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A Survey on Borderenergetic Graphs

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Abstract

A graph G of order n is said to be borderenergetic if its energy is equal to the energy of complete graph K_n. In this paper, we review some results concerning the ordinary borderenergetic, (signless) Laplacian and Seidel borderenergetic graphs.

1 Introduction

We first recall some definitions that will be kept throughout. Let $G = (V, E)$ be a simple graph with $n = |V(G)|$ vertices and $m = |E(G)|$ edges, and $A(G)$ denotes its adjacency matrix. The eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of $A(G)$ compose the eigenvalues of the graph G. If G has exactly s distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_s$ with multiplicities, respectively, $t_1, t_2 \ldots, t_s$ then we write $\mathrm{spec}(G) = \{[\lambda_1]^{t_1}, [\lambda_2]^{t_2}, \ldots, [\lambda_s]^{t_s}\}$. The nullity $\eta(G)$ of the graph G is the multiplicity of the eigenvalue zero in its adjacency spectrum. A graph is said to be integral if all eigenvalues of its adjacency matrix consist entirely of integers.

The Laplacian and signless Laplacian matrix of graph G are respectively, $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$, where $D(G) = (d_{ij})$ is the diagonal matrix whose entries are the degree of vertices, namely $d_{ii} = \deg(v_i)$ and $d_{ij} = 0$ for $i \neq j$.

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The first Zagreb index $M_1(G)$ is equal to the sum of squares of the degrees of the vertices. Two Zagreb indices $M_1(G) = \sum_{uv\in E}(d_u + d_v)$ and $M_2(G) = \sum_{uv\in E} d_u d_v$ are vertex-degree-based graph invariants that have been introduced in the 1970s, see [32,34].

The maximum degree and the minimum degree of G are denoted by Δ and δ, respectively. The girth of graph G is the length of a shortest cycle of G. A complete bipartite graph with a bipartition of sizes n_1 and n_2 is denoted by K_{n_1,n_2}. The line graph of G, denoted by $\mathcal{L}(G)$, is a graph such that the vertex set of $\mathcal{L}(G)$ is the edge set of G and two vertices u and v of $\mathcal{L}(G)$ are adjacent if the edges corresponding to u and v share a common end vertex.

The energy of the graph was introduced by Ivan Gutman [23] $\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|$, where λ_i's are eigenvalues of G. If $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ and $q_1 \geq q_2 \geq \cdots \geq q_n$ are the Laplacian and the signless Laplacian eigenvalues of G then two quantities $\mathcal{E}_L(G) = \sum_{i=1}^{n} \left|\mu_i - \bar{d}\right|$ and $\mathcal{E}_Q(G) = \sum_{i=1}^{n} \left|q_i - \bar{d}\right|$, where $\bar{d}$ is the average degree of G, are called the Laplacian and the signless Laplacian energy of G, respectively.

In 1966, Van Lint and Seidel [58] introduced a symmetric $(0, 1, -1)$-adjacency matrix for a graph G called the Seidel matrix of G of $S(G) = J - I - 2A(G)$, where J is the matrix with entries 1 in every position. In [35], Haemers defined the Seidel energy of G as $\mathcal{E}_S(G) = \sum_{i=1}^{n} |\rho_i|$, where ρ_i's are the Seidel eigenvalues of G.

The upper bound $\mathcal{E}(G) \leq \sqrt{2mn}$, was established by McClelland in [48]. In the mentioned paper, an approximate formula for energy of graphs was proposed:

$$\mathcal{E}(G) \approx a\sqrt{2mn}, \quad a \approx 0.9, \tag{1}$$

which was eventually demonstrated to be highly accurate in the case of molecular graphs [26,33]. An additional corroboration of this formula was the analogous lower

$$\mathcal{E}(G) \approx \sqrt{\frac{16}{27}}\sqrt{2mn},$$

that holds for certain molecular graphs, in particular, for hexagonal systems [24]. According to formula (1), the energy of a graph would be a monotonically increasing function with respect to the number of edges. If this formula could be applied to all graphs, then among graphs with n vertices, the complete graph K_n would have the greatest energy, equal to $E(K_n) = 2n - 2$. Counter examples for this naive conjecture were soon discovered [7]. Somewhat later [59], the first systematic construction of graphs with the property $\mathcal{E}(G) > E(K_n)$ were reported.

