MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **86** (2021) 343-362

ISSN 0340 - 6253

Extremal Cover Cost and Reverse Cover Cost of Trees*

Jing Huang

School of Mathematics, South China University of Technology, Guangzhou 510641, P. R. China

jhuangmath@foxmail.com

(Received April 28, 2020)

Abstract

For a connected graph $G = (V_G, E_G)$, the cover cost and the reverse cover cost of a vertex v in G are, respectively, defined as $CC_G(v) = \sum_{u\in V_G} H_{vu}$ and $RC_G(v) = \sum_{u\in V_G} H_{uv}$, where H_{uv} is the expected hitting time for random walk starting at u to visit v. Georgakopoulos and Wagner [J. Graph Theory 84 (2017) 311-326] characterized the unique tree with the maximum and the minimum cover cost (resp. reverse cover cost) among all n-vertex trees. In this paper, the second and the third largest and smallest cover cost (resp. reverse cover cost) of a vertex in an n-vertex trees are determined. All the corresponding extremal trees are also identified.

1 Introduction

Let $G = (V_G, E_G)$ be a tree with V_G the vertex set and E_G the edge set. The *neighborhood* of a vertex v, written by $N_G(v)$, is the set of vertices adjacent to v in G. The degree of v is defined to be $d_G(v) = |N_G(v)|$. The *distance* between vertices u and v, denoted by $d_G(u, v)$, is the length of a shortest path connecting them. For simplicity, when there is no danger of confusion, we omit the subscripts G for our notation. We follow the notation and terminologies in [4] except otherwise stated.

*J. Huang was partially supported by China Postdoctoral Science Foundation (Grant No. 2019M662883) and Guangdong Basic and Applied Basic Research Foundation (Grant No. 2019A1515110277).

In 1947, Wiener [27] introduced an important graph parameter related to distance, defined by $W(G) = \sum_{\{u,v\}\subseteq V_G} d(u,v)$ which is also called *Wiener index*. It has been extensively studied and has found applications in chemistry, communication theory, and elsewhere. To learn more about it, one may be referred to [10, 12, 17, 21, 29] and the references therein for details. The *centrality* (also known as the *transmission*) of a vertex x in G is defined as $D(v) = \sum_{u\in V_G} d(u,v)$. Then it is obvious that $W(G) = \frac{1}{2}\sum_{v\in V_G} D(v)$.

For a graph G, we define the random walk on G as the Markov chain $X_k, k \geq 0$, that from its current vertex u jumps to an adjacent vertex with probability $1/d(u)$. The *hitting time* (also known as the *first passage time*) T_v of the vertex v is the minimum number of steps that the random walk takes to reach that vertex:

$$T_v = \inf\{k \geq 0 : X_k = v\}.$$

The expected hitting time of random walks is an important parameter of graphs [2,20], which has been studied extensively. There are many results about the expected hitting time of random walks on graphs. For example, vertex transitive graphs (see [1,9,26]), edge transitive graphs (see [24,25]), distance regular graphs (see [3,11]), graphs with cutpoints (see [7]) and some others (see [5, 6, 16, 18, 28, 30]) related information.

The *cover cost* (see [14]) of a vertex v in G is defined as the sum of the expected hitting times from v to all other vertices, that is,

$$CC_G(v) = \sum_{u\in V_G} H_{vu}.$$

It is well known that the cover cost is closely related to the *cover time* of a graph (see [14]), which is defined as the expected number of the steps taken for random walk beginning at v to visit all vertices of the graph.

In analogy to $CC(v)$, Georgakopoulos and Wagner [15] proposed the *reverse cover cost* of a vertex v in G, which is defined as the sum of the expected hitting times from all other vertices to v, i.e.,

$$RC(v) = \sum_{u\in V_G} H_{uv}.$$

Georgakopoulos and Wagner [15] exhibited a beautiful relationship between the cover cost (resp. reverse cover cost) of a vertex, the Wiener index, and related graph invariants. Furthermore, the maximum and the minimum values together with the corresponding extremal graphs of the hitting time, the cover cost, and the reverse cover cost for trees were characterized. In 2019, Huang, Li, and Xie [19] determined the sharp upper and lower bounds on the cover cost (resp. reverse cover cost) of a vertex among all n-vertex unicyclic graphs and identified all the corresponding extremal graphs. Recently, Li and Wang [22] studied the cover cost and reverse cover cost of trees with given segment sequence. More specifically, they characterized the unique tree with the minimum cover cost (resp. reverse cover cost) and the maximal reverse cover cost among all trees with given segment sequence.

In this paper, we continue to study the extremal value of cover cost and reverse cover cost of trees. We determine the second and the third largest and smallest cover cost (resp. reverse cover cost) of a vertex among all n-vertex trees and characterize all the corresponding extremal trees, respectively. The paper is organized as follows. In Section 2, we recall some important known results. In Section 3, we respectively determine the unique n-vertex tree having the second and the third smallest, the second and the third largest cover cost, whereas in Section 4, we do the parallel work for the reverse cover cost.

2 Preliminaries

In this section, we give some preliminary results, which will be used to prove our main results. If $x \in V_T$, then $T - x$ denotes the graph obtained from T by deleting the vertex x and all its incident edges. If $xy \notin E_T$, then $T + xy$ is a graph obtained from T by adding an edge xy. For $xy \in E_T, T - xy$ denotes the graph obtained from T by deleting the edge xy.

The following lemma comes from [15], which gives a beautiful relationship between $CC(v), RC(v), D(v)$ and $W(T)$.

Lemma 2.1 ([15]). *Let T be a tree on n vertices with $x \in V_T$. Then*

(i) $CC_T(x) + D_T(x) = 2W(T)$;

(ii) $(2n-1)D_T(x) - RC_T(x) = 2W(T)$.

Let T be a tree with $x \in V_T$ and $e \in E_T$. Denote by $A_x(e)$ the set of vertices in the same connected component of $T - e$ as x and $B_x(e) = V_T \setminus A_x(e)$ the complement. Then the following Lemma is an equivalent form of Lemma 2.1 (i).

Lemma 2.2 ([15]). *Let T be a tree with $x \in V_T$. Then*

$$CC(x) = \sum_{e \in E_T} (2|A_x(e)| - 1)|B_x(e)|.$$

Lemma 2.3 ([15]). *Let T be a tree on n vertices with $r \in V_T$. Let $r_1, r_2, \ldots, r_k$ be all neighbors of r and let $T_1, T_2, \ldots, T_k$ be the associated branches. Then*

$$\begin{aligned} CC(r) &= 2\sum_{i=1}^{k} W(T_i) + 2\sum_{1\le i<j\le k} \big(D_{T_i}(r_i)|T_j| + D_{T_j}(r_j)|T_i| + 2|T_i||T_j|\big) \\ &\quad + \sum_{i=1}^{k} D_{T_i}(r_i) + n - 1. \end{aligned}$$

Let $\mathscr{T}_n$ be the set of all trees with n vertices. Denote by P_n and S_n the path and the star of order n, respectively. The *eccentricity* $\varepsilon(v)$ of a vertex v is defined as $\max_{u\in V_G} d(u,v)$, whereas a vertex with minimal eccentricity is called a *center* (vertex) of G. Let $S_{p,q}$ be the graph obtained from S_p and S_q by joining an edge between their centers and P'_n be the graph obtained from P_{n-1} by attaching a pendant edge to its penultimate vertex. A vertex is called a *pendant vertex* if it is of degree 1, whereas an edge is called a *pendant edge* if it contains a pendant vertex. A pendant vertex of a tree is also called a *leaf*.

