = n_{v_{\frac{p+3}{2}}} = \frac{p-1}{2}$. Thus

$$\begin{aligned} \sum_{uv\in E(H_1)} f(u,v) &= (p-1)\sqrt{\frac{\left(\frac{p-1}{2}\right) + \left(\frac{p+2q-3}{2}\right) - 2}{\left(\frac{p-1}{2}\right)\left(\frac{p+2q-3}{2}\right)}} + \sqrt{\frac{\left(\frac{p-1}{2}\right) + \left(\frac{p-1}{2}\right) - 2}{\left(\frac{p-1}{2}\right)\left(\frac{p-1}{2}\right)}} \qquad (3) \\ &= 2\,(p-1)\sqrt{\frac{p+q-4}{(p-1)(p+2q-3)}} + \frac{2\,\sqrt{p-3}}{p-1}. \qquad (4) \end{aligned}$$

Now, let $H_2 = G[C_q]$ be the graph induced by vertices $\{v_1, v_{p+1}, \ldots, v_n\}$. Considering the edge (v_1, v_n), we get $n_{v_1} = \frac{2p+q-3}{2}$ and $n_{v_n} = \frac{q-1}{2}$. Analogously to the previous case, $n_{v_j} = \frac{2p+q-3}{2}$ and $n_{v_{j+1}} = \frac{q-1}{2}$ for $j = \left\{p+1, \ldots, \frac{2p+q-3}{2}\right\}$. For $j \in \left\{\frac{2p+q+1}{2}, \ldots, n\right\}$ then $n_{v_j} = \frac{q-1}{2}$ and $n_{v_{j+1}} = \frac{2p+q-3}{2}$. The remaining edge $(u,v) = \left(v_{\frac{2p+q-1}{2}}, v_{\frac{2p+q+1}{2}}\right)$ has $n_u = n_v = \frac{q-1}{2}$. Thus,

$$\begin{aligned} \sum_{uv\in E(H_2)} f(u,v) &= (q-1)\sqrt{\frac{\left(\frac{q-1}{2}\right) + \left(\frac{2p+q-3}{2}\right) - 2}{\left(\frac{q-1}{2}\right)\left(\frac{2p+q-3}{2}\right)}} + \sqrt{\frac{\left(\frac{q-1}{2}\right) + \left(\frac{q-1}{2}\right) - 2}{\left(\frac{q-1}{2}\right)\left(\frac{q-1}{2}\right)}} \\ &= 2\,(q-1)\sqrt{\frac{p+q-4}{(2\,p+q-3)(q-1)}} + \frac{2\,\sqrt{q-3}}{q-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} ABC_{GG}(G) &= \sum_{uv\in H_1} f(u,v) + \sum_{uv\in H_2} f(u,v) \\ &= 2\,(p-1)\sqrt{\frac{p+q-4}{(p+2\,q-3)(p-1)}} + \frac{2\,\sqrt{p-3}}{p-1} \end{aligned}$$

$$+ \quad 2\,(q-1)\sqrt{\frac{p+q-4}{(2\,p+q-3)(q-1)}} + \frac{2\,\sqrt{q-3}}{q-1},$$

and the result follows.

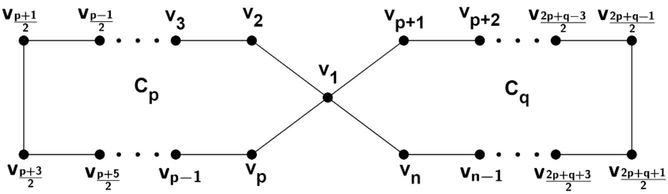

Figure 2. Vertex labeling of a graph $G \in B_1(n)$ where C_p and C_q are odd cycles

■

Lemma 3.2. *Let $G \in B_1(n)$ be a graph on $n = p+q-1$ vertices such that C_p and C_q are even cycles. Then*

$$ABC_{GG}(G) = 2\,p\sqrt{\frac{p+q-3}{p\,(p+2\,q-2)}} + 2\,q\sqrt{\frac{p+q-3}{q\,(2\,p+q-2)}}. \tag{5}$$

Proof. Let G be the graph labeled as Figure 3. Let $H_1 = G[C_p]$ be the graph induced by vertices $\{v_1, v_2, \ldots, v_p\}$ and consider the edge $(v_1, v_p) \in E(H_1)$. Note that $n_{v_1} = \frac{p+2q-2}{2}$ and $n_{v_p} = \frac{p}{2}$. Taking the advantage of the symmetry, we can observe that this same situation occurs p times which can be written as $n_{v_i} = \frac{p+2q-2}{2}$ and $n_{v_{i+1}} = \frac{p}{2}$ for $i \in \{1, \ldots, \frac{p}{2}\}$. For $i \in \{\frac{p}{2}, \ldots, p-1\}$, $n_{v_i} = \frac{p}{2}$ and $n_{v_{i+1}} = \frac{p+2q-2}{2}$. Thus,

$$\begin{aligned}\sum_{uv \in H_1} f(u,v) &= p\sqrt{\frac{\left(\frac{p}{2}\right) + \left(\frac{p+2q-2}{2}\right) - 2}{\left(\frac{p}{2}\right)\left(\frac{p+2q-2}{2}\right)}} \\ &= p\sqrt{\frac{\left(\frac{2p+2q-2-4}{2}\right)4}{p\,(p+2q-2)}} = 2\,p\sqrt{\frac{p+q-3}{p\,(p+2\,q-2)}}.\end{aligned}$$

Now, let $H_2 = G[C_q]$ be the induced graph by vertices $\{v_1, v_{p+1}, \ldots, v_n\}$. Considering edge (v_1, v_n), we get $n_{v_1} = \frac{2p+q-2}{2}$ and $n_{v_n} = \frac{q}{2}$. Analogously to the previous case, we have $n_{v_j} = \frac{2p+q-2}{2}$ and $n_{v_{j+1}} = \frac{q}{2}$ for $j \in \{p+1, \ldots, \frac{2p+q-2}{2}\}$. For $j \in \{\frac{2p+q}{2}, \ldots, n-1\}$, we obtain $n_{v_j} = \frac{q}{2}$ and $n_{v_{j+1}} = \frac{2p+q-2}{2}$. Thus,

$$\sum_{uv \in E(H_2)} f(u,v) = q\sqrt{\frac{\left(\frac{q}{2}\right) + \left(\frac{q+2p-2}{2}\right) - 2}{\left(\frac{q}{2}\right)\left(\frac{q+2p-2}{2}\right)}} = q\sqrt{\frac{\left(\frac{2p+2q-2-4}{2}\right)4}{q\,(p+2q-2)}} = 2\,q\sqrt{\frac{p+q-3}{q\,(2\,p+q-2)}}.$$