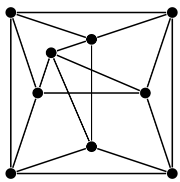
Graphs of order n with the property $\mathcal{E}(G) > 2n - 2$ are called named hyperenergetic [27]. Numerous classes of hyperenergetic graphs have been recognized; for details see the survey [30], and the recent paper [17]. The search for hyperenergetic graphs became purposeless after Nikiforov proved in [50] that for almost all n-vertex graphs

$$\mathcal{E}(G) = \left(\frac{4}{3\pi} + o(n)\right) n^{\frac{3}{2}},$$

implying that almost all graphs are hyperenergetic. The question that remained open was if there exist graphs of order n, other than K_n, satisfying the equality $\mathcal{E}(G) = 2n - 2$. This class of graphs are called the borderenergetic graphs. The aim of continuing this paper is to deal with this graph invariant.

2 Ordinary borderenergetic graphs

The first borderenergetic graph was discovered by Hou et al. in [39], but in that time it did not attract much attention. The earliest example of $E(G) = E(K_n)$ is a graph of order 9, that was reported in [39], see Figure 1. This borderenergetic graph is the line graph of the complete bipartite graph $K_{3,3}$ and its spectrum is $\mathrm{spec}(G) = \{[4]^1, [1]^4, [-2]^4\}$. Therefore, $E(\mathcal{L}(K_{3,3})) = 16 = 2(9) - 2 = E(K_9)$.



$\mathcal{L}(K_{3,3})$

Figure 1. The first borderenergetic graph, discovered in year 2001.

In the winter of 2014 when Professor Gutman visited Linan, China, He posed the problem to construct borderenergetic graphs to one of the author X. Li. Then, Li together with Gong and Xu was immediately devoted to work on this interesting problem, and they produced a joint paper with Gutman. So, Gong et al. [22] is the first paper to give the official name for borderenergetic graph, there they studied the graphs with the same energy as the complete graph K_n. A graph G on n vertices is said to be borderenergetic if its energy equals the energy of the complete graph K_n, namely if $\mathcal{E}(G) = E(K_n) = 2n-2$.

After that, more papers on the borderenergetic graphs and similar concepts, such as L-borderenergetic graphs, Q-borderenergetic graphs, etc. have been published. In 2017 Li was invited to write a survey paper on borderenergetic graphs by a Chinese journal Journal of Anhui University, see [11]. Since it is in Chinese, an English survey paper should be more popular for the readers working along with this subject. Our this survey paper is an updated version of [11].

In [22], it was shown that there exist borderenergetic graphs on order n for each integer $n \geq 7$. The number of borderenergetic graphs were determined for $n = 7, 8, 9$ in [22], $n = 10, 11$ in [45,51] and $n = 12$ in [19]. A family of non-regular and non-integral borderenergetic threshold graphs was discovered in [40]. In [12], the authors obtained three asymptotically tight bounds on the number of edges of borderenergetic graphs.

In [15], the authors proved that a borderenergetic graph is not bipartite when G is a sparse graph. Moreover, for a borderenergetic bipartite graph, they presented a lower bound for the largest eigenvalue and an upper bound for the middle eigenvalue, respectively.

In [56], the authors proposed a procedure for the construction of borderenergetic graphs and investigated three sequences of borderenergetic graphs. Li et al. in [46] studied the existence of borderenergetic chemical graphs (a graph is chemical if it has a maximum degree at most 4) and showed that there is no borderenergetic graph with maximum degree at most 3. They proved five necessary conditions for borderenergetic graphs with maximum degree 4, and as a result, they showed that there is no borderenergetic graph with maximum degree 4 and order $n \geq 22$. They also considered a problem concerning borderenergetic graphs with large minimum degrees. In continuing, they showed that there is no borderenergetic graph of order n with minimum degree $n-2$ and then they constructed two families of borderenergetic graphs with minimum degree $n-3$ and $n-4$, respectively, the former is for all integers $n \geq 7$ while the latter is for all even numbers $n \geq 8$. We refer the readers to [11,31] for more information.

Recently Deng et al. in [8] investigated the girth of a borderenergetic graph G in the case that G is a dense graph, and inferred that the girth is 3.

As shown in [3], the energy of an integral graph is necessarily an even integer. Since before the exploring of borderenergetic graphs, all discovered graphs with integer energy were integral and there was a conjecture as follows.

