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More Lower Bounds for Two Kirchhoffian Indices

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Abstract

Using standard tools of electric networks we find two new lower bounds for both the multiplicative and additive degree-Kirchhoff indices, $K^*(G)$ and $K^+(G)$, of a graph $G = (V, E)$ in terms of a small number of parameters: $|V|, |E|$, and the smallest and largest degrees. The bounds are attained by the complete graph and by a large family of strongly regular graphs.

1 Introduction

In what follows we will work with a simple connected graph $G = (V, E)$ whose vertex set is $V = \{1, 2, \ldots, n\}$, whose edge set is E and whose degree sequence is $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta$. Reference [30] contains all concepts from Graph Theory used here and not explicitly defined.

One of the most thoroughly studied indices or descriptors of a graph G is the Kirchhoff index, introduced in [18] by Klein and Randić, and defined by

$$K(G) = \sum_{i<j} R_{ij},$$

where R_{ij} denotes the effective resistance computed between the vertices i and j by means of Ohm's laws. The multiplicative degree-Kirchhoff index is a related resistive descriptor, proposed by Chen and Zhang in [5], and defined by

$$K^*(G) = \sum_{i<j} d_i d_j R_{ij}.$$

References [1], [4], [10], [13], [16], [21], [23] [24] and [33] are a sample of earlier articles related to the index $K^*(G)$. More recently a lot of interest has been placed in finding maximal graphs among certain families of graphs (see [9]) and also in computing the value of the index, and the number of spanning trees, in certain intricate graphs by finding the eigenvalues μ_i of the normalized Laplacian of these graphs (see [14], [19]), and exploiting the characterization

$$K^*(G) = 2|E| \sum_i \frac{1}{\mu_i} \tag{1}$$

in terms of these eigenvalues.

A close relative of the indices above is the additive degree-Kirchhoff index, introduced in [11] and defined as

$$K^+(G) = \sum_{i<j} (d_i + d_j) R_{ij}. \tag{2}$$

The articles [2], [8], [15], [17], [20] [26], [27], [31] and [32] are examples of research focused on finding bounds, extremal graphs and closed-form formulas for several families of graphs with respect to this index.

2 Bounds for the multiplicative degree–Kirchhoff index

In [23], through an expression equivalent to (1) but in terms of the eigenvalues of the transition probability matrix of the random walk on G, we proved the inequality

$$K^*(G) \geq 2|E| \left(n - 2 + \frac{1}{n} \right) \tag{3}$$

for an arbitrary G.

In [24] we improved (3) to

$$K^*(G) \geq 2|E| \left(n - 2 + \frac{1}{\Delta + 1} \right). \tag{4}$$

This bound was used to prove that the star graph S_n attains the minimum of $R^*(G)$ for all G.

In [1], using majorization, we found the bound

$$K^*(G) \geq 2|E| \left(\frac{1}{1+\frac{\sigma}{\sqrt{n-1}}} + \frac{(n-2)^2}{n-1-\frac{\sigma}{\sqrt{n-1}}} \right), \tag{5}$$

where $\sigma^2 = \frac{2}{n} \sum_{(i,j)\in E} \frac{1}{d_i d_j} = \frac{2}{n} R_{-1}(G)$, and $R_{-1}(G)$ is the generalized Randić index with $\alpha = -1$.

Later in [28], refining the arguments in [24], we improved (4) to

$$K^*(G) \geq n-1+2|E|(n-2). \tag{6}$$

The equalities in (3) through (6) are attained by the complete graph K_n. The one in (6) is also attained by the star graph S_n

Next we will present a couple of new inequalities for the multiplicative degree-Kirchhoff index, using some electric ideas that have been exploited in the literature in several different scenarios. The first bound involves only the parameters n, $|E|$ and δ; the second uses additionally the parameter Δ.

Theorem 1 *The multiplicative degree-Kirchhoff index of an n-vertex graph G satisfies*

$$K^*(G) \geq \delta^2(n-1) + 2\delta \left(\binom{n}{2} - |E| \right), \tag{7}$$

and

$$K^*(G) \geq \delta^2(n-1) + 2|E|(n-1-\Delta). \tag{8}$$

These lower bounds are attained by K_n.

