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The Steiner Wiener Index of Trees with Given Bipartition*

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Abstract

The Wiener index $W(G)$ of a connected graph G is defined as

$$W(G) = \sum_{\{u,v\} \in V(G)} d_G(u,v)\,,$$

where $d_G(u,v)$ is the distance between the vertices u and v in G. For $S \subseteq V(G)$, the Steiner distance $d(S)$ of the vertices of S, introduced by Chartrand et al. in 1989, is the minimum size of a connected subgraph of G whose vertex set contains S. For an integer $k \geq 1$, the Steiner k-Wiener index $SW_k(G)$ of G, introduced by Li, Mao, and Gutman, is $\sum_{S \subseteq V(G), |S|=k} d(S)$. Clearly, $SW_2(G) = W(G)$ for a connected graph G. Li, Mao, and Gutman proved that for any tree T,

$$SW_k(T) = \sum_{e \in E(T)} \sum_{i=1}^{k-1} \binom{n_1(e)}{i} \binom{n_2(e)}{k-i}.$$

Using Vandermonde's convolution formula, we reformulate it as

$$SW_k(T) = (n-1)\binom{n}{k} - \sum_{e \in E(T)} \left[\binom{n_1(e)}{k} + \binom{n_2(e)}{k}\right]$$

for any tree T of order n. Thereby, we determine the minimum and the maximum Steiner k-Wiener index of trees with given bipartition. This extends the results on Wiener index of trees with given bipartition due to Du (International Journal of Quantum Chemistry 112 (2012) 1598-1605).

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1 Introduction

All graphs in this paper are undirected, finite, and simple. We refer to [2] for graph theoretical notation and terminology not described here. Let G be a graph. For two vertices $u, v \in V(G)$, the *distance* $d(u,v) = d_G(u,v)$ between u and v is the length of a shortest path connecting u and v. For more details on this subject, one may see [3, 9]. The *Wiener index* $W(G)$ of a connected graph is defined by

$$W(G) = \sum_{u,v \in V(G)} d_G(u,v).$$

Mathematicians studied this graph invariant since the 1970s in [8]. Information on chemical applications of the Wiener index can be found in [19, 20]. The Steiner distance of a graph, introduced by Chartrand et al. in [5] in 1989, is a natural and nice generalization of the concept of the classical graph distance. For a graph $G = (V, E)$ and a set $S \subseteq V$, an *S-Steiner tree* or a *Steiner tree connecting S* (or simply, an *S-tree*) is a subgraph $H = (V', E')$ of G that is a tree with $S \subseteq V'$. The *Steiner distance* $d(S)$ among the vertices of S (or simply the *distance* of S) is the minimum size of a connected subgraph of G such that $S \subseteq V(H)$. It is clear that H must be a tree, and if $|S| = k$, then $d(S) \geq k-1$. For more details on Steiner distance, we refer to [1, 4–6, 9, 17].

As a generalization of Wiener index, the *Steiner k-Wiener index* of a connected graph G, denoted by $SW_k(G)$, was introduced by Li, Mao, Gutman in [11]:

$$SW_k(G) = \sum_{S \subseteq V(G), |S|=k} d(S).$$

It is clear that for a connected graph G of order n,

$$SW_1(G) = 0, \;\; SW_2(G) = W(G), \;\; SW_n(G) = n-1.$$

For various results on Steiner Wiener index, we refer to a survey paper [14] and [10, 11, 15, 16, 18].

A bipartite graph G is a graph whose vertices can be partitioned into two disjoint subsets $V_1(G)$ and $V_2(G)$ such that every edge connects a vertex in $V_1(G)$ to one in $V_2(G)$. If $|V_1(G)| = p$ and $|V_2(G)| = q$ with $p \geq q \geq 1$, then we say G has a (p,q)-bipartition. Let $H(r; x, y)$ be the tree obtained by attaching x and y pendant vertices, respectively, to the two end vertices of the path of order r, where $r \geq 1$, $x \geq y \geq 0$. For integers p, q

with $p \geq q \geq 1$, let $S(p,q) = H(2; p-1, q-1)$. Obviously, $S(p,q)$ has (p,q)-bipartition. Du [7] showed that for any tree T with (p,q)-bipartition,

$$W(S(p,q)) \leq W(T) \leq W(H(2q-1; x, y)),$$

where $x = \lceil \frac{p-q+1}{2} \rceil, y = \lfloor \frac{p-q+1}{2} \rfloor$.