Conjecture 2.1. *If $\mathcal{E}(G)$ is an integer, then the graph G is integral.*

Clearly this conjecture is wrong and the borderenergetic graphs G_i $(i = 1, 2, 3)$ of order 8, depicted in Figure 2, provide the counterexamples. The spectra of these graphs are

$$\begin{aligned}
\mathrm{spec}(G_1) &= \left\{[3+\sqrt{6}]^1, [1]^1, [3-\sqrt{6}]^1, [-1]^3, [-2]^2\right\},\\
\mathrm{spec}(G_2) &= \left\{[(5+\sqrt{17})/2]^1, [1]^2, [(5-\sqrt{17})/2]^1, [-1]^1, [-2]^3\right\},\\
\mathrm{spec}(G_3) &= \left\{[(7+\sqrt{33})/2]^1, [(7-\sqrt{33})/2]^1, [-1]^5, [-2]^1\right\}.
\end{aligned}$$

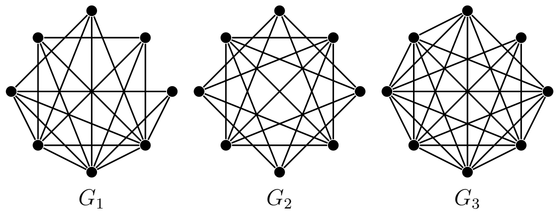


Figure 2. The three distinct non-integral borderenergetic graphs of smallest order 8 with $\mathcal{E}(G_i) = 14$ $(i = 1, 2, 3)$.

Here, we review some basic theorem, concerning the ordinary the borderenergetic graphs.

Theorem 2.2. *We have the following statements:*

1) [22] *There is no borderenergetic graph of order $n \leq 6$.*

2) [22] *For any $n \geq 7$, there exist borderenergetic graph of order n.*

3) [22] *There exists a unique borderenergetic graph of order* 7*. This graph is depicted in Figure* 3*.*

4) [22] *There are exactly* 6 *borderenergetic graphs of order* 8*, five of which have $\delta = 4 = n - 4$. These graphs are depicted in Appendix (Figure* 9*).*

5) [22] *There are exactly* 17 *borderenergetic graphs of order* 9*. Only four among them are integral and exactly one graph is with $\delta = 6 = n - 3$. These graphs are depicted in Appendix (Figure* 10*).*

6) [45, 51] *There are exactly* 49 *borderenergetic graphs of order* 10*, among which* 37 *are non-co-spectral.*

7) [51] *There are exactly* 158 *borderenergetic graphs of order* 11*, of which* 157 *are connected.*

8) [19] *There are exactly* 572 *connected borderenergetic graphs of order* 12*. In Appendix (Table* 3*) the distributions of* 12*-vertex borderenergetic graphs together with the number of edges are shown.*

9) [22] *For each integer* n ($n \geq 13$)*, there exists a non-complete borderenergetic graph of order* n*.*

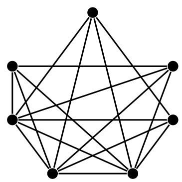


Figure 3. The smallest non-complete borderenergetic graph.

Two graphs G and G' with the same energy are called equienergetic, see [2, 5]. Evidently, two co-spectral graphs or borderenergetic graphs of the same order are equienergetic but generally there are numerous families of mutually non-co-spectral graphs that are equienergetic [44]. In Theorem 2.2 the large families of mutually equienergetic graphs is reported.

In follow, we investigate non-complete borderenergetic graphs by means of tensor product, line graph, strongly regular graphs, the union of graphs, and complements, respectively.

The tensor product of two graphs G_1 and G_2 is the graph $G_1 \otimes G_2$ with vertex set $V(G_1) \times V(G_2)$, in which two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if both $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$. The number of vertices G denote by $|V(G)|$.

Theorem 2.3. [22] *Let* G *be a borderenergetic graph. Suppose that* G *is obtained from the tensor product of two integral graphs* H_1 *and* H_2*. Then both* $|V(H_1)|$ *and* $|V(H_2)|$ *are odd numbers.*

Theorem 2.4. [22] *The line graph of a Petersen graph is a connected non-complete borderenergetic graph.*

Consider a regular graph G which is neither complete nor empty. Then G is said to be a strongly regular graph with parameters (n, k, a, c) if it is k-regular with order n, every pair of adjacent vertices has a common neighbors, and every pair of distinct non-adjacent vertices has c common neighbors. As known [21] the eigenvalues of a strongly regular graph with parameters (n, k, a, c) are k with multiplicity 1, $\theta = [(a-c)+\sqrt{\Delta}]/2$ with multiplicity m_θ and $\tau = [(a-c)-\sqrt{\Delta}]/2$ with multiplicity m_τ, where $\Delta = (a-c)^2 + 4(k-c)$and m_θ, m_τ satisfy the equations

$$m_\theta + m_\tau = n - 1,$$

$$\theta m_\theta + \tau m_\tau = -k.$$

Usually, a strongly regular graph with $m_\theta = m_\tau$ is called a conference graph.