The following two lemmas is about the maximum and minimum, the second largest and smallest of Wiener index among all n-vertex trees.

Lemma 2.4 ([13]). *Let T be a tree on n vertices. Then*

$$(n-1)^2 \le W(T) \le \frac{n^3-n}{6}.$$

The lower bound holds with equality if and only if $T \cong S_n$, whereas the upper bound holds with equality if and only if $T \cong P_n$.

Lemma 2.5 ([8]). *Let T be a tree with $T \in \mathscr{T}_n \setminus \{S_n, P_n\}$. Then*

$$n^2 - n - 2 \le W(T) \le \frac{n^3 - 7n + 18}{6}.$$

The lower bound holds with equality if and only if $T \cong S_{2,n-2}$, whereas the upper bound holds with equality if and only if $T \cong P'_n$.

3 Trees with the second (resp. third) smallest and largest cover cost

The minimum and maximum cover cost among all trees with n vertices have been determined by Georgakopoulos and Wagner [15] (see Theorem 3.1 in the following). In this section, we investigate the second (resp. third) smallest and largest cover cost of among all trees with n vertices.

Theorem 3.1 ([15]). *Let T be a tree on $n \geq 2$ vertices with $v \in V_T$. Then*

$$2n^2 - 6n + 5 \leq CC(v) \leq \frac{n^3 - n}{3} - \left\lfloor \frac{n^2}{4} \right\rfloor .$$

The lower bound holds with equality if and only if $T \cong S_n$ with v as one of its leaves, whereas the upper bound holds with equality if and only if $T \cong P_n$ with v as a center of it.

Let T be a tree on n vertices with $v \in V_T$. Assume that $\varepsilon(v) = t$, then it is obvious that

$$\begin{aligned} D(v) &\leq 1 + 2 + \cdots + t + (n-t-1)t \\ &= \frac{t(t+1)}{2} + (n-t-1)t \\ &= \frac{-t^2 + (2n-1)t}{2}. \end{aligned} \tag{3.1}$$

Note that $f(x) = \frac{-x^2+(2n-1)x}{2}$ is a increasing function of x for $1 \leq x \leq n-1$. Then the following two lemmas about the maximum and the second largest of $D(v)$ among all n-vertex trees is a direct consequence of (3.1).

Lemma 3.2. *Let T be a tree on $n \geq 2$ vertices with $v \in V_T$. Then $D(v) \leq \frac{n(n-1)}{2}$ with equality if and only if $T \cong P_n$ with $d(v) = 1$.*

Lemma 3.3. *Let T be a tree on $n \geq 4$ vertices with $v \in V_T$, and assume that we do not have $T \cong P_n$ with $d(v) = 1$. Then $D(v) \leq \frac{(n-2)(n+1)}{2}$ with equality if and only if $T \cong P_n'$ with v being the farthest leaf from the vertex of degree 3.*

Lemma 3.4. *Let T be a tree obtained from $P_{n_1}, P_{n_2}, P_{n_3}$ and an isolated vertex v by joining an edges between v and v_1, v_2, v_3, respectively, where v_i is one of the leaves of $P_{n_i} (i = 1, 2, 3)$. Put $T^* = T - bc + cd$, where c (resp. d) is the leaf of P_{n_3} (resp. P_{n_2}) other than v_3 (resp. v_2) and b is the neighbor of c. If $n_2 \geq n_3$, then $CC_{T^*}(v) > CC_T(v)$.*

Proof. It is routine to check that

$$D_{T^*}(v) - D_T(v) = d_{T^*}(v, c) - d_T(v, c) = n_2 - n_3 + 1$$

and

$$W(T^*) - W(T) = D_{T^*}(c) - D_T(c) = n_1(n_2 - n_3 + 1).$$

Then it follows from Lemma 2.1 that

$$\begin{aligned} CC_{T^*}(v) - CC_T(v) &= 2(W(T^*) - W(T)) - (D_{T^*}(v) - D_T(v)) \\ &= (2n_1 - 1)(n_2 - n_3 + 1) > 0. \end{aligned}$$

This completes the proof. ■

Now we are ready to determine the second and the third smallest cover cost among all trees with n vertices.

Theorem 3.5. *Let T be a tree on $n \geq 4$ vertices with $v \in V_T$, and assume that we do not have $T \cong S_n$ with v one of its leaves. Then $CC(v) \geq 2n^2 - 5n + 2$ with equality if and only if $T \cong S_{2,n-2}$ with v the farthest leaf from the vertex of degree $n-2$.*

Proof. Choose a tree T and $v \in V_T$ such that $CC(v)$ is as small as possible, where we do not have $T \cong S_n$ with $d(v) = 1$. If $T \cong S_n$ with v as its center, then it follows from Lemma 2.1 that

$$CC(v) = 2W(S_{2,n-2}) - D(v) = 2n^2 - 5n + 3 > 2n^2 - 5n + 2.$$

Therefore, $T \ncong S_n$.

Then we show that v is a leaf of T. Suppose to the contrary that $d(v) = h \geq 2$. Let $N_T(v) = \{v_1, v_2, \ldots, v_h\}$ and F_i be the connected component of $T - vv_i$ containing v_i, $i = 1, 2, \ldots, h$. Without loss of generality, assume that $|F_1| \geq |F_2| \geq \cdots \geq |F_h|$. After a short calculation, we obtain

$$D_T(v) = (D_{F_1}(v_1) + |F_1|) + (D_{F_2}(v_2) + |F_2|) + \cdots + (D_{F_h}(v_h) + |F_h|)$$

and

$$D_T(v_2) = (D_{F_1}(v_1) + 2|F_1|) + D_{F_2}(v_2) + \cdots + (D_{F_h}(v_h) + 2|F_h|) + 1,$$

Therefore,

$$D(v_2) - D(v) = |F_1| + |F_3| + \cdots + |F_h| - |F_2| + 1 \geq |F_1| - |F_2| + 1 > 0.$$

Combining with Lemma 2.1, one has $CC(v_2) < CC(v)$, a contradiction to the choice of v.