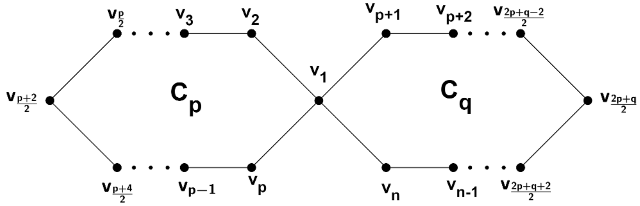

Figure 3. Vertex labeling of a graph $G \in B_1(n)$ where C_p and C_q are even cycles

Therefore,

$$ABC_{GG}(G) = \sum_{uv \in H_1} f(u,v) + \sum_{uv \in H_2} f(u,v) = 2\,p\sqrt{\frac{p+q-3}{p\,(p+2\,q-2)}} + 2\,q\sqrt{\frac{p+q-3}{q\,(2\,p+q-2)}},$$

and the result follows. ■

Lemma 3.3. *Let $G \in B_1(n)$ be a graph on $n = p+q-1$ vertices such that C_p is an odd cycle and C_q is an even cycle. Then*

$$ABC_{GG}(G) = \frac{2\sqrt{p-3}}{p-1} + 2\,(p-1)\sqrt{\frac{p+q-4}{(p+2\,q-3)(p-1)}} + 2\,q\sqrt{\frac{p+q-3}{q\,(2\,p+q-2)}}. \qquad (6)$$

Proof. Let G be the graph labeled as Figure 4. Let $H_1 = G[C_p]$ be the graph induced by vertices $\{v_1, v_2, \ldots, v_p\}$ and consider the edge $(v_1, v_p) \in E(H_1)$. Note that $n_{v_1} = \frac{p+2q-3}{2}$ and $n_{v_p} = \frac{p-1}{2}$. Taking the advantage of the symmetry, we can observe that this same situation occurs $p-1$ times which can be written as $n_{v_i} = \frac{p+2q-3}{2}$ and $n_{v_{i+1}} = \frac{p-1}{2}$ for $i \in \{1, \ldots, \frac{p-1}{2}\}$. For $i \in \left\{\frac{p+3}{2}, \ldots, p-1\right\}$, then $n_{v_i} = \frac{p-1}{2}$ and $n_{v_{i+1}} = \frac{p+2q-3}{2}$. The remaining edge $\left(v_{\frac{p+1}{2}}, v_{\frac{p+3}{2}}\right)$ has $n_{v_{\frac{p+1}{2}}} = n_{v_{\frac{p+3}{2}}} = \frac{p-1}{2}$. Thus,

$$\begin{aligned}\sum_{uv \in H_1} f(u,v) &= (p-1)\sqrt{\frac{\left(\frac{p-1}{2}\right) + \left(\frac{p+2q-3}{2}\right) - 2}{\left(\frac{p-1}{2}\right)\left(\frac{p+2q-3}{2}\right)}} + \sqrt{\frac{\left(\frac{p-1}{2}\right) + \left(\frac{p-1}{2}\right) - 2}{\left(\frac{p-1}{2}\right)\left(\frac{p-1}{2}\right)}} \\ &= 2\,(p-1)\sqrt{\frac{p+q-4}{(p-1)(p+2q-3)}} + \frac{2\sqrt{p-3}}{p-1}.\end{aligned}$$

Now, let $H_2 = G[C_q]$ be the graph induced by vertices $\{v_1, v_{p+1}, \ldots, v_n\}$. Considering the edge (v_1, v_n), we get $n_{v_1} = \frac{2p+q-2}{2}$ and $n_{v_n} = \frac{q}{2}$. Analogously to the previous case, $n_{v_j} = \frac{2p+q-2}{2}$ and $n_{v_{j+1}} = \frac{q}{2}$ for $j \in \left\{p+1, \ldots, \frac{2p+q-2}{2}\right\}$. For $j \in \left\{\frac{2p+q}{2}, \ldots, n-1\right\}$ then $n_{v_j} = \frac{q}{2}$ and $n_{v_{j+1}} = \frac{2p+q-2}{2}$. Thus,

$$\sum_{uv \in E(H_2)} f(u,v) = q\sqrt{\frac{\left(\frac{q}{2}\right) + \left(\frac{2p+q-2}{2}\right) - 2}{\left(\frac{q}{2}\right)\left(\frac{2p+q-2}{2}\right)}} = 2\,q\sqrt{\frac{p+q-3}{q\,(2\,p+q-2)}}.$$

Therefore,

$$ABC_{GG}(G) = \sum_{uv\in H_1} f(u,v) + \sum_{uv\in H_2} f(u,v) = 2\,(p-1)\sqrt{\tfrac{p+q-4}{(p+2\,q-3)(p-1)}} + \tfrac{2\,\sqrt{p-3}}{p-1} +$$

$$+2\,q\sqrt{\tfrac{p+q-3}{q\,(2\,p+q-2)}},$$

and the result follows.

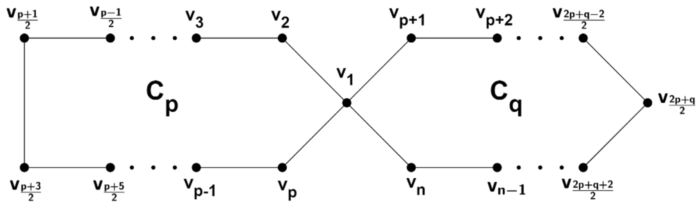

Figure 4. Vertex labeling of a graph $G \in B_1(n)$ where C_p is an odd cycle and C_q is an even cycle

■

4 Minimizing $ABC_{GG}(G)$ for all $G \in B_1(n)$

In order to characterize the graphs with minimal Graovac-Ghorbani index in $B_1(n)$, we performed computational experiments to bicyclic graphs with no pendent vertices up to 16 vertices. These graphs were generated by using Nauty-Traces package [10] and the ABC_{GG} indices were computed in BlueJ software [12, 13].

