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A Very Short Proof for a Lower Bound for Energy of Graphs

Mohammad Reza Oboudi

Department of Mathematics, College of Sciences, Shiraz University, Shiraz, 71457-44776, Iran
mr_oboudi@yahoo.com, mr_oboudi@shirazu.ac.ir

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Abstract

Let G be a simple graph with n vertices $v_1,\dots,v_n$ and m edges. Let $A(G) = [a_{ij}]_{n\times n}$ be the adjacency matrix of G, that is $a_{ij} = 1$ if v_i and v_j are adjacent, and $a_{ij} = 0$, otherwise. Let $\lambda_1,\dots,\lambda_n$ be the eigenvalues of $A(G)$. The energy of G, denoted by $\mathcal{E}(G)$, is defined as $|\lambda_1| + \cdots + |\lambda_n|$. Let G be a connected graph with minimum degree $\delta(G)$. In [X. Ma, *MATCH Commun. Math. Comput. Chem.* **81** (2019) 393–404] with a long proof, it is shown than $\mathcal{E}(G) \geq 2\delta(G)$ and that the equality holds if and only if G is a regular complete multipartite graph. In this paper we prove the same result in a few lines.

1 Introduction

Throughout this paper we will consider only simple graphs (finite and undirected, without loops and multiple edges). Let $G = (V(G), E(G))$ be a simple graph. For every vertex $v \in V(G)$, the *degree* of v is the number of edges incident with v and is denoted by $deg_G(v)$. An *isolated* vertex is a vertex with degree zero. A *regular* graph is a graph such that all of its vertices have the same degree. The *complete* graph with n vertices is denoted by K_n. Let $t \geq 2$ and $n_1,\dots,n_t$ be some positive integers. By $K_{n_1,\dots,n_t}$ we mean the *complete multipartite graph* with parts size $n_1,\dots,n_t$. In particular, the *complete bipartite graph* with part sizes m and n is denoted by $K_{m,n}$.

Let G be a graph with vertex set $\{v_1,\dots,v_n\}$. The *adjacency matrix* of G, $A(G) = [a_{ij}]$, is the $n \times n$ matrix such that $a_{ij} = 1$ if v_i and v_j are adjacent, and $a_{ij} = 0$, otherwise. Since $A(G)$ is symmetric, all of the eigenvalues of $A(G)$ are real. By the eigenvalues of G we mean those of its adjacency matrix. We denote the eigenvalues of G by $\lambda_1(G) \geq \cdots \geq \lambda_n(G)$. The energy of graphs was defined by Ivan Gutman in 1978. For example, since the eigenvalues of the complete graph K_n are $n-1$ (with multiplicity 1) and -1 (with multiplicity $n-1$), so $\mathcal{E}(K_n) = 2n-2$. See [3–6, 8, 9] for more details. In [7] with a long proof it is shown that if G is a connected graph with minimum degree $\delta(G)$, then $\mathcal{E}(G) \geq 2\delta(G)$, and the equality holds if and only if G is a regular complete multipartite graph. In this paper we state a very short proof for this result.

2 A very short proof for the lower bound

In this section we prove the main result of the paper. We need the following well known results.

Theorem 1. [1] *Let G be a graph with n vertices and m edges. Then*

$$\lambda_1(G) \geq \frac{2m}{n},$$

and the equality holds if and only if G is regular.

Theorem 2. [2, 10] *A graph has exactly one positive eigenvalue if and only if its non-isolated vertices form a complete multipartite graph.*

Remark 1. *Let G be a graph with n vertices and m edges. Assume that $\lambda_1, \ldots, \lambda_n$ are all eigenvalues of G. It is well known that $\lambda_1 + \cdots + \lambda_n = 0$ and $\lambda_1^2 + \cdots + \lambda_n^2 = 2m$. These equalities show that if G has at least one edge, then G has at least one positive eigenvalue. Now assume that G has at least one edge and $p \geq 1$ be the number of positive eigenvalues of G. Suppose that $\lambda_1, \ldots, \lambda_p$ be the positive eigenvalues of G and $\lambda_{p+1}, \ldots, \lambda_n$ be the non-positive eigenvalues of G. Since $\lambda_1 + \cdots + \lambda_n = 0$, we find that $\mathcal{E}(G) = \lambda_1 + \cdots + \lambda_p - \lambda_{p+1} - \cdots - \lambda_n = 2\lambda_1 + \cdots + 2\lambda_p$.*

Now we state a very short prove for the following result that has been proved in [7] with long proof.

Theorem 3. [7] *Let G be a connected graph with minimum degree $\delta(G)$. Then*

$$\mathcal{E}(G) \geq 2\delta(G),$$

and the equality holds if and only if $G = K_1$ or G is a regular complete multipartite graph.

Proof. Let n and m be the number of vertices and the number of edges of G, respectively. If $m = 0$, then there is nothing to prove. So let $m \geq 1$. Thus G has at least one positive eigenvalue. Let p be the number of positive eigenvalues of G and $\lambda_1 \geq \cdots \geq \lambda_p$ be the positive eigenvalues of G. Since $\sum_{v \in V(G)} deg(v) = 2m$, $2m \geq n\delta(G)$ and the equality holds if and only if G is regular. Using Remark 1 and Theorem 1 we find that

$$\mathcal{E}(G) = 2\lambda_1 + \cdots + 2\lambda_p \geq 2\lambda_1 \geq \frac{4m}{n} \geq 2\delta(G). \tag{1}$$

Thus $\mathcal{E}(G) \geq 2\delta(G)$ and the equality holds if and only if $p = 1$ and G is regular. Since G is connected, by Theorem 2, the equality holds if and only if G is a complete multipartite graph. In other words, $G = K_{q,\ldots,q}$ for some positive integer q. The proof is complete. ■

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