Since $T \ncong S_n$, $\varepsilon_T(v) \geq 3$. Let $vv_1v_2v_3$ be a path of length 3 in T. If $T \ncong S_{p,q}$ for any $p+q=n$, then put $T' = T - \{v_2w : w \in N(v_2) \setminus v_1\} + \{v_1w : w \in N(v_2) \setminus v_1\}$. Thus we do not have $T' \cong S_n$ with $d(v)=1$. It is routine to check that

$$W(T) - W(T') = (|A_{v_1}(v_1v_2)| - 1)(|A_{v_2}(v_1v_2)| - 1), \quad D_T(v) - D_{T'}(v) = |A_{v_2}(v_1v_2)| - 1.$$

Note that $\{v, v_1\} \subseteq A_{v_1}(v_1v_2)$, $\{v_2, v_3\} \subseteq A_{v_2}(v_1v_2)$. Then $|A_{v_1}(v_1v_2)| \geq 2$ and $|A_{v_2}(v_1v_2)| \geq 2$. Again by Lemma 2.1, we have

$$CC_T(v) - CC_{T'}(v) = (2|A_{v_1}(v_1v_2)| - 3)(|A_{v_2}(v_1v_2)| - 1) > 0,$$

that is, $CC_{T'}(v) < CC_T(v)$, a contradiction to the choice of T.

Therefore, $T \cong S_{p,q}$ with v as a leaf of it. Without loss of generality, assume that $v \in V_{S_p}$. Then

$$W(T) = (n-1)(n-2) + pq, \quad D(v) = 2p + 3q - 4.$$

Again by Lemma 2.1,

$$\begin{aligned} CC(v) &= 2pq - 2p - 3q + 2n^2 - 6n + 8 && (3.2)\\ &= -2p^2 + (2n+1)p + 2n^2 - 9n + 8 && \big(\text{since } p+q=n\big)\\ &\geq \min\big\{2n^2 - 5n + 2, 2n^2 - 4n - 2\big\} && \big(\text{since } 2 \leq p \leq n-2\big)\\ &= 2n^2 - 5n + 2 && (3.3) \end{aligned}$$

for $n \geq 4$. The equality in (3.3) holds if and only if $p = 2$. Consequently, $CC(v) \geq 2n^2 - 5n + 2$ with equality if and only if $T \cong S_{2,n-2}$ with v being the farthest leaf from the vertex of degree $n-2$ and the proof is complete. ∎

Theorem 3.6. *Let T be a tree on $n \geq 6$ vertices with $v \in V_T$. If neither $T \cong S_n$ with $d(v) = 1$ nor $T \cong S_{2,n-2}$ with v the farthest leaf from the vertex of degree $n-2$, then $CC(v) \geq 2n^2 - 5n + 3$ with equality if and only if $T \cong S_n$ with v as its center.*

Proof. Choose a tree T with $v \in V_T$ such that $CC(v)$ is as small as possible, where neither $T \cong S_n$ with $d(v) = 1$ nor $T \cong S_{2,n-2}$ with v the farthest leaf from the vertex of degree $n-2$.

If $T \cong S_n$ with v as its center, then by Lemma 2.1, one has

$$CC(v) = 2W(S_n) - D(v) = 2(n-1)^2 - (n-1) = 2n^2 - 5n + 3.$$

In the following, assume that $T \in \mathscr{T}_n \setminus \{S_n\}$ and $v \in V_T$. If $T \cong S_{2,n-2}$, then v is one of the leaves adjacent to the vertex of degree $n-2$. After a direct calculation, we have

$CC(v) = 2W(S_{2,n-2}) - D(v) = 2(n-2)(n+1) - 2(n-1) = 2n^2 - 4n - 2 > 2n^2 - 5n + 3$

for $n \geq 6$. Therefore, $T \ncong S_{2,n-2}$.

Consequently, as a similar proof in the former theorem, we have $T \cong S_{p,q}$ with v as a leaf of it for some $p + q = n$ and $3 \leq p \leq n-3$. Without loss of generality, assume that $v \in V_{S_p}$. Then by (3.2), we have

$$\begin{aligned} CC(v) &= 2pq - 2p - 3q + 2n^2 - 6n + 8 \\ &= -2p^2 + (2n+1)p + 2n^2 - 9n + 8 \\ &\geq \min\left\{2n^2 - 3n - 7, 2n^2 - 2n - 13\right\} \\ &> 2n^2 - 5n + 3 \end{aligned}$$

for $n \geq 6$ and we are done.

■

The following two results give the second and third largest values of cover cost for trees with given order; all the corresponding extremal graphs are also characterized.

Theorem 3.7. *Let T be a tree on $n \geq 6$ vertices with $v \in V_T$, and assume that we do not have $T \cong P_n$ with v being its center. Then*

$$CC(v) \leq \begin{cases} \frac{4n^3 - 3n^2 - 4n - 9}{12}, & \text{if } n \text{ is odd;} \\ \frac{4n^3 - 3n^2 - 4n - 24}{12}, & \text{if } n \text{ is even} \end{cases}$$

with equality if and only if $T \cong P_n$ with v being the vertex of distance $\lceil \frac{n}{2} \rceil - 2$ from one of its end vertices.

Proof. Choose a tree T and $v \in V_T$ such that $CC(v)$ is as large as possible, where we do not have $T \cong P_n$ with v being its center. Let $N(v) = \{v_1, v_2, \ldots, v_k\}$ and T_i be the connected component of $T - vv_i$ containing v_i with $|T_i| = n_i, i = 1, 2, \ldots, k$. Without loss of generality, assume that $n_1 \geq n_2 \geq \cdots \geq n_k$. In view of Lemma 2.3, we have

$$\begin{aligned} CC(v) &= 2\sum_{i=1}^{k} W(T_i) + 2\sum_{1 \leq i < j \leq k} \left(D_{T_i}(v_i)n_j + D_{T_j}(v_j)n_i + 2n_in_j\right) \\ &\quad + \sum_{i=1}^{k} D_{T_i}(v_i) + n - 1. \end{aligned} \tag{3.4}$$

Together with Lemmas 2.4-2.5 and Lemmas 3.2-3.3, we obtain either $T_i \cong P_{n_i}$ with $d_{T_i}(v_i) = 1$ or $T_i \cong P'_{n_i}$ with v_i being the farthest leaf from the vertex of degree 3. In addition, if $T_i \cong P'_{n_i}$ with v_i being the farthest leaf from the vertex of degree 3 for some i, then $T_j \cong P_{n_j}$ with $d_{T_j}(v_j) = 1$ for any $j \neq i$. Then we proceed by distinguishing the following two cases to complete the proof.