4.1 When n is odd

We first consider the case when both cycles have odd length. Since $n \geq 9$ and n is odd, we have that $n = 2k - 1$, where $k \geq 5$ and $k \in \mathbb{N}$.

Case (i) : C_p and C_q are odd cycles.

From the symmetry of the process of removing vertices from cycle C_p and adding them to C_q, we get that: if k is odd, $3 \leq p \leq \frac{n+1}{2}$ and $\frac{n+1}{2} \leq q \leq n-2$; if k is even, $3 \leq p \leq \frac{n-1}{2}$ and $\frac{n+3}{2} \leq q \leq n-2$. From these inequalities, we obtain that: if k is odd, then $3 \leq p \leq k$ and $k \leq q \leq 2k-3$; if k is even, then $3 \leq p \leq k-1$ and $k+1 \leq q \leq 2k-3$.

Let $x \in \mathbb{N}$ be the number of vertices removed from C_q and added to C_p. Note that in this process, x should be even to keep both cycles of odd length and that $p = 3 + x$ and $q = 2k - 3 - x$. The following facts are true:

(F1) If $k \geq 5$ is odd, then $0 \leq x \leq k - 3$;

(F2) If $k \geq 5$ is even, then $0 \leq x \leq k - 4$.

Let $G^1(p, q)$ be a graph of order n belonging to $B_1(p, q)$ with fixed p and q such that $n = p + q - 1$. Our proof initially considers the graph with $p = 3$ and $q = n - 2$, which we denote by $G^1_0(3, n-2)$. Removing two vertices from cycle C_{n-2} of the graph $G^1_0(3, n-2)$ and adding them to C_3, we obtain the graph $G^1_1(5, n-4)$. If n and k are odd, we prove that $G^1_0(3, n-2)$ has minimal ABC_{GG} among all graphs $G^1_x(3 + x, n - 2 - x)$, where $0 \leq x \leq (n-5)/2$. If n is odd and k is even, we prove that $G^1_0(3, n-2)$ has minimal ABC_{GG} among all graphs $G^1_x(3 + x, n - 2 - x)$, where $0 \leq x \leq (n-7)/2$.

For instance, for $n = 17$ we have $k = 9$ and all graphs in $B_1(17)$ are: $G^1_0(3, 15)$, $G^1_2(5, 13)$, $G^1_4(7, 11)$, $G^1_6(9, 9)$. In the other hand, if $n = 19$, we have $k = 10$ and all graphs in $B_1(19)$ are: $G^1_0(3, 17)$, $G^1_2(5, 15)$, $G^1_4(7, 13)$, $G^1_6(9, 11)$. In both cases, we will prove in the next lemmas that graph $G^1_0(3, n-2)$ has minimal ABC_{GG}.

Recall Equation (2) of Lemma 3.1 and rewrite it as:

$$\begin{aligned} f_1(x) :&= ABC_{GG}(G) = 2(x+2)\sqrt{\frac{2k-4}{(4k-x-6)(x+2)}} + \\ &+ 2\,(2\,k-x-4)\sqrt{\frac{2k-4}{(2k+x)(2\,k-x-4)}} + \frac{2\,\sqrt{2\,k-x-6}}{2\,k-x-4} + \frac{2\,\sqrt{x}}{x+2}. \end{aligned} \tag{7}$$

Next, we prove that the first three terms of $f_1(x)$ is an increasing function in x.

Lemma 4.1. *Let $n = 2k - 1$ such that $k \geq 5$ and let $x \geq 0$. Then, the function*

$$\begin{aligned} g_1(x) &= 2(x+2)\sqrt{\frac{2k-4}{(4k-x-6)(x+2)}} + 2\,(2\,k-x-4)\sqrt{\frac{2k-4}{(2k+x)(2\,k-x-4)}} \\ &+ \frac{2\,\sqrt{2\,k-x-6}}{2\,k-x-4} \end{aligned}$$

is increasing in x.

Proof. We will use Facts (F1) and (F2) to prove our result. We split $g_1(x)$ into two cases:

Case 1. Let $h(x)=\frac{2\sqrt{2\,k-x-6}}{2\,k-x-4}$. We have that $h'(x)$ is given by

$$h'(x)=\frac{2\,k-x-8}{(2\,k-x-4)^2\sqrt{2\,k-x-6}}.$$

Assume that $k\geq 5$ is odd. From (F1), we have $2\,k-x-8\geq 0$ and $2\,k-x-6\geq 0$. Now, assume that $k\geq 5$ is even. From (F2), we have $2\,k-x-8\geq 0$ and $2\,k-x-6\geq 0$. Then $h'(x)>0$, which means that $h(x)$ is increasing in x for all $k\geq 5$.

Case 2. Let $m(x)=2(x+2)\sqrt{\frac{2k-4}{(4k-x-6)(x+2)}}+2\,(2\,k-x-4)\sqrt{\frac{2k-4}{(2k+x)(2\,k-x-4)}}$. We have that $m'(x)$ is given by

$$\begin{aligned}
m'(x) &= \frac{8(k-1)(k-2)}{(x+2)(4\,k-x-6)^2\sqrt{\frac{2k-4}{(x+2)(4\,k-x-6)}}}-\frac{8(k-1)(k-2)}{(2k-x-4)(2k+x)^2\sqrt{\frac{2k-4}{(2k-x-4)(2k+x)}}}\\
&= \frac{8(k-1)(k-2)}{\sqrt{(2k-4)}\left(\frac{(x+2)(4\,k-x-6)^2}{\sqrt{(x+2)(4\,k-x-6)}}\right)}-\frac{8(k-1)(k-2)}{\sqrt{(2k-4)}\left(\frac{(2k-x-4)(2k+x)^2}{\sqrt{(2k-x-4)(2k+x)}}\right)}
\end{aligned}$$

Define $d_1(x)$ and $d_2(x)$ as

$$d_1(x)=\frac{(x+2)(4\,k-x-6)^2}{\sqrt{(x+2)(4\,k-x-6)}}=(4k-x-6)\sqrt{(4k-x-6)(x+2)},$$

$$d_2(x)=\frac{(2k-x-4)\,(2k+x)^2}{\sqrt{(2k-x-4)(2k+x)}}=(2k+x)\sqrt{(2k+x)(2k-x-4)}.$$