Theorem 2.5. [22] *Let G be a conference graph. If G is integral and non-complete borderenergetic, then G has parameters $(9, 4, 1, 2)$.*

The union of two vertex disjoint graphs G and H is denoted by $G \cup H$. The union of k vertex-disjoint copies of a graph H is sometimes denoted by kH. The join $G + H$ is the graph obtained from $G \cup H$ by connecting all vertices from $V(G)$ with all vertices from $V(H)$. The complement of a graph G, denoted by $\overline{G}$, has the same vertices as G where two vertices in $\overline{G}$ are adjacent if and only if they are not adjacent in G.

A class of non-complete connected $(n-3)$-regular borderenergetic graphs has been found by Gong et al. in [22].

Theorem 2.6. [22] *Let p, q and r be non-negative integers, and let $p + q = 2$. Then $\overline{pC_4 \cup qC_6 \cup rC_3}$ is borderenergetic.*

Corollary 2.7. [22] *For each integer n $(n \geq 7)$, there exists a connected non-complete borderenergetic graph of order n.*

We now show how to construct connected non-complete $(n-1-k)$-regular $(k > 2)$ borderenergetic graphs by using some k-regular graphs of small order.

Theorem 2.8. [12] *Let G be a k-regular integral graph of order n with t non-negative eigenvalues. If $\mathcal{E}(G) = 2(n-t+k)$, then $\mathcal{E}(\overline{G}) = 2(n-1)$, where $\overline{G}$ is complement of graph G.*

Theorem 2.9. [12] *Let k be an even integer. Let $G = pG_1 \cup qK_{k+1}$ be a disconnected k-regular graph consisting of p copies of G_1 and q copies of K_{k+1}, where G_1 is a connected k-regular integral graph with $k+2$ vertices, having t_1 non-negative eigenvalues, and satisfying $\mathcal{E}(G_1) = 2k + 4 - 2t_1 + \frac{2k}{p}$, $p|2k$, $p \geq 1$, $q \geq 1$. Then G is a connected non-complete borderenergetic graph.*

Corollary 2.10. [12]

1) *For integer n ($n > 12$) satisfying $5|(n-12)$, there exists a connected non-complete $(n-5)$-regular borderenergetic graph of order n.*
2) *For integer n ($n > 16$) satisfying $7|(n-16)$, there exists a connected non-complete $(n-7)$-regular borderenergetic graph of order n.*

Theorem 2.11. [56] *Let $G = G^{(0)}$ is any r-regular borderenergetic graph of order n. Then*

1) *For $p \neq 0$, $G + \overline{K}_p$ is borderenergetic if and only if $p = n - r$.*
2) *Consider an infinite sequence of graph $\mathcal{G} = \{G^{(0)}, G^{(1)}, \ldots, G^{(k)}, \ldots\}$ such that each $G^{(k)} = G^{k-1} + \overline{K}_{n-r}$ is of order $n + k(n-r)$, where $k \geq 1$. For each $r \geq 1$, $G^{(k)}$ is non-co-spectral and borderenergetic graph with $K_{n+k(n-r)}$.*

2.1 Borderenergetic graphs with maximum or minimum degrees

Many results on graph energy are closely related to their maximum or minimum degrees. For instance, Nikiforov [50] obtained the following result: Let G be a graph of order n with at least n edges and no isolated vertices. If G is C_4-free and $\Delta(G) \leq 3$, then $\mathcal{E}(G) > n$. In [43], Li et al. proved that there are exactly 4 connected graphs with $\Delta(G) \leq 3$ whose energies are equal to the number of vertices.

Here, consider the borderenergetic graphs with maximum degree $\Delta \leq 4$. This kind of graphs are also addressed as chemical graphs.