Case 1. $T_i \cong P_{n_i}$ with $d_{T_i}(v_i) = 1$ for $1 \leq i \leq k$. By Lemma 2.2, we have

$$CC(v) = \sum_{e \in E_T} (2|A_v(e)| - 1)|B_v(e)|. \tag{3.5}$$

If $k \geq 4$, then put $T' = T - vv_2 + av_2$, where a is a leaf of T_k other than v_k. Obviously, $T' \ncong P_n$. Note that for any edge $e \in E_T \setminus (E_{T_k} \cup \{vv_k\})$, the sizes of $A_v(e)$ and $B_v(e)$ are not modified, whereas for $e \in E_{T_k} \cup \{vv_k\}$, its contribution to (3.5) changes from $(2A-1)B$ to $(2(A-n_2)-1)(B+n_2)$, where $A = |A_v(e)|, B = |B_v(e)|$ (as defined for T before the modification). Thus in view of (3.5), we have

$$\begin{aligned} CC_{T'}(v) - CC_T(v) &= \sum_{e \in E_{T_k} \cup \{vv_k\}} [(2(A-n_2)-1)(B+n_2) - (2A-1)B] \\ &= \sum_{e \in E_{T_k} \cup \{vv_k\}} \left[2n_2\left(A - B - n_2 - \frac{1}{2}\right)\right] \end{aligned} \tag{3.6}$$

It is routine to check that $A \geq n_1 + n_2 + \cdots + n_{k-1} + 1$ and $B \leq n_k$. Then

$$A - B \geq n_1 + n_2 + \cdots + n_{k-1} - n_k + 1 \geq n_1 + n_2 - n_k + 1 \geq n_2 + 1.$$

Combining with (3.6), one has $CC_{T'}(v) > CC_T(v)$, which leads to a contradiction.

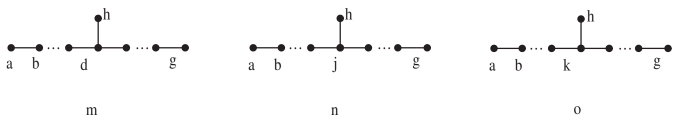

Figure 1. Graphs H_1, H_2 and H_3 considered in Theorem 3.7.

If $k = 3$, then in a similar way as in the proof above, we know that $n_1 = \frac{n-1}{2}$ when n is odd and $n_1 \in \left\{\frac{n}{2} - 1, \frac{n}{2}\right\}$ when n is even. In view of Lemma 3.4, $CC_T(v) \leq CC_{H_1}(u_1)$ when n is odd and $CC_T(v) \leq \max\{CC_{H_2}(u_2), CC_{H_3}(u_3)\}$ when n is even, where H_1, H_2, H_3 are the graphs as depicted in Fig. 1 and $u_1 \in V_{H_1}, u_2 \in V_{H_2}, u_3 \in V_{H_3}$ are the vertices labeled $\frac{n+1}{2}$, $\frac{n}{2}$ and $\frac{n+2}{2}$, respectively. By some direct calculations, it is not difficult to find that

$$CC_{H_1}(u_1) = \frac{4n^3 - 9n^2 + 26n - 33}{12} < \frac{4n^3 - 3n^2 - 4n - 9}{12}$$

for $n \geq 5$ and

$$\max\left\{CC_{H_2}(u_2), CC_{H_3}(u_3)\right\} = \frac{4n^3 - 9n^2 + 26n - 24}{12} < \frac{4n^3 - 3n^2 - 4n - 24}{12}$$

for $n \geq 6$. Note that $H_i \ncong P_n (i = 1, 2, 3)$, which leads a contradiction again.

Consequently, $k = 1$ or $k = 2$. Both lead to $T \cong P_n$. Since $CC(v)$ is as large as possible and v is not a center of P_n, v is the vertex at distance $\lceil \frac{n}{2} \rceil - 2$ from one of its end vertices by applying Lemma 2.1. By some direct calculations, one has

$$D_{P_n}(v) = \begin{cases} \frac{n^2+3}{4}, & \text{if } n \text{ is odd;} \\ \frac{n^2+4}{4}, & \text{if } n \text{ is even.} \end{cases}$$

Together with Lemmas 2.1 and 2.4, we have

$$CC_{P_n}(v) = 2W(P_n) - D_{P_n}(v) = \begin{cases} \frac{4n^3-3n^2-4n-9}{12}, & \text{if } n \text{ is odd;} \\ \frac{4n^3-3n^2-4n-24}{12}, & \text{if } n \text{ is even.} \end{cases}$$

Case 2. $T_i \cong P'_{n_i}$ with v_i being the farthest leaf from the vertex of degree 3 for some i and $T_s \cong P_{n_s}$ with $d_{T_s}(v_s) = 1$ for any $s \neq i$. Let $\tilde{T}$ be the tree satisfying $\tilde{T}_j \cong P'_{n_j}$ with v_j being the farthest leaf from the vertex of degree 3 for some j and $\tilde{T}_s \cong P_{n_s}$ with $d_{\tilde{T}_s}(v_s) = 1$ for any $s \neq j$. Then it is routine to check that

$$\begin{aligned} W(T_i) - W(\tilde{T}_i) &= 3 - n_i, \quad D_{T_i}(v_i) - D_{\tilde{T}_i}(\tilde{v}_i) = -1, \\ W(T_j) - W(\tilde{T}_j) &= n_j - 3, \quad D_{T_j}(v_j) - D_{\tilde{T}_j}(\tilde{v}_j) = 1 \end{aligned}$$

and $W(T_s) = W(\tilde{T}_s), D_{T_s}(v_s) = D_{\tilde{T}_s}(\tilde{v}_s)$ for $s \in \{1, 2, \ldots, k\} \setminus \{i, j\}$. By Lemma 2.3, one has $CC_T(v) = CC_{\tilde{T}}(\tilde{v})$. Therefore, without loss of generality, assume that $T_1 \cong P'_{n_1}$ with v being the farthest leaf from the vertex of degree 3 and $T_s \cong P_{n_s}$ with $d_{T_s}(v_s) = 1$ for any $s \neq 1$.