If $d_2(x)\geq d_1(x)$, then $m'(x)\geq 0$ and $m(x)$ is increasing in x. Let $t(x)=d_2(x)^2-d_1(x)^2$. By algebraic manipulations, we have that

$$\begin{aligned}
t(x) &= d_2(x)^2-d_1(x)^2\\
&= (2k+x)^3(2k-x-4)-(x+2)(4k-x-6)^3\\
&= 16\,k^4+16\,k^3x-4\,kx^3-x^4-32\,k^3-48\,k^2x-24\,kx^2-4\,x^3\\
&\quad -(64\,k^3x-48\,k^2x^2+12\,kx^3-x^4+128\,k^3-384\,k^2x+\\
&\quad +168\,kx^2-20\,x^3-576\,k^2+720\,kx-144\,x^2+864\,k-432\,x-432)\\
&= 16\,k^4-48\,k^3x+48\,k^2x^2-16\,kx^3-160\,k^3+336\,k^2x-\\
&\quad -192\,kx^2+16\,x^3+576\,k^2-720\,kx+\\
&\quad +144\,x^2-864\,k+432\,x+432\\
t(x) &= 16(k-1)(k-x-3)^3.
\end{aligned}$$

Note that $x = k-3$ is the root of $t(x)$. Using Facts (F1) and (F2), we get $t(x) > 0$. We have that $t(x) = d_2(x)^2 - d_1(x)^2 = (d_2(x) - d_1(x))(d_2(x) + d_1(x)) \geq 0$, then $d_2(x) - d_1(x) \geq 0$. Thus, $m'(x) > 0$, which means that $m(x)$ is increasing in x for all $k \geq 5$.
Therefore, from Cases 1 and 2, we have that $g_1'(x) > 0$, and so $g_1(x)$ is increasing in x for all $k \geq 5$. ■

Lemma 4.2. *Let $n = 2k-1$ such that $k \geq 5$. Let $x \geq 0$ and*

$$\begin{aligned} f_1(x) &= 2(x+2)\sqrt{\frac{2k-4}{(4k-x-6)(x+2)}} + 2\,(2\,k-x-4)\sqrt{\frac{2k-4}{(2k+x)(2\,k-x-4)}} \\ &+ \frac{2\,\sqrt{2\,k-x-6}}{2\,k-x-4} + \frac{2\,\sqrt{x}}{x+2}. \end{aligned}$$

Then, we have that $f_1(0)$ is the minimal value of $f_1(x)$.

Proof. Let $g_1(x)$ be defined by

$$\begin{aligned} g_1(x) &= 2(x+2)\sqrt{\frac{2k-4}{(4k-x-6)(x+2)}} + 2\,(2\,k-x-4)\sqrt{\frac{2k-4}{(2k+x)(2\,k-x-4)}} \\ &+ \frac{2\,\sqrt{2\,k-x-6}}{2\,k-x-4}. \end{aligned}$$

From Lemma 4.1 we have that $g_1(x)$ is increasing in x for all $k \geq 5$, which implies that

$$g_1(x) \geq g_1(0).$$

So, $f_1(x) \geq g_1(x) \geq g_1(0) = f_1(0)$, and the result follows. ■

Note that $f_1(0) = ABC_{GG}(G_0^1(3, n-2))$ and Lemma 4.2 implies that $f_1(x)$ is minimized to the graph $G_0^1(3, n-2)$ among all graphs $G_i^1(3, n-2)$ when C_p and C_q have odd lengths.

Case (ii) : C_p and C_q are even cycles.

From the symmetry of the process of removing vertices from cycle C_p and adding them to C_q, we get that: if k is odd, $4 \leq p \leq \frac{n-1}{2}$ and $\frac{n+3}{2} \leq q \leq n-3$; if k is even, $4 \leq p \leq \frac{n+1}{2}$ and $\frac{n+1}{2} \leq q \leq n-3$. From these inequalities, we get: if k is odd, $4 \leq p \leq k-1$ and $k+1 \leq q \leq 2k-4$; if k is even, $4 \leq p \leq k$ and $k \leq q \leq 2k-4$.

Let $x \in \mathbb{N}$ be the number of vertices removed from C_q and added to C_p. Note that in this process, x should be even to keep both cycles of even length and that $p = 4 + x$ and $q = 2k - 4 - x$. The following facts are true:

(F3) If $k \geq 5$ is odd, then $0 \leq x \leq k-5$;

(F4) If $k \geq 5$ is even, then $0 \leq x \leq k-4$.

We will prove that for $k \geq 11$, the graph $G_0^1(4, n-3)$ has minimal ABC_{GG} index among all graph when C_p and C_q are both even cycles.

So, we can rewrite Equation (5) of Lemma 3.2 as:

$$f_2(x) := 2\,(x+4)\sqrt{\frac{2\,k-3}{(4\,k-x-6)(x+4)}} + 2\,(2\,k-x-4)\sqrt{\frac{2\,k-3}{(2\,k+x+2)(2\,k-x-4)}}. \qquad (8)$$

Lemma 4.3. *Let $n = 2k-1$ with $k \geq 5$. Let $x \geq 0$ and*

$$f_2(x) = 2\,(x+4)\sqrt{\frac{2\,k-3}{(4\,k-x-6)(x+4)}} + 2\,(2\,k-x-4)\sqrt{\frac{2\,k-3}{(2\,k+x+2)(2\,k-x-4)}}.$$

If $k \geq 11$, then $f_2(0)$ is the minimal value of $f_2(x)$. If k is odd and $5 \leq k \leq 10$, then $f_2(k-5)$ is the minimal value of $f_2(x)$. If k is even and $5 \leq k \leq 10$, then $f_2(k-4)$ is the minimal value of $f_2(x)$.