In this paper, we extend the above results to Steiner k-Wiener index by showing the following result for any $k \geq 2$: for a tree T with (p,q)-bipartition,

$$SW_k(S(p,q)) \leq SW_k(T) \leq SW_k(H(2q-1; x, y)),$$

where $x = \lceil \frac{p-q+1}{2} \rceil, y = \lfloor \frac{p-q+1}{2} \rfloor$.

2 Minimum Steiner Wiener index of trees

We start with some useful notation. For a graph $G = (V(G), E(G))$ and an edge $e = xy \in E(G)$, let

$$N_1(e) = \{u |\ u \in V(G), d(u,x) < d(u,y)\},\ N_2(e) = \{u |\ u \in V(G), d(u,x) > d(u,y)\},$$

and let $n_1(e) = |N_1(e)|$ and $n_2(e) = |N_2(e)|$, respectively. The *k-Steiner transmission* $s\sigma_k(G,v)$ of a vertex $v \in V(G)$,

$$s\sigma_k(G,v) = \sum_{S \subseteq V(G), |S|=k, v \in S} d(S).$$

Lemma 2.1. *Let G be a graph obtained from graph H_1 and graph H_2 by identifying a vertex v of H_1 and a vertex u of H_2. Then*

$$SW_k(G) = \sum_{i=1}^{2} SW_k(H_i) + \sum_{i=1}^{k-2} s\sigma_{i+1}(H_1,v) s\sigma_{k-i}(H_2,u) + \sum_{i=1}^{k-1} s\sigma_{i+1}(H_1,v) s\sigma_{k-i+1}(H_2,u).$$

Proof. Let G be a graph as defined in the statement of the theorem. Let us consider an arbitrary $S \subseteq V(G)$ with $|S| = k$. One can see that each S can be classified into four parts:

(I) $S \subseteq V(H_1)$,

(II) $S \subseteq V(H_2)$,

(III) $S \cap V(H_1) \neq \emptyset$, $S \cap V(H_2) \neq \emptyset$ and $u(v) \in S$,

(IV) $S \cap V(H_1) \neq \emptyset$, $S \cap V(H_2) \neq \emptyset$ and $u(v) \notin S$.

It is clear that

(1) type-I S contribute $SW_k(H_1)$ to $SW_k(G)$;

(2) type-II S contribute $SW_k(H_2)$ to $SW_k(G)$;

(3) for type-III S, since $u \in S$, we shall choose $k-1$ vertices from $V(H_1) \cup V(H_2) \setminus \{u\}$: i vertices from $V(H_1)$ and $k-i-1$ vertices from $V(H_2)$, where i run over all elements in $\{1, 2, \ldots, k-2\}$. Hence, such S contribute $\sum_{i=1}^{k-2} s\sigma_{i+1}(H_1, v) s\sigma_{k-i}(H_2, u)$ to $SW_k(G)$ in all.

(4) for type-IV S, since $S \cap V(H_1) \neq \emptyset$, $S \cap V(H_2) \neq \emptyset$ and $u(v) \notin S$, we choose k vertices except $u(v)$: i vertices from $V(H_1)$ and $k-i$ vertices from $V(H_2)$, where i run over all elements in $\{1, 2, \ldots, k-1\}$. Hence, such S contribute $SW_k(G)$ by $\sum_{i=1}^{k-1} s\sigma_{i+1}(H_1, v) s \sigma_{k-i+1}(H_2, u)$ in all.

Summing up the above, we arrive at our conclusion. ■

Corollary 2.1. *Let G, H be two nontrivial connected graphs with $u, v \in V(G)$, $w \in V(H)$. Let GuH (GvH, respectively) be the graph obtained from G and H by identifying u (v, respectively) with w. For any $2 \leq k \leq |V(G)|$, if $\sum_{i=2}^{k} s\sigma_k(G, u) < \sum_{i=2}^{k} s\sigma_k(G, v)$, then*

$$SW_k(GuH) < SW_k(GvH).$$

Proof. It is immediate from the Lemma 2.1 above. ■

Let us recall the classical result of Wiener.

Theorem 2.1. *(Wiener [13]) For any tree T,*

$$W(T) = \sum_{e \in E(T)} n_1(e) n_2(e).$$

Li, Mao and Gutman [11] established the following theorem on Steiner k-Wiener index of a tree, which generalizes the above result.