If $k \geq 3$, then put $T'' = T - vv_2 + bv_2$, where b is a leaf of T_k other than v_k. It is obvious that $T'' \ncong P_n$. Just similar as the proof in Case 1, we obtain $CC_{T''}(v) > CC_T(v)$, leading to a contradiction to the choice of T. Consequently, $k \leq 2$ and then $T \cong P'_n$. Since $CC(v)$ is as large as possible, v must be the vertex at distance $\lfloor \frac{n}{2} \rfloor - 2$ from the vertex of degree 3 by Lemma 2.1 again. It is not hard to get that

$$D_{P'_n}(v) = \begin{cases} \frac{n^2-5}{4}, & \text{if } n \text{ is odd;} \\ \frac{n^2-4}{4}, & \text{if } n \text{ is even.} \end{cases}$$

Together with Lemmas 2.1 and 2.5, we have

$$\begin{aligned} CC_{P_n'}(v) &= 2W(P_n') - D_{P_n'}(v) \\ &= \begin{cases} \frac{4n^3-3n^2-28n+87}{12}, & \text{if } n \text{ is odd;} \\ \frac{4n^3-3n^2-28n+84}{12}, & \text{if } n \text{ is even} \end{cases} \\ &< \begin{cases} \frac{4n^3-3n^2-4n-9}{12}, & \text{if } n \text{ is odd;} \\ \frac{4n^3-3n^2-4n-24}{12}, & \text{if } n \text{ is even} \end{cases} \end{aligned} \tag{3.7}$$

for $n \geq 5$ and we are done. ■

Theorem 3.8. *Let T be a tree on $n \geq 8$ vertices with $v \in V_T$. If neither $T \cong P_n$ with v as its center nor $T \cong P_n$ with v being the vertex of distance $\lceil \frac{n}{2} \rceil - 2$ from one of its end vertices, then*

$$CC(v) \leq \begin{cases} \frac{4n^3-3n^2-4n-45}{12}, & \text{if } n \text{ is odd;} \\ \frac{4n^3-3n^2-4n-72}{12}, & \text{if } n \text{ is even} \end{cases}$$

with equality if and only if $T \cong P_n$ with v being the vertex of distance $\lceil \frac{n}{2} \rceil - 3$ from one of its end vertices.

Proof. Choose a tree T and $v \in V_T$ such that $CC(v)$ is as large as possible, where neither $T \cong P_n$ with v as its center nor $T \cong P_n$ with v being the vertex of distance $\lceil \frac{n}{2} \rceil - 2$ from one of its end vertices. Let $N(v) = \{v_1, v_2, \ldots, v_k\}$ and T_i be the connected component of $T - vv_i$ containing v_i with $|T_i| = n_i, i = 1, 2, \ldots, k$. Without loss of generality, assume that $n_1 \geq n_2 \geq \cdots \geq n_k$. Equality 3.4 together with Lemmas 2.3, 2.4-2.5 and 3.2-3.3 yields that either $T_i \cong P_{n_i}$ with $d_{T_i}(v_i) = 1$ or $T_i \cong P'_{n_i}$ with v_i being the farthest leaf from the vertex of degree 3. Furthermore, if $T_i \cong P'_{n_i}$ with v_i being the farthest leaf from the vertex of degree 3 for some i, then $T_j \cong P_{n_j}$ with $d_{T_j}(v_j) = 1$ for any $j \neq i$.

Case 1. $T_i \cong P_{n_i}$ with $d_{T_i}(v_i) = 1$ for $1 \leq i \leq k$. If $k \geq 4$, then put $T' = T - vv_2 + hv_2$, where h is a leaf of T_k other than v_k. Obviously, $T' \ncong P_n$. Similarly, $CC_{T'}(v) > CC_T(v)$, which is impossible.

Figure 2. Graphs H_4 and H_5 considered in Theorem 3.8.

If $k = 3$, then in a similar way as in the proof of the above theorem, we have $n_1 \in \{\frac{n-1}{2}, \frac{n-3}{2}\}$ when n is odd and $n_1 \in \{\frac{n}{2} - 1, \frac{n}{2}, \frac{n}{2} - 2\}$ when n is even. Thus, as a consequence of Lemma 3.4, one has $CC_T(v) \leq \{CC_{H_1}(u_1), CC_{H_4}(u_4)\}$ when n is odd and $CC_T(v) \leq \max\{CC_{H_2}(u_2), CC_{H_3}(u_3), CC_{H_5}(u_5)\}$ when n is even, where H_1, H_2, H_3, H_4, H_5 are the graphs as depicted in Fig. 1-Fig. 2 and $u_1 \in V_{H_1}, u_2 \in V_{H_2}, u_3 \in V_{H_3}, u_4 \in V_{H_4}, u_5 \in V_{H_5}$ are the vertices labeled $\frac{n+1}{2}, \frac{n}{2}, \frac{n+2}{2}, \frac{n-1}{2}, \frac{n-2}{2}$, respectively. It can be obtained by some direct calculations that

$$\max\left\{CC_{H_1}(u_1), CC_{H_4}(u_4)\right\} = \frac{4n^3-9n^2+26n-33}{12} < \frac{4n^3-3n^2-4n-45}{12}$$

for $n \geq 7$ and

$$\max\left\{CC_{H_2}(u_2), CC_{H_3}(u_3), CC_{H_5}(u_5)\right\} = \frac{4n^3-9n^2+26n-24}{12} < \frac{4n^3-3n^2-4n-72}{12}$$

for $n \geq 8$. Note that $H_i \ncong P_n$ for $1 \leq i \leq 5$, a contradiction is obtained again.

Consequently, $k \leq 2$. Lead to the fact that $T \cong P_n$. Since neither $T \cong P_n$ with v as its center nor $T \cong P_n$ with v being the vertex of distance $\lceil \frac{n}{2} \rceil - 2$ from one of its end vertices and $CC(v)$ is as large as possible, v is the vertex at distance $\lceil \frac{n}{2} \rceil - 3$ from one of its end vertices by applying Lemma 2.1. By some direct calculations, one has

$$D_{P_n}(v) = \begin{cases} \frac{n^2+15}{4}, & \text{if } n \text{ is odd;} \\ \frac{n^2+24}{4}, & \text{if } n \text{ is even.} \end{cases}$$

Together with Lemmas 2.1 and 2.4, we have

$$CC_{P_n}(v) = 2W(P_n) - D_{P_n}(v) = \begin{cases} \frac{4n^3-3n^2-4n-45}{12}, & \text{if } n \text{ is odd;} \\ \frac{4n^3-3n^2-4n-72}{12}, & \text{if } n \text{ is even.} \end{cases}$$

Case 2. $T_i \cong P'_{n_i}$ with v_i being the farthest leaf from the vertex of degree 3 for some i and $T_s \cong P_{n_s}$ with $d_{T_s}(v_s) = 1$ for any $s \neq i$. Just similar as the proof in the above theorem, we have $k \leq 2$ and then $T \cong P'_n$. Since $CC(v)$ is as large as possible, it can be obtained from lemma 2.1 that v must be the vertex at distance $\lfloor \frac{n}{2} \rfloor - 2$ from the vertex of degree 3. By virtue of (3.7), one has

$$\begin{aligned} CC_{P'_n}(v) &= \begin{cases} \frac{4n^3-3n^2-28n+87}{12}, & \text{if } n \text{ is odd;} \\ \frac{4n^3-3n^2-28n+84}{12}, & \text{if } n \text{ is even} \end{cases} \\ &< \begin{cases} \frac{4n^3-3n^2-4n-45}{12}, & \text{if } n \text{ is odd;} \\ \frac{4n^3-3n^2-4n-72}{12}, & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

for $n \geq 8$ and the proof is complete.