Theorem 2.12. [46]

1) *There is no non-complete borderenergetic graph with $\Delta = 2$ or 3.*
2) *Let G be a non-complete borderenergetic graph of order n with $\Delta = 4$. Then G must have the following properties:*

i) $|E(G)| = 2n$ *or* $2n-1$;

ii) $|G| \leq 21$;

iii) G *is non-bipartite;*

iv) $\eta(G) = 0$.

3) *Let* G *be a* 4*-regular non-complete borderenergetic graph of order* n *and* H *be a maximal bipartite subgraph of* G. *Then* $|E(G)| - |E(H)| \geq 3$.

Corollary 2.13. [31] *If* $m = 2n$, *then all borderenergetic graphs with* $\Delta = 4$ *are* 4*-regular. If* $m = 2n-1$, *then the borderenergetic graphs with* $\Delta = 4$ *are either* 4*-regular with one edge deleted* (*namely, with* $n-2$ *vertices of degree* 4 *and two vertices of degree* 3) *or* 4*-regular with an additional vertex of degree* 2 (*namely, with* $n-1$ *vertices of degree* 4 *and one vertex of degree* 2).

Corollary 2.14. [31] *The number of borderenergetic graphs with* $\Delta = 4$ *is finite.*

In what follows, we consider borderenergetic graphs with large minimum degree.

Theorem 2.15. [46] *No borderenergetic graph has minimum degree* $n-2$. *Besides, for each integer* $n \geq 7$, *there exists a connected non-complete borderenergetic graph of order* n *with minimum degree* $n-3$ *and for each even integer* $n \geq 8$, *there exists a non-complete borderenergetic graph of order* n *with minimum degree* $n-4$.

2.2 Borderenergetic bipartite graphs

By use a computer search, all borderenergetic graphs with order $7 \leq n \leq 11$ have been found [22, 45, 51] and it can be seen that all such graphs are not bipartite. Here, some properties of bipartite borderenergetic graphs are surveyed.

Theorem 2.16. [15] *Let* G *be a borderenergetic graph and suppose* $m < \frac{2(n-1)^2}{2}$. *Then* G *is not bipartite.*

Theorem 2.17. [15] *Let* G *be a borderenergetic graph. If* G *is bipartite, then the numbers of positive eigenvalues and negative eigenvalues of* $A(G)$ *are not less than* $\frac{(n-1)^2}{m}$, *respectively.*

Corollary 2.18. [15] *Let* G *be a connected borderenergetic graph. If* G *is a* k*-cyclic graph with* $k \leq n-3$, *then* G *is not bipartite.*

Theorem 2.19. [15] *Let G be a borderenergetic bipartite graph with $\eta(G) = 0$. Then*

1) $\lambda_1 \geq \sqrt{\frac{2(n-1)^2 - mn + 2m}{n}}$,

2) $\lambda_{\frac{n}{2}} \leq \sqrt{\frac{4(n-1)^2 - 4m}{n(n-2)}}$.

Theorem 2.20. [31] *A bipartite borderenergetic graph of order n must possess at least $\lceil m \rceil$ edges where*

$$m = \frac{1}{8}\left(n^2 + 6n - 8 - \sqrt{n^4 - 20n^3 + 84n^2 - 128n + 64}\right).$$

Recall that for large values of n, the number of edges is asymptotically equal to $\frac{16}{n}(n-1)^2$.

Theorem 2.21. [8] *Let G be a 2-connected non-complete borderenergetic graph. If it satisfies*

$$\frac{1}{n}\sum_{i=1}^{n} d_i^2(G) \geq \frac{1}{2n^2}(4\sqrt{2}\sqrt{n^6 - 11n^5 + 44n^4 - 84n^3 + 83n^2 - 41n + 8} + n^4 - 9n^3 + 33n^2 - 41n + 16), \quad (2)$$

then the girth of the borderenergetic graph G is 3.

A natural corollary from Theorem 2.21 is that such a 2-connected non-complete borderenergetic graph satisfying the condition inequality (2) is not bipartite.

Now, consider the girth of a borderenergetic graph G of order n when the order n of G is large enough

Theorem 2.22. [8] *Let G be a 2-connected non-complete borderenergetic graph of order n. If the order n of G is large enough and G satisfies*

$$d_i^2(G) \geq O(2\sqrt{2}n), \quad (1 \leq i \leq n),$$

then the girth of the borderenergetic graph G is 3.