Theorem 2.2. *(Li, Mao and Gutman [11]) Let $k \geq 2$ be an integer. For any tree T,*

$$SW_k(T) = \sum_{e \in E(T)} \sum_{i=1}^{k-1} \binom{n_1(e)}{i} \binom{n_2(e)}{k-i}.$$

Using the well-known Vandermonde's convolution formula we reformulate the above theorem in the following way.

Theorem 2.3. *Let k be an integer such that $2 \leq k \leq n$. For any tree T of order n,*

$$SW_k(T) = (n-1)\binom{n}{k} - \sum_{e \in E(T)} \left[\binom{n_1(e)}{k} + \binom{n_2(e)}{k}\right].$$

Proof. Vandermonde's convolution formula says that

$$\sum_{i=0}^{k} \binom{n_1}{i}\binom{n_2}{k-i} = \binom{n_1+n_2}{k}.$$

By Theorem 2.2,

$$\begin{aligned}
SW_k(T) &= \sum_{e \in E(T)} \sum_{i=1}^{k-1} \binom{n_1(e)}{i}\binom{n_2(e)}{k-i} \\
&= \sum_{e \in E(T)} \left[\sum_{i=0}^{k} \binom{n_1(e)}{i}\binom{n_2(e)}{k-i} - \binom{n_1(e)}{k} - \binom{n_2(e)}{k}\right] \\
&= \sum_{e \in E(T)} \left[\binom{n_1(e)+n_2(e)}{k} - \binom{n_1(e)}{k} - \binom{n_2(e)}{k}\right] \\
&= (n-1)\binom{n}{k} - \sum_{e \in E(T)} \left[\binom{n_1(e)}{k} + \binom{n_2(e)}{k}\right].
\end{aligned}$$

■

As usual, we use P_n and $S_n(\cong K_{1,n-1})$ to denote the path and stars of order n, respectively. Li, Mao, Gutman determined the Steinter k-Wiener indices of two special types of graphs.

Proposition 2.1. *(Li, Mao, Gutman [11]) Let P_n be the path of order n $(n \geq 3)$, and let k be an integer with $2 \leq k \leq n$. Then*

$$SW_k(P_n) = (k-1)\binom{n+1}{k+1}.$$

Proposition 2.2. *(Li, Mao, Gutman [12]) Let T be a graph obtained from a path P_t and a star S_{n-t+1} by identifying a pendant vertex of P_t and the center v of S_{n-t+1}, where $1 \leq t \leq n-1$ and $k \leq n$. Then*

$$SW_k(T) = t\binom{n-1}{k} - \binom{t}{k+1} - \binom{n}{k+1} + \binom{n-t+1}{k+1} + (k+1)\binom{n}{k}.$$

Theorem 2.4. *Let $T = H(r; x, y)$ be the tree obtained by attaching x and y pendant vertices, respectively, to the two end vertices of the path of order r, where $r \geq 1$, $x \geq y \geq 0$.*

Then

$$SW_k(T) = f(x) + f(y) - 2\binom{r}{k+1} - (k-1)\binom{r+1}{k+1} + (k+r-1)\sum_{i=1}^{k-1}\binom{x}{i}\binom{y}{k-i}$$
$$+\sum_{i=1}^{k-2}\sum_{j=1}^{k-i-1}\binom{x}{i}\binom{r}{j}\binom{y}{k-i-j}(k+r-j-1),$$

where

$$f(t) = r\binom{t+r-1}{k} - \binom{t+r}{k+1} + \binom{t+1}{k+1} + (k+1)\binom{t+r}{k}.$$

Proof. Label the vertices of the path of order r by $v_1, v_2, \ldots, v_r$ successively. The x (y, respectively) pendant vertices which joined to v_1 (v_r, respectively) denoted by $u_1, u_2, \ldots, u_x$ ($w_1, w_2, \ldots, w_y$, respectively). Let

$$V_1 = \{v_i |\ i = 1, 2, \ldots, r\}, V_2 = \{u_i |\ i = 1, 2, \ldots, x\}, V_3 = \{w_i |\ i = 1, 2, \ldots, y\}.$$

Let us consider an arbitrary set $S \subseteq V(T)$ with $|S| = k$. One can see that each S can be classified into four types:

(I) $S \subseteq V_1 \cup V_2$,

(II) $S \subseteq V_1 \cup V_3$,

(III) $S \subseteq V_2 \cup V_3, S \cap V_2 \neq \emptyset$, and $S \cap V_3 \neq \emptyset$,

(IV) $S \subseteq V_1 \cup V_2 \cup V_3$, and $S \cap V_i \neq \emptyset, i = 1, 2, 3$.