■

4 Trees with the second (resp. third) smallest and largest reverse cover cost

Trees with the minimum and the maximum reverse cover cost have been determined by Georgakopoulos and Wagner [15] (see Theorem 4.1 in the following). In this section, the second (and the third) smallest and largest reverse cover cost among all trees with n vertices are discussed.

Theorem 4.1 ([15]). *Let T be a tree on $n \geq 2$ vertices with $v \in V_T$. Then*

$$n - 1 \leq RC_T(v) \leq \frac{n(n-1)(4n-5)}{6}.$$

The lower bound holds with equality if and only if $T \cong S_n$ with v being a center of it, whereas the upper bound holds with equality if and only if $T \cong P_n$ with v being one of its end vertices.

Lemma 4.2. *Let x be a leaf of an n-vertex tree T and put $T_0 = T - x$. Then $RC_T(x) = RC_{T_0}(y) + (n-1)(2n-3)$, where y is the unique neighbor of x in T.*

Proof. By short calculations, we obtain

$$D_T(x) = D_{T_0}(y) + n - 1, \quad W(T) = W(T_0) + D_T(x) = W(T_0) + D_{T_0}(y) + n - 1.$$

Together with Lemma 2.1, one has

$$\begin{aligned} RC_T(x) &= (2n-1)D_T(x) - 2W(T) \\ &= (2n-3)D_{T_0}(y) - 2W(T_0) + (n-1)(2n-3) \\ &= RC_{T_0}(y) + (n-1)(2n-3), \end{aligned}$$

as desired.

■

The graph transformation in the next lemma will be used repeatedly in the following proofs.

Lemma 4.3. *Let T be a tree with $u \in V_T$ and $d_T(u) \geq 2$. Denote $T' = T - uu_1 + vu_1$, where $v, u_1 \in N_T(u)$. Then $RC_{T'}(v) < RC_T(v)$.*

Proof. Let T be a tree on n vertices. By a direct calculation, we have

$$D_T(v) - D_{T'}(v) = |A_{u_1}(uu_1)|$$

and

$$W(T) - W(T') = |A_{u_1}(uu_1)|(|A_v(uv)| - |A_u(uv)| + |A_{u_1}(uu_1)|).$$

In view of Lemma 2.1, one has

$$\begin{aligned} RC_T(v) - RC_{T'}(v) &= (2n-1)(D_T(v) - D_{T'}(v)) - 2(W(T) - W(T')) \\ &= |A_{u_1}(uu_1)|(2n - 2|A_v(uv)| + 2|A_u(uv)| - 2|A_{u_1}(uu_1)| - 1) \\ &= |A_{u_1}(uu_1)|(4|A_u(uv)| - 2|A_{u_1}(uu_1)| - 1). \end{aligned}$$

Note that $1 \leq |A_{u_1}(uu_1)| \leq |A_u(uv)| - 1$. Then the above equality indicates that $RC_{T'}(v) < RC_T(v)$, as desired.

■

We are now in a position to establish the second smallest reverse cover cost of a vertex among all n-vertex trees.

Theorem 4.4. *Let T be a tree on $n \geq 5$ vertices with $v \in V_T$, and assume that we do not have $T \cong S_n$ with v being its center. Then $RC(v) \geq n + 4$ with equality if and only if $T \cong S_{2,n-2}$ with $d(v) = n - 2$.*

Proof. Choose a tree T with $v \in V_T$ such that $RC(v)$ is as small as possible, where we do not have $T \cong S_n$ with $d(v) = n - 1$. In order to complete the proof, it suffices to show the following two claims.

Claim 1. *$\varepsilon_T(v) = 2$.*

Proof of Claim 1. Suppose to the contrary that there exists a vertex $u \in V_T$ such that $d(v, u) = 3$. Assume that vw_1w_2u is the unique path connecting v and u. Put $T' = T - w_1w_2 + vw_2$. Then $T' \ncong S_n$. By Lemma 4.3, we have $RC_{T'}(v) < RC_T(v)$, a contradiction to the choice of T. ■

Claim 2. *There exists a unique vertex, say v', such that $d(v, v') = 2$.*

Proof of Claim 2. Suppose to the contrary that there exist $u_1, u_2 \in V_T$ such that $d(v, u_1) = d(v, u_2) = 2$. Assume that vw_1u_1 (resp. vw_2u_2) is the unique path connecting v and u_1 (resp. u_2), here $w_1 = w_2$ is allowed. Let $T'' = T - w_1u_1 + vu_1$. Then it is obvious that $T'' \ncong S_n$. Again by Lemma 4.3, we know $RC_{T''}(v) < RC_T(v)$, a contradiction to the choice of T. ■

By Claims 1-2, we have $T \cong S_{2,n-2}$ with $d(v) = n-2$. Then Lemma 2.1 together with a direct calculation yields that

$$RC(v) = (2n-1)D(v) - 2W(S_{2,n-2}) = n(2n-1) - 2(n-2)(n+1) = n+4.$$

This completes the proof. ■

Let S'_n be the graph obtained from S_{n-2} by attaching two pendant edges to two of its leaves, respectively. Then the unique tree with the third reverse cover cost among all n-vertex trees is characterized in the following.

Theorem 4.5. *Let T be a tree on $n \geq 5$ vertices with $v \in V_T$. If neither $T \cong S_n$ with $d(v) = n-1$ nor $T \cong S_{2,n-2}$ with $d(v) = n-2$, then $RC(v) \geq n+9$ with equality if and only if $T \cong S'_n$ with $d(v) = n-3$.*

Proof. Choose a tree T with $v \in V_T$ such that $RC(v)$ is as small as possible, where neither $T \cong S_n$ with $d(v) = n-1$ nor $T \cong S_{2,n-2}$ with $d(v) = n-2$. Then we complete the proof by showing the following two claims.

Claim 1. $\varepsilon_T(v) = 2$.

Proof of Claim 1. Suppose to the contrary that there exists a vertex $u \in V_T$ such that $d(v,u) = 3$. Assume that vw_1w_2u is the unique path connecting v and u. Let n_1, n_2, n_3, n_4 be the order of the components of $T - \{vw_1, w_1w_2, w_2u\}$ containing v, w_1, w_2 and u, respectively. Obviously, $n_1 + n_2 + n_3 + n_4 = n$. Let $T' = T - w_2u + w_1u$. Then $T' \notin \{S_n, S_{2,n-2}\}$. On the other hand, it is routine to check that

$$D_T(v) - D_{T'}(v) = n_4, \quad W(T) - W(T') = n_4(n_1 + n_2 - n_3).$$

The above equalities together with Lemma 2.1 give

$$\begin{aligned} RC_T(v) - RC_{T'}(v) &= (2n-1)(D_T(v) - D_{T'}(v)) - 2(W(T) - W(T')) \\ &= n_4(2n - 2n_1 - 2n_2 + 2n_3 - 1) \\ &= n_4(4n_3 + 2n_4 - 1) > 0. \end{aligned}$$

Consequently, $RC_{T'}(v) < RC_T(v)$, a contradiction to the choice of T again. ■

Claim 2. *There are exactly two vertices with distance* 2 *from* v *in* T.