2.3 Borderenergetic threshold graphs

Threshold graphs were first introduced by Chvátal and Hammer [6]. The spectral properties of threshold graphs were studied in [4, 41, 42]. A graph G is threshold (or degree maximal graph) if and only if it can be obtained from a single vertex by iterating the operations of adding a new vertex that is either connected to no other vertex (an isolated vertex) or connected to every other vertex (a cone vertex). The sequence of these operations is called the building sequence of the respective threshold graph. In view of this, we may represent a threshold graph on n vertices using a binary sequence $b = b_1b_2\cdots b_n$. Here b_i is 0 if the vertex v_i was added as an isolated vertex, and b_i is 1 if v_i was added as a cone vertex. In our representation, b_1 is always zero. We write 0^s (resp. 1^s) if there are s repeated 0's (resp. 1's) in the building sequence. For example, we write $0^21^201^3$ for 00110111.

Recently, Jacobs et al. in [42] considered the eigenvalues and energies of threshold graphs. They showed that if $4|n$ and $n \geq 8$, then there is an n-vertex threshold graph equienergetic with the complete graph K_n. In addition, if $9|n$, then there are two n-vertex threshold graphs equienergetic to K_n and these are non-co-spectral.

Theorem 2.23. [42] *For $m \geq 1$, the following threshold graphs of order n are borderenergetic:*

1) $01^{2m+1}0^{2m}1^{4m+2}$, $n = 8m + 4$,

2) $01^{2m}0^{2m-1}1^{4m}$, $n = 8m$,

3) $01^{4m}0^{2m-1}1^{3m}$, $n = 9m$,

4) $01^{m}0^{2m-1}1^{6m}$, $n = 9m$.

Theorem 2.24. [40]

1) *For each $n \geq 3$ and $p \geq 1$, there exist $n-1$ pairwise non-co-spectral borderenergetic threshold graphs on pn^2 vertices.*

2) *Let $p \geq 2$. Then the threshold graph $G = 01^p0^{p-1}1^{2p}$ is borderenergetic.*

Corollary 2.25. [42] *For each $n \geq 3$, there exist $n-1$ threshold graphs on n^2 vertices, pairwise non-co-spectral and borderenergetic.*

In [40], the authors listed all borderenergetic threshold graphs of order $8 \leq n \leq 23$ and all borderenergetic threshold graphs of the form $0^p1^q0^s1^t$ of order n vertices, $n = p + q + s + t \leq 100$.

Theorem 2.26. [40] *There are no borderenergetic threshold graphs $0^p1^q0^t$ and 0^p1^q ($p > 1$).*

2.4 Bounds on the size of borderenergetic graphs

It is a well-known fact that the number of vertices and the number of edges are two main structural aspects of a graph that have an effect on the values of energy (for details see [25, 28, 29] and the references quoted therein). This fact can be stated as follows.

Empirical rule 2.1. •

1) *If two graphs have an equal number of vertices and equal number of edges, then their energies do not differ significantly.*

2) *If two graphs of the same order are equienergetic, then the number of their edges do not differ significantly.*

The fact is that Rule 2.1 was tested and verified on numerous examples of molecular graphs [26, 33, 48]. The molecular graphs contain a relatively small number of edges, usually $m \leq \frac{3}{2}n$. From the study of borderenergetic graphs, it becomes evident that in the case of graphs with a larger number of edges, Rule 2.1 could be seriously breaked, especially it's part (2).

In continuing this section, three asymptotically tight bounds on the edge number of borderenergetic graphs are given. We first state the definition of the r-degree of a vertex and a previously known bound for graph energy, valid for general graphs. For an integer $r \geq 0$, the r-degree $d_r(v_i)$ of a vertex $v_i \in G$ is defined as the number of walks of length r starting at v_i. Clearly, one has $d_0(v_i) = 1$, $d_1(v_i) = d_i$ and

$$d_{r+1}(v_i) = \sum_{w \in N(v_i)} d_r(w),$$

where $N(v_i)$ is the set of all neighbors of the vertex v_i.

Theorem 2.27. [12] *Let G be a borderenergetic graph. Then*

$$m \geq \left\lceil \frac{1}{2} \frac{\sum_{v_i \in V(G)} d_{r+1}^2(v_i)}{\sum_{v_i \in V(G)} d_r^2(v_i)} + \frac{1}{2(n-1)} \left(2(n-1) - \sqrt{\frac{\sum_{v_i \in V(G)} d_{r+1}^2(v_i)}{\sum_{v_i \in V(G)} d_r^2(v_i)}} \right)^2 \right\rceil. \qquad (3)$$

If G is $(n-3)$-regular, then the bound in inequality (3) is asymptotically tight.