By Proposition 2.2, it is clear that

(1) type-I S contribute to $SW_k(T)$ by $r\binom{x+r-1}{k} - \binom{r}{k+1} - \binom{x+r}{k+1} + \binom{x+1}{k+1} + (k+1)\binom{x+r}{k}$;

(2) type-II S contribute to $SW_k(T)$ by $r\binom{y+r-1}{k} - \binom{r}{k+1} - \binom{y+r}{k+1} + \binom{y+1}{k+1} + (k+1)\binom{y+r}{k}$;

(3) for type-III S, we shall choose k vertices from $V_2 \cup V_3$: i vertices from V_2 and $k-i$ vertices from V_3, where i run over all elements in $\{1, 2, \ldots, k-1\}$. Hence, such S contribute $SW_k(T)$ by $(k+r-1)\sum_{i=1}^{k-1}\binom{x}{i}\binom{y}{k-i}$ in all.

(4) for type-IV S, since $S \subseteq V_1 \cup V_2 \cup V_3$, and $S \cap V_i \neq \emptyset, i = 1, 2, 3$, we choose k vertices: i vertices from V_2, j vertices from V_1 and $k-i-j$ vertices from V_3, where i run over all elements in $\{1, 2, \ldots, k-2\}$, j run over all elements in $\{1, 2, \ldots, k-i-1\}$. Hence, such S contribute $SW_k(T)$ by $\sum_{i=1}^{k-2}\sum_{j=1}^{k-i-1}\binom{x}{i}\binom{r}{j}\binom{y}{k-i-j}(k+r-j-1)$ in all.

(5) If $S \subseteq V_1$, then we compute $d(S)$ twice in type-I and type-II. By Proposition 2.1, we can get $\sum_{s\subseteq V_1} d(S) = (k-1)\binom{r+1}{k+1}$

Summing up the above (1) to (4) and subtracted (5), the result follows. ■

Theorem 2.5. *Let p, q be two integers with $p \geq q \geq 1$ and let $k \in \{2, \ldots, p+q\}$. For any tree T with (p, q)-bipartition,*

$$SW_k(T) \geq (k-1)[\sum_{i=0}^{k-2} \tbinom{p-1}{i}\tbinom{q-1}{k-i-2} + \tbinom{p-1}{k-1} + \tbinom{q-1}{k-1}] + 2k \sum_{i=1}^{k-2} \tbinom{p-1}{i}\tbinom{q-1}{k-i-1}$$
$$+ (k+1) \sum_{i=1}^{k-1} \tbinom{p-1}{i}\tbinom{q-1}{k-i},$$

with equality if and only if $T \cong S(p, q)$.

Proof. Let T be a tree with the minimum Steiner k-Wiener index among all trees with (p, q)-bipartition. It suffices to show that diameter of T is 3. Suppose, on the contrary, that the diameter of T is at least 4. Let $P = u_1 u_2 \ldots u_t$ be a longest path of T. Trivially, both u_1 and u_t are leaves of T. Let S be the set of neighbors of u_4 in T different from u_3. Let H_1 (H_2, respectively) be the component of $T - S$ ($T - \{u_3 u_4\}$, respectively) containing u_4. Then, T can be obtained from H_1 and H_2 by identifying $u_4 \in V(H_1)$ with $u_4 \in V(H_2)$. Let T' be the tree obtained from H_1 and H_2 by identifying $u_2 \in V(H_1)$ with $u_4 \in V(H_2)$. Clearly, T' also has (p, q)-bipartition. Let H_3 be the component of $T - \{u_2 u_3\}$ containing u_2. Denote the order of H_i by n_i for each $i \in \{1, 2, 3\}$. A simple calculation shows that

$$s\sigma_k(H_1, u_4) - s\sigma_k(H_1, u_2) = 2\tbinom{n_3-1}{k-1} + \sum_{i=1}^{k-2} \tbinom{n_3-1}{i}\tbinom{n_1-n_3-1}{k-i-1} > 0.$$

It follows that $\sum_{i=2}^{k} S\sigma_i(H_1, u_4) > \sum_{i=2}^{k} S\sigma_i(H_1, u_2)$, and by Corollary 2.1, we have

$$SW_k(T) > SW_k(T'),$$

a contradiction. This shows that the diameter of T is 3, and thus $T \cong S(p, q)$. ■

3 Maximum Steiner Wiener index of trees

In this section, we determine the maximum Steiner Wiener index of trees with given bipartition.