Proof of Claim 2. Suppose to the contrary that there exist $u_1, u_2, u_3 \in V_T$ such that $d(v, u_1) = d(v, u_2) = d(v, u_3) = 2$. Assume that vw_1u_1, vw_2u_2 and vw_3u_3 are, respectively, the unique path connecting v and u_1, u_2, u_3, here $w_1 = w_2 = w_3$ or $w_1 = w_2 \neq w_3$ is allowed. Let $T'' = T - w_1u_1 + vu_1$. It is obvious that $T'' \notin \{S_n, S_{2,n-2}\}$. By Lemma 4.3, $RC_{T''}(v) < RC_T(v)$, a contradiction again. ■

It follows from Claims 1-2 that either $T \cong S'_n$ with $d(v) = n - 3$ or $T \cong S_{3,n-3}$ with $d(v) = n - 3$. By a direct calculation, we get

$$W(S'_n) = n^2 - 5 \quad D_{S'_n}(v) = D_{S_{3,n-3}}(v) = n + 1, \quad W(S_{3,n-3}) = n^2 - 7.$$

In view of Lemma 2.1,

$$RC_{S'_n}(v) = n + 9 < n + 13 = RC_{S_{3,n-3}}(v).$$

This completes the proof. ■

The second largest of reverse cover cost together with its corresponding extremal tree among all trees n vertices are given in the following.

Theorem 4.6. *Let* T *be a tree on* $n \geq 5$ *vertices with* $v \in V_T$*, and assume that we do not have* $T \cong P_n$ *with* v *one of its leaves. Then*

$$RC(v) \leq \frac{4n^3 - 9n^2 + 5n - 30}{6}.$$

The equality holds if and only if $T \cong P'_n$ *with* v *being the farthest leaf from the vertex of degree* 3.

Proof. We proceed by induction on n. For $n = 5$, it is routine to check that $\mathscr{T}_5 = \{P_5, P'_5, S_5\}$. Let P_5, P'_5, S_5 be the graphs with $a_i \in V_{P_5}$ $(i = 1, 2), a_j \in V_{P'_5}$ $(3 \leq j \leq 6), a_l \in V_{S_5}$ $(l = 7, 8)$ as depicted in Fig. 2. By a direct calculation, we have

$$\begin{array}{llll} RC_{P_5}(a_1) = 14, & RC_{P_5}(a_2) = 23, & RC_{P'_5}(a_3) = 45, & RC_{P'_5}(a_4) = 22, \\ RC_{P'_5}(a_5) = 9, & RC_{P'_5}(a_6) = 36, & RC_{S_5}(a_7) = 4, & RC_{S_5}(a_8) = 32. \end{array} \tag{4.1}$$

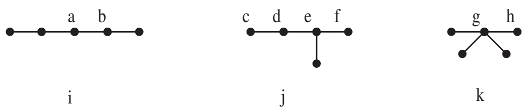

Figure 3. Graphs P_5, P'_5 and S_5 considered in Theorem 4.6.

Then $RC(v) \leq 45$ with equality if and only if $T \cong P_5'$ and $v = a_3$, which implies that the assertion holds for $n = 5$. Suppose the assertion holds for every tree T with order smaller than n which does not satisfy $T \cong P_m$ with v one of its leaves, where m is the order of T. Choose an n-vertex tree $T \ncong P_n$ or $T \cong P_n$ with $d(v) \neq 1$ such that $RC(v)$ is as large as possible.

First we prove that if $T \cong P_n$, then v cannot be a neighbor of one of its leaves. Otherwise, assume that $N(v) = \{v_1, v_2\}$, where v_2 is a leaf of P_n. Denote $T' = T - vv_2 + v_1v_2$. Then $T' \cong P_n'$ and it is routine to check that

$$W(T') - W(T) = D_{T'}(v_2) - D_T(v_2) = 3 - n, \quad D_{T'}(v) - D_T(v) = d_{T'}(v, v_2) - d_T(v, v_2) = 1.$$

Therefore, $RC_{T'}(v) - RC_T(v) = 4n - 7 > 0$, i.e., $RC_{T'}(v) > RC_T(v)$, which contradicts the choice of T.

Next we show that v is a leaf of T. If $d(v) \geq 2$ with $w_1, w_2 \in N(v)$, let $T'' = T - vw_1 + w_1w_2$. Then we do not have $T'' \cong P_n$ with $d(v) = 1$. In view of Lemma 4.3, one has $RC_{T''}(v) > RC_T(v)$, a contradiction to the choice of T.

Now assume that u is the unique neighbor of v in T. Put $T_0 = T - v$. By the induction hypothesis, one has

$$\begin{aligned} RC_T(v) &= RC_{T_0}(u) + (n-1)(2n-3) && (4.2)\\ &\leq \frac{4(n-1)^3 - 9(n-1)^2 + 5(n-1) - 30}{6} + (n-1)(2n-3) && (4.3)\\ &= \frac{4n^3 - 9n^2 + 5n - 30}{6}, \end{aligned}$$

where (4.2) is a immediate consequence of Lemma 4.2 and (4.3) holds with equality if and only if $T_0 \cong P_{n-1}'$ and u is the farthest leaf from the vertex of degree 3. Consequently, $T \cong P_n'$ with v being the farthest leaf from the vertex of degree 3 and we are done.

■

Denote by P_n'' the graph obtained from P_{n-1} by attaching a pendant edge to its third last vertex. Then we have the following result about the third largest reverse cover cost among all n-vertex trees.

Theorem 4.7. *Let T be a tree on $n \geq 5$ vertices with $v \in V_T$. If neither $T \cong P_n$ with $d(v) = 1$ nor $T \cong P_n'$ with v being the farthest leaf from the vertex of degree 3. Then*

$$RC(v) \leq \frac{4n^3 - 9n^2 + 5n - 84}{6}.$$

The equality holds if and only if $T \cong P''_n$ with v being the leaf of distance $n-4$ from the vertex of degree 3.

Proof. We proceed by induction on n. For $n = 5$, it follows from the proof of Theorem 4.6 that $RC(v) \leq 36$ with equality if and only if $T \cong P''_5$ and $v = a_6$, which implies that the assertion holds for $n = 5$. Suppose the assertion holds for every tree T with order smaller than n which satisfies neither $T \cong P_m$ with $d(v) = 1$ nor $T \cong P'_m$ with v being the farthest leaf from the vertex of degree 3, where m is the order of T.