Proof. The functions $f_2'(x)$ and $f_2''(x)$ are given by

$$\begin{aligned} f_2'(x) &= \frac{2\,(2\,k-1)(2\,k-3)}{(4\,k-x-6)^2\,(x+4)\sqrt{\frac{2\,k-3}{(4\,k-x-6)(x+4)}}} \\ &\quad - \frac{2\,(2\,k-1)(2\,k-3)}{(2\,k+x+2)^2(2\,k-x-4)\sqrt{\frac{2\,k-3}{(2\,k+x+2)(2\,k-x-4)}}} \end{aligned}$$

$$\begin{aligned} f_2''(x) &= \frac{2\,(2\,k-1)(2\,k-3)^2(2\,k-2x-7)}{(2\,k+x+2)^4(2\,k-x-4)^3\left(\frac{2\,k-3}{(2\,k+x+2)(2\,k-x-4)}\right)^{\frac{3}{2}}} \\ &\quad - \frac{2(2\,k-1)(2\,k-3)^2(2\,k-2x-9)}{(x+4)^3\,(4\,k-x-6)^4\left(\frac{2\,k-3}{(x+4)(4k-x-6)}\right)^{\frac{3}{2}}} \end{aligned}$$

Taking $f_2'(x) = 0$, we obtain $4(k-x-4)(4kx^2 - 3x^2 - 8k^2x + 38kx - 24x + 4k^3 - 44k^2 + 100k - 52) = 0$, which has critical points $x_1 = k-4$, $x_2 = k-4+\frac{(3k-2)}{\sqrt{(4k-3)}}$ and $x_3 = k-4-\frac{(3k-2)}{\sqrt{(4k-3)}}$. Suppose that x_3 is integer. In this case, $(4k-3)$ should be a perfect square, that is, $4k-3 = m^2$. Thus, $\frac{(3k-2)}{\sqrt{4k-3}} = \frac{\frac{3m^2}{4}+\frac{1}{4}}{m} = \frac{3m}{4} + \frac{1}{4m}$ cannot be integer for $m > 1$, and we get a contradiction. So, the only critical points are $x = 0$ and $x = k-4$.

We get that

$$f_2''(k-4) = \frac{4\,(2\,k-1)(2\,k-3)^2}{(3\,k-2)^4 k^3 \left(\frac{2\,k-3}{k(3\,k-2)}\right)^{\frac{3}{2}}},$$

is positive, and so $x = k-4$ is a minimum of the function $f_2(x)$.

Let $k \geq 5$ and k is odd. In this case, $0 \leq x \leq k-5$ and we need to prove whether $f_2(0) > f_2(k-5)$ or $f_2(0) < f_2(k-5)$. Let

$$m(k) = f_2(k-5) - f_2(0) \;=\; 2\,(k+1)\sqrt{\frac{2\,k-3}{3(k+1)(k-1)}} + 2\,(k-1)\sqrt{\frac{2\,k-3}{(3\,k-1)(k-1)}} - \\ -2\,(k-2)\sqrt{\frac{2\,k-3}{(k+1)(k-2)}} - 2\sqrt{2}$$

The derivatives of $m(k)$ are given as follows:

$$m'(k) = \frac{2\sqrt{3}\left(k^2-2\,k+2\right)}{3\sqrt{(2\,k-3)(k+1)(k-1)}(k-1)} + \frac{2\left(3\,k^2-2\,k-2\right)}{\sqrt{(3\,k-1)(2\,k-3)(k-1)}(3\,k-1)} - \frac{2\,k^2+4\,k-13}{\sqrt{(2\,k-3)(k+1)(k-2)}(k+1)}$$

$$m''(k) = \frac{4\,k^4+16\,k^3-156\,k^2+316\,k-191}{2\sqrt{(2\,k-3)(k+1)(k-2)}(2\,k-3)(k+1)^2(k-2)} - \frac{2\sqrt{3}\left(k^4-4\,k^3+12\,k^2-10\,k-2\right)}{3\sqrt{(2\,k-3)(k+1)(k-1)}(2\,k-3)(k+1)(k-1)^2} - \\ - \frac{2\left(9\,k^4-12\,k^3-36\,k^2+78\,k-38\right)}{\sqrt{(3\,k-1)(2\,k-3)(k-1)}(3\,k-1)^2(2\,k-3)(k-1)}$$

We know that $m(k)$ and $m'(k)$ are continuous in this interval. Using a numerical method, we get $k_1 = 5$ and $k_2 \simeq 10.3147$ as exact real roots for $m(k)$. Using a numerical method for $m'(k)$, we obtain $k_1' \simeq 7.5248$. Evaluating $m''(k)$ for k_1', we have that $m''(7.5248) > 0$, and k_1' is an absolute minimum. Since $m(k) < 0$ for $5 < k \leq 10$, $m(k_1') < 0$ and $m(11) > 0$, then, we have that $m(k) \geq 0$ for all $k \geq 10.3147$. Therefore, $m(k) > 0$ for all $k \geq 11$, which implies that $f_2(x) \geq f_2(0)$, and the graph $G_0(4, n-3)$ minimizes $f_2(x)$. Now, let k be odd and $5 \leq k \leq 10$. In this case, $m(k) \leq 0$, which implies that $f_2(k-5)$ is minimum and the graph $G_{k-5}^1(p,q) = G_{k-5}^1(k-1, k+1)$ minimizes $f_2(x)$.

If k is even, since $0 \leq x \leq k-4$, we need to prove that $f_2(0) < f_2(k-4)$. Replacing the extremes $x_1 = 0$ and $x_2 = k-4$ in $f_2(x)$ we obtain

$$f_2(0) = 2\,(k-2)\sqrt{\frac{2\,k-3}{(k+1)(k-2)}} + 2\sqrt{2},$$

$$f_2(k-4) = 4\,k\sqrt{\frac{2\,k-3}{k(3\,k-2)}}.$$

Consider the function

$$h(k) \;=\; f_2(k-4) - f_2(0) = 4\,k\sqrt{\frac{2\,k-3}{k(3\,k-2)}} - 2\,(k-2)\sqrt{\frac{2\,k-3}{(k+1)(k-2)}} - 2\sqrt{2}$$

The derivatives of $h(k)$ are given as follows:

$$h'(k) = \frac{4\left(3k^2-4k+3\right)}{\sqrt{(3k-2)(2k-3)k}(3k-2)} - \frac{2k^2+4k-13}{\sqrt{(2k-3)(k+1)(k-2)}(k+1)}$$

$$h''(k) = \frac{4k^4+16k^3-156k^2+316k-191}{2\sqrt{(2k-3)(k+1)(k-2)}(2k-3)(k+1)^2(k-2)} - \frac{12\left(3k^4-8k^3+18k^2-18k+3\right)}{\sqrt{(3k-2)(2k-3)k}(3k-2)^2(2k-3)k}.$$