Theorem 3.1. *(Li, Mao, Gutman [11]) Let T be a tree of order n, and let k be an integer with $2 \leq k \leq n$. Then*

$$(n-1)\tbinom{n-1}{k-1} \leq SW_k(T) \leq (k-1)\tbinom{n+1}{k+1},$$

with the left side of equality if and only if $T \cong S_n$, and with the right side of equality if and only if $T \cong P_n$.

Theorem 3.2. *Let p and q be two integers with $p \geq q \geq 1$. If T is a tree with a (p,q)-bipartition, then*

$$SW_k(T) \leq f(x) + f(y) - 2\binom{2q-1}{k+1} - (k-1)\binom{2q}{k+1} + (k+2q-2)\sum_{i=1}^{k-1}\binom{x}{i}\binom{y}{k-i}$$
$$+\sum_{i=1}^{k-2}\sum_{j=1}^{k-i-1}\binom{x}{i}\binom{2q-1}{j}\binom{y}{k-i-j}(k+2q-j-2),$$

where

$$f(t) = (2q-1)\binom{t+2q-2}{k} - \binom{t+2q-1}{k+1} + \binom{t+1}{k+1} + (k+1)\binom{t+2q-1}{k},$$

and $x = \lceil \frac{p-q+1}{2} \rceil, y = \lfloor \frac{p-q+1}{2} \rfloor$,
with equality if and only if $T \cong H(2q-1; x, y)$, where $x = \lceil \frac{p-q+1}{2} \rceil, y = \lfloor \frac{p-q+1}{2} \rfloor$.

Proof. Let T be a tree such that $SW_k(T)$ is maximized among all trees with (p,q)-bipartition. We show that $T \cong H(2q-1; x, y)$, where $x = \lceil \frac{p-q+1}{2} \rceil, y = \lfloor \frac{p-q+1}{2} \rfloor$. If $p = q$ or $p = q+1$, then by Theorem 3.1, T is a path. Since $H(2q-1; x, y) \cong P_{p+q}$, the result follows. Next assume that $p > q+1$. Take a longest path $P = u_1 u_2 \ldots u_t$ of T. Clearly both u_1 and u_t are leaves of T. To show $T \cong H(2q-1; x, y)$, where $x = \lceil \frac{p-q+1}{2} \rceil, y = \lfloor \frac{p-q+1}{2} \rfloor$, we first show that u_2 and u_{t-1} are only two vertices with degree greater than two. Suppose that there exists other vertex which has degree at least 3 and is distinct from u_2 and u_{t-1}. Let u_i be the vertex of P with $d(u_i) > 2$ such that i is the smallest integer greater than 3. Let H_1 (H_2, H_3 respectively) be the component of $T - \{u_i u_{i+1}\}$ ($T - \{u_{i-2} u_{i-3}\}$, $T - \{u_{i-1}, u_{i+1}\}$ respectively), containing u_{i+1} (u_{i-3}, u_i respectively). Let S be the set of neighbors of u_i in T different from u_{i-1}, u_{i+1}. We denote the order of H_i by n_i for $i \in \{1,2,3\}$.

First assume that $n_1 > n_2$. Let $T' = T \setminus \{u_i v|\ v \in S\}) \cup \{u_{i-2} v|\ v \in S\}$. By Theorem 2.3, we have

$$SW_k(T') - SW_k(T) = \binom{n_1+n_3+1}{k} + \binom{n_2+1}{k} - \binom{n_1+1}{k} - \binom{n_2+n_3+1}{k}$$
$$+\binom{n_1+n_3}{k} + \binom{n_2+2}{k} - \binom{n_1+2}{k} - \binom{n_2+n_3}{k} > 0,$$

contradicting the choice of T.

Next we consider the case when $n_1 \leq n_2$. Let u_i be the vertex with $d(u_i) > 2$ such that i is the largest integer less than $t-1$. By a similar argument as above, let H_1' (H_2', H_3' respectively) be the component of $T - \{u_{i+2} u_{i+3}\}$ ($T - \{u_{i-1} u_i\}$, $T - \{u_{i-1}, u_{i+1}\}$ respectively), containing u_{i+3} (u_{i-1}, u_i respectively). Let S be the set of neighbors of u_i

in T different from u_{i-1}, u_{i+1}. We denote the order of H_i' by n_i' for $i \in \{1,2,3\}$. Then $n_1' \le n_1 - 2, n_2' \ge n_2 + 2$, so $n_1' < n_2'$. Let $T' = T \setminus \{u_i v | \ v \in S\}) \cup \{u_{i+2} v | \ v \in S\}$. By Theorem 2.3, we have

$$SW_k(T') - SW_k(T) = \binom{n_2'+n_3'+1}{k} + \binom{n_1'+1}{k} - \binom{n_2'+1}{k} - \binom{n_1'+n_3'+1}{k} + \binom{n_2'+n_3'}{k} + \binom{n_1'+2}{k} - \binom{n_2'+2}{k} - \binom{n_1'+n_3'}{k} > 0,$$

contradicting the choice of T.