Choose an n-vertex tree T with $v \in V_T$ such that $RC(v)$ is as large as possible, where neither $T \cong P_n$ with $d(v) = 1$ nor $T \cong P'_n$ with v being the farthest leaf from the vertex of degree 3. In a similar way as in the proof of Theorem 4.6, we know that if $T \cong P_n$, then v cannot be a neighbor of one of its leaves.

Next we prove that if $T \cong P'_n$, then v cannot be the vertex which is adjacent to the farthest leaf from the vertex of degree 3. Otherwise, let v_1 be one of the leaves of P'_n adjacent to the vertex of degree 3. By a direct calculation, we have

$$D(v_1) = \frac{n^2 - 3n + 6}{2} > \frac{n^2 - 3n + 2}{2} = D(v).$$

In view of Lemma 2.1, $RC(u) > RC(v)$, a contradiction to the choice of v.

Then we show that v is a leaf of T. If $d(v) \geq 2$ with $w_1, w_2 \in N(v)$, put $T' = T - vw_1 + w_1w_2$. Then neither $T' \cong P_n$ with $d(v) = 1$ nor $T' \cong P'_n$ with v being the farthest leaf from the vertex of degree 3. It follows from Lemma 4.3 that $RC_{T'}(v) > RC_T(v)$, which leads to a contradiction to the choice of T.

Now assume that u is the unique neighbor of v and denote $T_0 = T - v$. By induction, one has

$$\begin{aligned} RC_T(v) &= RC_{T_0}(u) + (n-1)(2n-3) && (4.4) \\ &\leq \frac{4(n-1)^3 - 9(n-1)^2 + 5(n-1) - 84}{6} + (n-1)(2n-3) && (4.5) \\ &= \frac{4n^3 - 9n^2 + 5n - 84}{6}, \end{aligned}$$

where (4.4) is a direct consequence of Lemma 4.2 and (4.5) holds with equality if and only if $T_0 \cong P'_{n-1}$ and u is the leaf of distance $n-5$ from the vertex of degree 3. Therefore, $T \cong P'_n$ with v being the leaf of distance $n-4$ from the vertex of degree 3.

This completes the proof. ■

References

[1] D. J. Aldous, Hitting times for random walks on vertex–transitive graphs, *Math. Proc. Cambridge Phil. Soc.* **106** (1989) 179–191.

[2] D. J. Aldous, J. A. Fill, *Reversible Markov Chains and Random Walks on Graphs*, Univ. California, Berkeley, 1993.

[3] N. L. Biggs, Potential theory on distance–regular graphs, *Comb. Prob. Comput.* **2** (1993) 243–255.

[4] J. A. Bondy, U. S. R. Murty, *Graph Theory*, Springer, New York, 2008.

[5] X. Chang, H. Xu, Chung–Yau invariants and graphs with symmetric hitting times, *J. Graph Theory* **85** (2017) 691–705.

[6] H. Y. Chen, Hitting times for random walks on subdivision and triangulation graphs, *Lin. Multilin. Algebra* **66** (2018) 117–130.

[7] H. Y. Chen, F. J. Zhang, The expected hitting times for graphs with cutpoints, *Stat. Prob. Lett.* **66** (2004) 9–17.

[8] H. Y. Deng, The trees on $n \geq 9$ vertices with the first to seventeenth greatest Wiener indices are chemical trees, *MATCH Commun. Math. Comput. Chem.* **57** (2007) 393–402.

[9] L. Devroye, A. Sbihi, Random walks on highly symmetric graphs, *J. Theor. Prob.* **4** (1990) 497–514.

[10] A. A. Dobrynin, R. Entringer, I. Gutman, Wiener index of trees: theory and applications, *Acta Appl. Math.* **66** (2001) 211–249.

[11] P. G. Doyle, J. L. Snell, *Random Walks and Electrical Networks*, Math. Assoc. Am., Washington, 1984.

[12] R. C. Entringer, Distance in graphs: trees, *J. Comb. Math. Comb. Comput.* **24** (1997) 65–84.

[13] R. C. Entringer, D. E. Jackson, D. A. Snyder, Distance in graphs, *Czech. Math. J.* **26** (1976) 283–296

[14] A. Georgakopoulos, A tractable variant of cover time, preprint 2012.

[15] A. Georgakopoulos, S. Wagner, Hitting times, cover cost, and the Wiener index of a tree, *J. Graph Theory* **84** (2017) 311–326.

[16] Z. L. Guo, S. C. Li, X. Liu, X. L. Mei, Expected hitting times for random walks on the diamond hierarchical graphs involving some classical parameters, *Lin. Multilin. Algebra*, accepted.

[17] I. Gutman, S. C. Li, W. Wei, Cacti with n vertices and t cycles having extremal Wiener index, *Discr. Appl. Math.* **232** (2017) 189–200.

[18] J. Huang, S. C. Li, Expected hitting times for random walks on quadrilateral graphs and their applications, *Lin. Multilin. Algebra* **66** (2018) 2389–2408.

[19] J. Huang, S. C. Li, Z. Xie, Further results on the expected hitting and the related invariants of graphs, *Discr. Math.* **342** (2019) 78–95.

[20] J. G. Kemeny, J. L. Snell, *Finite Markov Chains*, Princeton, Van Nostrand, 1960.

[21] M. Knor, R. Škrekovski, A. Tepeh, Mathematical aspects of Wiener index, *Ars Math. Contemp.* **11** (2016) 327–352.

[22] S. C. Li, S. J. Wang, Extremal cover cost and reverse cover cost of trees with given segment sequence, *Discr. Math.* **343** (2020) #111791.

[23] L. Lovász, Random walks on graphs, *Bolyai Soc. Math. Stud.* **2** (1993) 1–46.

[24] J. L. Palacios, J. M. Renom, Random walks on edge transitive graphs, *Stat. Prob. Lett.* **37** (1998) 29–34.

[25] J. L. Palacios, J. M. Renom, P. Berrizbeitia, Random walks on edge transitive graphs (II), *Stat. Prob. Lett.* **43** (1999) 25–32.

[26] A. Van Slijpe, Random walks on the triangular prism and other vertex transitive graphs, *J. Comput. Appl. Math.* **15** (1986) 383–394.

[27] H. Wiener, Structural determination of paraffin boiling points, *J. Am. Chem. Soc.* **69** (1947) 17–20.

[28] C. Y. Wang, Z. L. Guo, S. C. Li, Expected hitting times for random walks on the k-triangle graph and their applications, *Appl. Math. Comput.* **338** (2018) 698–710.

[29] K. Xu, M. Liu, K. C. Das, I. Gutman, B. Furtula, A survey on graphs extremal with respect to distance–based topological indices, *MATCH Commun. Math. Comput. Chem.* **71** (2014) 461–508.

[30] H. Xu, S. T. Yau, Discrete Green's functions and random walks on graphs, *J. Comb. Theory A* **120** (2013) 483–499.