Proof. According to Foster's first sum rule (see [12]) we have $\sum_{i<j:d(i,j)=1} R_{ij} = n-1$, and this together with the inequality (see [6])

$$R_{ij} \geq \frac{1}{d_i} + \frac{1}{d_j},$$

that holds when i and j are not neighbors, allows us to get

$$\begin{aligned} K^*(G) = \sum_{i<j:d(i,j)=1} d_i d_j R_{ij} + \sum_{i<j:d(i,j)>1} d_i d_j R_{ij} &\geq \delta^2(n-1) + \sum_{i<j:d(i,j)>1} d_i d_j R_{ij} \\ &\geq \delta^2(n-1) + \sum_{i<j:d(i,j)>1} d_i d_j \left(\frac{1}{d_i} + \frac{1}{d_j} \right) \\ &= \delta^2(n-1) + \sum_{i<j:d(i,j)>1} (d_i + d_j). \end{aligned} \tag{9}$$

Now we may choose to bound $d_i + d_j \geq 2\delta$ in (9), and using this we get the bound (7).

Alternatively, we may write

$$\sum_{i<j:d(i,j)>1}(d_i+d_j) = \sum_{i<j}(d_i+d_j) - \sum_{i<j:d(i,j)=1}(d_i+d_j)$$

$$= 2|E|(n-1) - \sum_{i<j:d(i,j)=1}(d_i+d_j) \geq 2|E|(n-1) - 2|E|\Delta,$$

and using this in (9) we get (8).

It is clear that for $G = K_n$ both lower bounds are equal to $(n-1)^3 = K^*(K_n)$ ■

If all degrees in either (7) or (8) are equal to d we obtain the following result that was first found in [22]:

Theorem 2 *If the graph is d-regular the multiplicative degree-Kirchhoff index satisfies*

$$K^*(G) \geq d(n^2 - n - d). \tag{10}$$

We will prove now that the equality in (10) (and thus also in (7) and (8)) is attained, in addition to K_n, by a particular set of strongly regular graphs. The discussion of this family of graphs follows, and improves, the material in [25]. Denote by $N(i)$ the set of all neighbors of the vertex i, and by D the diameter of the graph, that is, $D = \max_{i,j}\{d(i,j) : i,j \in V\}$. Then we can prove

Theorem 3 *If G is d-regular, has diameter $D = 2$ and satisfies $|N(i) \cap N(j)| = d$ for all non-neighboring vertices, then the bound (10) is attained by G.*

Proof. Suppose that i and j are not neighbors; since they share d neighbors, by deleting all other edges but the ones between i, j and their neighbors, and applying the monotonicity principle ([7]), the effective resistance R_{ij} is bounded above by the resistance of a circuit built with d paths of length 2 laid out in parallel between i and j, which is $\dfrac{2}{d}$. Using also Foster's first sum rule we obtain

$$K^*(G) = d^2K(G) = d^2\left(n-1+\sum_{i<j:d(i,j)=2}R_{ij}\right) \leq d^2\left(n-1+\sum_{i<j:d(i,j)=2}\frac{2}{d}\right)$$

$$= d^2(n-1) + 2d\sum_{i<j:d(i,j)=2}1 = d^2(n-1) + 2d\left(\binom{n}{2} - \frac{nd}{2}\right),$$

which is the same as (10) ■

The conditions in the previous proposition are met by strongly regular graphs (see [3]) with parameters (n, d, ν, d), for some ν. Such graphs include the class of complete r-partite graph $K_{s,s,\ldots,s}$, for $r \geq 2, s \geq 2$, with $d = (r-1)s$ and $\nu = (r-2)s$. We suspect there are no other strongly regular graphs satisfying the condition that the second and fourth parameters are equal, and no other diameter 2 regular graphs satisfying theorem 3.

Remark 1. The new bounds (7) and (8) are not comparable to (6), and they are not comparable to one another. To see this, notice first that (6) attains the actual value $K^*(S_n)$, for $n \geq 3$, whereas (7) and (8) do not. Conversely, for G a d-regular graph, the bound (6) becomes

$$K^*(G) \geq n - 1 + nd(n-2),$$

which is worse than (10) as long as $d < n-1$. In other words, (6) is outperformed by (7) and (8) on regular graphs other than the complete graph.

Also, the bound given by (8) is strictly smaller than the one given by (7) when $\Delta = n-1$ and the graph is not the complete graph. Finally, for the linear graph on n vertices (7) gives the bound $(n-1)^2$ and (8) gives the bound $(n-1)(2n-5)$. Therefore (7) and (8) are not comparable.

3 Bounds for the additive degree–Kirchhoff index

In [27] we showed that for all G

$$K^+(G) \geq K^+(K_n) = 2(n-1)^2. \tag{11}$$

Later in [2] we showed a cornucopia of lower bounds in different scenarios, working with properties on electric networks and majorization: bounds in terms of the inverse index, bounds with a fixed number of pendant vertices, bounds dependent on the sum $H = \sum_{i<j} \frac{d_i}{d_j}$, etc. An example of these is:

$$K^+(G) \geq n(n-4) + 2|E| \sum_{i=1}^{n} \frac{1}{d_i}. \tag{12}$$

Several of these bounds depended on a large number of parameters and were attained only by the complete graph K_n.