This shows $T \cong H(r; x, y)$.

Claim 1. It is clear that the following statements are equivalent:

(1) $x + y = p - q + 1$.

(2) $t = 2q + 1$.

(3) The diameter of T is $2q$.

(4) All the leaves of T belongs to the same color class.

Proof of Claim 1. The equivalence of (1), (2), (3) can be easily deduced from the fact that the diameter of T is $t - 1$, and $x + y + t - 2 = p + q$. Furthermore, either (2) or (3) implies (4).

Now, we use (4) implies (1). Let $V(T) = V_1 \cup V_2$, where V_i is a color classes, $i = 1, 2$ and $S = \{v | \ v \in V(T), d(v) = 1\}$. Without loss of generality, suppose $|V_1| = p, |V_2| = q$. Since $p > q + 1$ and S is contained in the same color class, $S \subset V_1$. Thus, $x + y = |S| = |V_1| - |V_2| + 1 = p - q + 1$.

This completes the proof of the claim.

Next we show that (1) of Claim 1: $x + y = p - q + 1$.

If $x+y < p-q+1$, then $t = p+q-(x+y)+2 > 2q+1$. Thus $|\{u_i| \ i \text{ is even }\}| \ge q+1$ and $|\{u_i| \ i \text{ is odd }\}| \ge q + 1$. It follows that $|V_1| \ge q + 1$ and $|V_2| \ge q + 1$, contradicting the assumption that T has (p, q)-bipartition.

Assume now that $x + y = l > p - q + 1$. Let $S_1 = \{u| \ u \in N(u_2), d(u) = 1\}, S_2 = \{w| \ w \in N(u_{t-1}), d(w) = 1\}$. Since (1) and (4) are equivalent by Claim 1, S_1 and S_2 are contained in the different color classes. Let $T' = H(r; x_1, y_1)$, where $x_1+y_1 = p-q+1, x_1 \le x, y_1 < y$. By Theorem 2.3, we have

$$\begin{aligned}SW_k(T') - SW_k(T) &= l[\tbinom{p+q-1}{k} + \tbinom{1}{k}] + \tbinom{x+1}{k} + \tbinom{p+q-x-1}{k} \\ &+ \tbinom{x+2}{k} + \tbinom{p+q-x-2}{k} + \ldots + \tbinom{p+q-y-1}{k} + \tbinom{y+1}{k} \\ &- (p-q+1)[\tbinom{p+q-1}{k} + \tbinom{1}{k}] - \tbinom{x_1+1}{k} - \tbinom{p+q-x_1-1}{k} \\ &- \tbinom{x_1+2}{k} - \tbinom{p+q-x_1-2}{k} - \ldots - \tbinom{p+q-y_1-1}{k} - \tbinom{y_1+1}{k} \\ &= [\tbinom{p+q-1}{k} + \tbinom{1}{k} - \tbinom{x_1+1}{k} - \tbinom{p+q-x_1-1}{k}] + \ldots \\ &+ [\tbinom{p+q-1}{k} + \tbinom{1}{k} - \tbinom{x}{k} - \tbinom{p+q-x}{k}] \\ &+ [\tbinom{p+q-1}{k} + \tbinom{1}{k} - \tbinom{p+q-y}{k} - \tbinom{y}{k}] + \ldots \\ &+ [\tbinom{p+q-1}{k} + \tbinom{1}{k} - \tbinom{p+q-y_1-1}{k} - \tbinom{y_1+1}{k}] > 0 \ .\end{aligned}$$

contradicting the choice of T.

By the equivalence of (1) and (3) of Claim 1, it follows that the diameter of T is $2q$. That is, $T \cong H(2q-1; x, y)$, where $x + y = p - q + 1$. Furthermore, by Theorem 2.4, we have $x = \lceil \frac{p-q+1}{2} \rceil$, $y = \lfloor \frac{p-q+1}{2} \rfloor$.

The proof of the theorem is completed. ■

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