We know that $h(k)$ and $h'(k)$ are continuous in this interval. Using a numerical method, we get $k_1 = \frac{21+5\sqrt{17}}{4} \simeq 10.4039$ and $k_2 = 4$ (with multiplicity 2) as exact real roots for $h(k)$. Making $h'(k) = 0$, we obtain

$$\left(36k^7 - 312k^6 + 172k^5 + 1600k^4 - 2595k^3 + 1546k^2 - 332k + 72\right)(2k-3)(k-4) = 0,$$

and the critical points are $k'_1 = 4$ and $k'_2 = 7.3648$. Evaluating $h''(k)$ in k'_2, we have that $h''(7.3648) > 0$, and k_2 is an absolute minimum. Since $h(3) < 0, h(8) < 0, h(k'_2) < 0$ and $h(11) > 0$, we have that $h(k) \geq 0$ for all $k \geq \frac{21+5\sqrt{17}}{4}$. Therefore, $h(k) > 0$ for all $k \geq 11$, which implies that $f_2(x) \geq f_2(0)$, and the graph $G_0^1(4, n-3)$ minimizes $f_2(x)$. Now, let k be even and $5 \leq k \leq 10$. In this case, $h(k) < 0$, which implies that $f_2(k-4)$ is minimum and the graph $G_{k-4}^1(p,q) = G_{k-4}^1(k,k)$ minimizes $f_2(x)$. ■

The next result shows that $G_0^1(3, n-2)$ has minimal ABC_{GG} among all graphs G in $B_1(n)$ when n is odd.

Lemma 4.4. *Let $G \in B_1(n)$ with odd n and $n \geq 9$. Then,*

$$ABC_{GG}(G) \geq ABC_{GG}(G_0^1(3, n-2)).$$

Proof. Let $n = 2k-1$ such that $n \geq 9$. First, suppose that $k \geq 11$. From Lemmas 4.2 and 4.3, we should prove that $f_1(0) < f_2(0)$. Considering $h(k) = f_2(0) - f_1(0)$, we have that

$$h(k) = 2(k-2)\sqrt{\frac{2k-3}{(k+1)(k-2)}} + 2\sqrt{2} - \left(\frac{2\sqrt{2}(k-2)}{\sqrt{k}} + 2\sqrt{\frac{2k-4}{2k-3}} + \frac{\sqrt{2k-6}}{k-2}\right)$$

$$h(k) = 2(k-2)\sqrt{\frac{2k-3}{(k+1)(k-2)}} + \frac{\sqrt{2}}{5} + \frac{9\sqrt{2}}{5} - \left(\frac{2\sqrt{2}(k-2)}{\sqrt{k}} + 2\sqrt{\frac{2k-4}{2k-3}} + \frac{\sqrt{2k-6}}{k-2}\right)$$

$$h(k) = \underbrace{2(k-2)\sqrt{\frac{2k-3}{(k+1)(k-2)}} + \frac{\sqrt{2}}{5} - \frac{2\sqrt{2}(k-2)}{\sqrt{k}}}_{i} + \underbrace{\frac{9\sqrt{2}}{5} - 2\sqrt{\frac{2k-4}{2k-3}} - \frac{\sqrt{2k-6}}{k-2}}_{ii}$$

Case (i). Let $m(k)$ be defined as

$$m(k) = 2(k-2)\sqrt{\frac{2k-3}{(k+1)(k-2)}} + \frac{\sqrt{2}}{5} - \frac{2\sqrt{2}(k-2)}{\sqrt{k}}$$

$$= \frac{\sqrt{k}\sqrt{2k-3}(10k-20)-\sqrt{2}\sqrt{k-2}\sqrt{k+1}(10k-\sqrt{k}-20)}{5\sqrt{k-2}\sqrt{k}\sqrt{k+1}}.$$

Let $t(k) = d_1(k)^2 - d_2(k)^2$, where $d_1(k) = \sqrt{k}\sqrt{2k-3}(10k-20)$ and $d_2(k) = -\sqrt{2}\sqrt{k-2}\sqrt{k+1}(10k-\sqrt{k}-20)$. We need to verify if $t(k) > 0$. Then, we have that,

$$\begin{aligned} t(k) &= d_1(k)^2 - d_2(k)^2 \\ &= k(2k-3)(10k-20)^2 - 2(k-2)(k+1)(10k-\sqrt{k}-20)^2 \\ &= 2(k-2)(20k^2\sqrt{k} - 51k^2 - 20k\sqrt{k} + 299k - 40\sqrt{k} - 400) \end{aligned}$$

Let $\sqrt{k} = u \geq 0$ and $r(k) = 20k^2\sqrt{k} - 51k^2 - 20k\sqrt{k} + 299k - 40\sqrt{k} - 400$. Then, $r(u) = 20u^5 - 51u^4 - 20u^3 + 299u^2 - 40u - 400$. We have that $r(u) \geq 0$ for all $u \geq u_1 \simeq 1.42596$ (the unique real root). Then, $r(k) \geq 0$ for all $k \geq k_1 \simeq 2.03336$. Since $k \geq 5$, we get $t(k) > 0$ and consequently $m(k) > 0$ for all $k \geq 5$.

Case (ii). Let $g(k)$ be defined as

$$\begin{aligned} g(k) &= \frac{9\sqrt{2}}{5} - 2\sqrt{\frac{2k-4}{2k-3}} - \frac{\sqrt{2\,k-6}}{k-2} \\ &= \frac{\sqrt{2}\sqrt{2k-3}(9k-5\sqrt{k-3}-18) - 10\sqrt{k-2}\sqrt{2}(k-2)}{5(k-2)\sqrt{2k-3}}. \end{aligned}$$

Let $t(k) = d_1(k)^2 - d_2(k)^2$, where $d_1(k) = 2(2k-3)(9k-5\sqrt{k-3}-18)^2$ and $d_2(k) = -10\sqrt{k-2}\sqrt{2}(k-2)$. We need to check whether $t(k) \geq 0$. Note that

$$\begin{aligned} t(k) &= d_1(k)^2 - d_2(k)^2 \\ &= 2(2k-3)(9k-5\sqrt{k-3}-18)^2 - 200(k-2)^3 \\ &= 2(62k^3 - 180k^2\sqrt{k-3} - 241k^2 + 630k\sqrt{k-3} + 195k - 540\sqrt{k-3} + 53). \end{aligned}$$

By making the variable change $u = \sqrt{k-3}$ we can prove that $t(k) \geq 0$ for all $k \geq 3$, which implies that $g(k) \geq 0$. Therefore, from Cases (i) and (ii), we have that $h(k) > 0$.