In [31] Yang and Klein obtained the inequality

$$K^+(G) \geq \left(\delta + \frac{2|E|}{n}\right)\left(n - 1 + \frac{n}{\delta} - \frac{\delta+1}{n-1}\right), \tag{13}$$

in terms of three parameters, and where the equality is attained by the complete graph K_n.

Similar results were obtained in [26], such as the following inequalities

$$K^+(G) \geq \frac{4|E|}{\Delta}\left(n - 2 + \frac{1}{\Delta+1}\right), \tag{14}$$

and

$$K^+(G) \geq (n-1)^2 + n\left(n - 2 + \frac{1}{\Delta+1}\right) \geq 2(n-1)^2, \tag{15}$$

where the latter obviously improves (11), and both are attained by K_n.

A common trait of the inequalities in [31] and [26] for $K^+(G)$ is that they were obtained as corollaries of other inequalities for $K^*(G)$.

Here the approach is different: we work with properties of electric networks applied directly to $K^+(G)$ and find two lower bounds, both of which depend only on the parameters n, $|E|$, δ and Δ, and which are attained by K_n and the same family of strongly regular graphs as in the previous section.

Theorem 4 *The additive degree-Kirchhoff index of an n-vertex graph G satisfies*

$$K^+(G) \geq 2\delta\left[(n-1) + \frac{2}{\Delta}\left(\binom{n}{2} - |E|\right)\right], \tag{16}$$

and

$$K^+(G) \geq 2\delta(n-1) + \frac{4|E|}{\Delta}(n - 1 - \Delta). \tag{17}$$

These lower bounds are attained by K_n and by all the graphs in theorem 3.

Proof. With similar arguments to those in the proof of theorem 1 we get

$$K^+(G) = \sum_{i<j:d(i,j)=1} (d_i + d_j)R_{ij} + \sum_{i<j:d(i,j)>1} (d_i + d_j)R_{ij}$$

$$\geq 2\delta(n-1) + \sum_{i<j:d(i,j)>1} (d_i + d_j)\left(\frac{1}{d_i} + \frac{1}{d_j}\right) \geq 2\delta(n-1) + \frac{2}{\Delta}\sum_{i<j:d(i,j)>1} (d_i + d_j).$$

Now we may choose either of the two bounds used in theorem 1 for the summation $\sum_{i<j:d(i,j)>1}(d_i + d_j)$, and (16) and (17) follow.

For $G = K_n$ both lower bounds are equal to $2(n-1)^2 = K^+(K_n)$

If all degrees in either (16) or (17) are equal to d we obtain

$$K^+(G) \geq 2(n^2 - n - d), \tag{18}$$

and since in this case $K^+(G) = 2dK(G)$, the argument in theorem 3, with the necessary changes being made, yields that the equality is attained by the family of d-regular graphs with diameter 2 satisfying $|N(i) \cap N(j)| = d$ ■

Remark 2. The same examples in remark 1 show that (16) and (17) are not comparable. It would be a bit tedious to go through all the comparisons between the new bounds and all the previous ones in order to show that they are not comparable. Suffice to say that when the graph is d-regular the bounds (12), (13), (14) and (15) become

$$K^+(G) \geq 2(n^2 - 2n), \tag{19}$$

$$K^+(G) \geq 2d\left(n - 1 + \frac{n}{d} - \frac{d+1}{n-1}\right), \tag{20}$$

$$K^+(G) \geq 2(n^2 - 2n + \frac{n}{d+1}), \tag{21}$$

and

$$K^+(G) \geq 2n^2 - 4n + 1 + \frac{n}{d+1}, \tag{22}$$

respectively, *all of which are weaker than* (18). It is immediate to see this for (19), (21) and (22). Perhaps (20) deserves a short explanation: in the interval $2 \leq d \leq n-1$, the parabola $f(d) = d(n^2 - n - 1 - d)$ attains its maximum value at the right end of the interval, i.e., $f(d) \leq f(n-1)$ for all d, or in other words

$$d(n^2 - n - 1 - d) \leq (n-2)(n-1)n.$$

Multiplying by $\frac{2}{n-1}$ both sides of the prior equation, and then adding $2n - 2d$ to both sides we get

$$2d\left(n - 1 + \frac{n}{d} - \frac{d+1}{n-1}\right) \leq 2(n^2 - n - d),$$

as claimed.

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