Let k be even such that $5 \leq k \leq 10$. From Lemmas 4.2 and 4.3, we should prove that $f_1(0) < f_2(k-4)$. Let $h(k) = f_2(k-4) - f_1(0)$ defined as

$$h(k) = 4\,k\sqrt{\frac{2\,k-3}{(3\,k-2)k}} - \frac{2\,\sqrt{2}(k-2)}{\sqrt{k}} - 2\sqrt{\frac{2k-4}{2\,k-3}} - \frac{\sqrt{2\,k-6}}{k-2}.$$

By replacing each k in $h(k)$ we obtain that $h(k) \geq 0$.

Let k be odd such that $5 \leq k \leq 10$. From Lemmas 4.2 and 4.3, we should prove that $f_1(0) < f_2(k-5)$. Let $h(k) = f_2(k-5) - f_1(0)$ defined as

$$\begin{aligned} h(k) &= 2\,(k+1)\sqrt{\frac{2\,k-3}{3(k+1)(k-1)}} + 2\,(k-1)\sqrt{\frac{2\,k-3}{(3\,k-1)(k-1)}} - \frac{2\,\sqrt{2}(k-2)}{\sqrt{k}} - \\ &\quad -2\sqrt{\frac{2k-4}{2\,k-3}} - \frac{\sqrt{2\,k-6}}{k-2}. \end{aligned}$$

By replacing each k in $h(k)$ we obtain that $h(k) \geq 0$.

Now, the proof is complete. ■

4.2 When n is even

Note that when $n = 2k$ such that $n = p + q - 1$. We should conclude that one of the cycles has odd length and the other has even length. Recall the graph $G_0^1(3, n-2)$. From the symmetry of the process of removing vertices from cycle C_q and adding them to C_q, we get that $3 \leq p \leq n-3$ and $4 \leq q \leq n-2$, which implies that

(F5) $3 \leq p \leq 2k-3$ and $4 \leq q \leq 2k-2$.

Let $x \in \mathbb{N}$ be the number of vertices removed from C_q and added to C_p. Note that in this process, x should be even to keep both cycles of even length such that $p = 3 + x$, $q = 2k - 2 - x$ and

$$0 \leq x \leq 2k-6.$$

Next, we prove that $G_0^1(3, n-2)$ has minimal $ABC_{GG}(G)$ among all graphs $G \in B_1(n)$ with n even. Equation (6) of Lemma 3.3 can be rewritten as a function of n and x:

$$f_2(x) = 2\,(n-x-2)\sqrt{\frac{n-2}{(n+x+2)(n-x-2)}} + 2\,(x+2)\sqrt{\frac{n-3}{(2\,n-x-4)(x+2)}} + \frac{2\,\sqrt{x}}{x+2}.$$

Note that $f(0)$ is equal to $ABC_{GG}(G_0^1(3, n-2))$.

Lemma 4.5. *Let $n \geq 5$ and $x \geq 0$. Let $g_2(x)$ be defined as*

$$g_2(x) = 2\,(n-x-2)\sqrt{\frac{n-2}{(n+x+2)(n-x-2)}} + 2\,(x+2)\sqrt{\frac{n-3}{(2\,n-x-4)(x+2)}}.$$

Then, $g_2(x)$ has its minimum value in $g_2(0)$.

Proof. Let $n \geq 5, x_1 = x$ and $x_2 = x + 2$. In order to prove that $g_2(x)$ is increasing in x, we need to prove that $g_2(x_2) - g_2(x_1) \geq 0$. Take $h(x) = g_2(x_2) - g_2(x_1)$. Note that

$$\begin{aligned} h(x) &= -2\,(n-x-2)\sqrt{\frac{n-2}{(n+x+2)(n-x-2)}} + 2\,(n-x-4)\sqrt{\frac{n-2}{(n+x+4)(n-x-4)}} \\ &+ 2\,(x+4)\sqrt{\frac{n-3}{(2\,n-x-6)(x+4)}} - 2\,(x+2)\sqrt{\frac{n-3}{(2\,n-x-4)(x+2)}}. \end{aligned}$$

One can prove that

$$h(0) = 2\,(n-4)\sqrt{\frac{n-2}{(n+4)(n-4)}} + 2\,\sqrt{2} - \frac{2\,(n-2)}{\sqrt{n+2}} - 2\,\sqrt{\frac{n-3}{n-2}} \geq 0,$$

and

$$h(n-6) = -2\,(n-4)\sqrt{\frac{n-3}{(n+2)(n-4)}} + 2\,(n-2)\sqrt{\frac{n-3}{(n-2)n}} - 2\,\sqrt{2} + 2\,\sqrt{\frac{n-2}{n-1}} \geq 0.$$

By using numerical analysis, we get that polynomial $h(x)$ has no root for $0 \leq x \leq n-6$. So, $h(x) \geq 0$, which implies that $g_2(x)$ is increasing and the result follows. ■

Lemma 4.6. *Let $n \geq 5$ and $x \geq 0$. Let $f_2(x)$ be defined as*

$$f_2(x) = 2\,(n-x-2)\sqrt{\frac{n-2}{(n+x+2)(n-x-2)}} + 2\,(x+2)\sqrt{\frac{n-3}{(2\,n-x-4)(x+2)}} + \frac{2\,\sqrt{x}}{x+2}. \quad (9)$$

Then, $f_2(x)$ has its minimum value in $f_2(0)$ for all $n \geq 5$.

Proof. Note that $f_2(x) \geq g_2(x)$. From Lemma 4.5, $g_2(x) \geq g_2(0)$. Therefore, $f_2(x) \geq g_2(x) \geq g_2(0) = f_2(0)$, and the result follows. ■

Next, we state the main result of this paper.

Theorem 4.7. *Let $G \in B_1(n)$ be a graph of order $n \geq 9$. If n is odd, then*

$$ABC_{GG}(G) \geq \frac{2(n-3)}{\sqrt{n-1}} + 2\sqrt{\frac{n-3}{n-2}} + \frac{2\sqrt{n-5}}{n-3}.$$

If n is even, then

$$ABC_{GG}(G) \geq 2\,\sqrt{\frac{n-3}{n-2}} + \frac{2\,(n-2)}{\sqrt{n+2}}.$$

Equality holds in both cases if and only if $G \cong G_0^1(3, n-2)$.

Proof. Let $G \in B_1(n)$. Suppose that n is odd. From Lemma 4.4, $ABC_{GG}(G) = f(x) \geq f(0) = ABC_{GG}(G_0^1(3, n-2))$. Now, suppose that n is even. From Lemma 4.6, $ABC_{GG}(G) = f''(x) \geq f''(0) = ABC_{GG}(G_0^1(3, n-2))$ and the result follows.

■

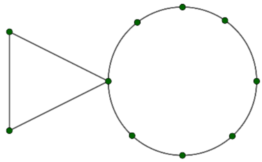

Figure 5. Graph of $B_1(n)$ family with minimal value of the ABC_{GG} index for $n = 10$.

In Figure 5, the extremal graph $G_0^1(3, 8)$ is displayed.

Note that Theorem 4.7 states the extremal graphs for $n \geq 9$. In Figure 6, we display all graphs up to 10 vertices that are extremal to the ABC_{GG} index in the family $B_1(n)$ obtained by exhaustive computational search.

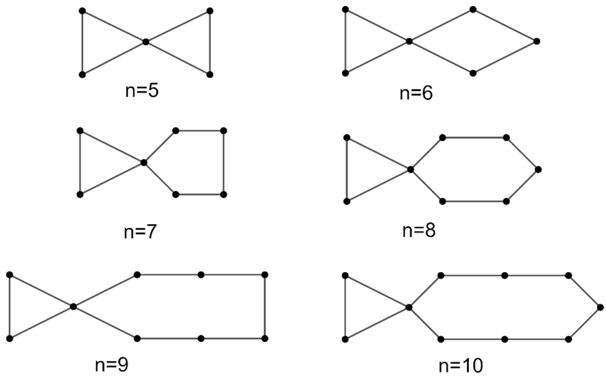

Figure 6. Graph of $B_1(n)$ family with minimal value of the ABC_{GG} index for $5 \leq n \leq 10$.

5 Conclusion

We finish this paper by presenting two conjectures related to the ABC_{GG} index for any bicyclic graph. The following conjectures were motivated by computational experiments for all bicyclic graphs up to 16 vertices. The computational routines in Python are freely available at `https://github.com/20445/ProjetoTeste`.

Let $\mathcal{B}'_n$ be the family of all bicyclic graphs on n vertices. The next conjecture states a lower bounds to the ABC_{GG} index among all graphs in $\mathcal{B}'_n$. It is worth mentioning that the extremal graphs belong to the family $\mathcal{B}_n$, that is, the bicylic graphs with no pendent vertices. This fact makes the study of all graphs in $B_1(n), B_2(n)$, and $B_3(n)$ useful to prove the general case.

Conjecture 5.1. *Let $G \in \mathcal{B}'_n$ be a bicyclic graph of order $n \geq 9$. If n is odd, then*

$$ABC_{GG}(G) \geq 2(n+1)\sqrt{\frac{n-2}{n^2-1}}.$$

If n is even, then

$$ABC_{GG}(G) \geq \frac{6}{n}\sqrt{n-2} + 2(n-2)\sqrt{\frac{1}{n+2}}.$$

For n odd, equality holds if and only if $G \cong B_3(4, 2, n-1)$. For n even, equality holds if and only if $G \cong B_3(6, 3, n-2)$.

Figure 7 displays the extremal graphs of Conjecture 5.1 according to the partity of n.

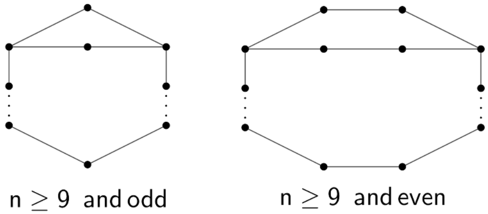

Figure 7. Bicyclic graphs with minimal value of ABC_{GG} index for $n \geq 9$.

Next, we present a conjecture about the upper bound to the ABC_{GG} index for all bicyclic graphs. Let H be the graph obtained by adding $n-4$ pendent vertices to one vertex of degree 3 of the complete graph K_4 minus an edge. Figure 8 displays the graphs with maximal ABC_{GG} index for $n \geq 4$, and the graph H is the last graph for $n \geq 8$.

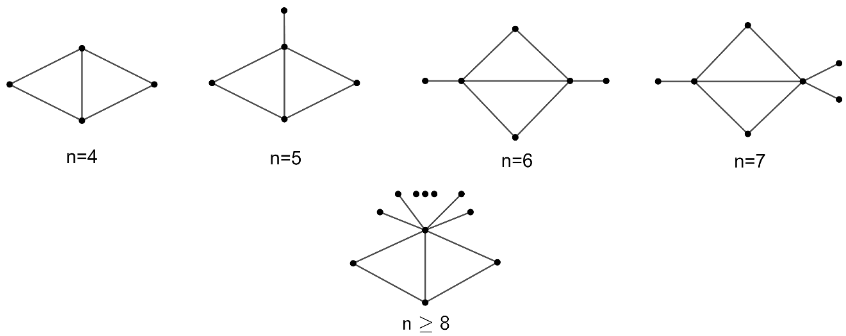

Figure 8. Bicyclic graphs with maximal value of ABC_{GG} index for $n \geq 4$.

Conjecture 5.2. *Let $G \in \mathcal{B}'_n$ a bicyclic graph with order $n \geq 8$. Then,*

$$ABC_{GG}(G) \leq (n-4)\sqrt{\frac{n-2}{n-1}} + \sqrt{\frac{n-4}{n-3}} + 2\sqrt{\frac{n-3}{n-2}} + \sqrt{2}.$$

Equality holds if and only if G is isomorphic to H.

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