For simplicity, in the following, we replace the notation $d_2(v_i)$ and $d_3(v_i)$ by t_i and σ_i for $v_i \in V(G)$, respectively.

Corollary 2.28. [12] *Let G be a borderenergetic graph of order n. Then*

$$m \geq \left\lceil \left(2(n-1) - \sqrt{\frac{1}{n}\sum_{i=1}^{n} d_i^2} \right)^2 \Big/ 2(n-1) + \frac{1}{2n}\sum_{i=1}^{n} d_i^2 \right\rceil . \tag{4}$$

If G is $(n-3)$-regular, then the bound in inequality (4) *is asymptotically tight.*

Corollary 2.29. [12] *Let G be a borderenergetic graph. Then*

$$m \geq \left\lceil \frac{1}{2}\sum_{i=1}^{n} t_i^2 \Big/ \sum_{i=1}^{n} d_i^2 + \frac{1}{2(n-1)} \left(2(n-1) - \sqrt{\sum_{i=1}^{n} t_i^2 \Big/ \sum_{i=1}^{n} d_i^2} \right)^2 \right\rceil . \tag{5}$$

If G is $(n-3)$-regular, then the bound in inequality (5) *is asymptotically tight*

Corollary 2.30. [12] *Let G be a borderenergetic graph. Then*

$$m \geq \left\lceil \frac{1}{2}\sum_{i=1}^{n} \sigma_i^2 \Big/ \sum_{i=1}^{n} t_i^2 + \frac{1}{2(n-1)} \left(2(n-1) - \sqrt{\sum_{i=1}^{n} \sigma_i^2 \Big/ \sum_{i=1}^{n} t_i^2} \right)^2 \right\rceil . \tag{6}$$

If G is $(n-3)$-regular, then the bound in inequality (6) *is asymptotically tight.*

Theorem 2.31. [31] *A borderenergetic graph of order n must possess at least $2n-2$ edges.*

3 Laplacian borderenergetic graphs

An analogous concept as borderenergetic graphs, called Laplacian borderenergetic graphs was proposed in [54]. That is, a graph G of order n is Laplacian borderenergetic or L-borderenergetic for short, if $\mathcal{E}_L(G) = \mathcal{E}_L(K_n) = 2n-2$.

In [9], Deng et al. showed a kind of L-borderenergetic threshold graphs. They continued to characterize this kind of graphs and obtained some interesting properties on their structures [14]. Also, they presented some asymptotically bounds on the order and size of L-borderenergetic graphs. Furthermore, they showed that all trees, cycles, the complete bipartite graphs, and many 2-connected graphs are not L-borderenergetic. They proved that in [10] there is no a 2-connected L-borderenergetic graphs of order $n \geq 5$ with maximum degree 3, which improves the result in [14]. Also, by surveying the L-borderenergetic graphs with maximum degree 4, they presented two asymptotically tight bounds on their sizes. In [13], they mainly surveyed a class join of graphs and checked that whether they are L-borderenergetic or not. In [15], Laplacian borderenergetic bipartite graphs is observed and some asymptotically tight bounds on their first Zagreb indices are shown.

Lu et al. in [47] presented all non-complete L-borderenergetic graphs of order $4 \leq n \leq 7$ and they constructed one connected non-complete L-borderenergetic graph on n vertices for each integer $n \geq 4$, which extends the result in [54] and completely confirms the existence of non-complete L-borderenergetic graphs. Particularly, they proved that there are at least $\frac{n}{2} + 4$ non-complete L-borderenergetic graphs of order n for any even integer $n \geq 6$. Tao et al. in [52] considered the extremal number of edges of non-complete L-borderenergetic graph, then use a computer search to find out all the L-borderenergetic graphs on no more than 10 vertices. By applying computer search, Elumalai et al. in [18] corrected the number of L-borderenergetic graphs of order 9 and 10, which was reported in [52]. In [36, 37] they constructed sequences of Laplacian borderenergetic non-complete graphs by means of graph operations, and all the non-complete and pairwise non-isomorphic L-borderenergetic. Recently, Vaidya et al. in [57] investigated a sequence of L-borderenergetic graphs and also devised a procedure to find sequence of L-borderenergetic graphs from the known L-borderenergetic graph.

In this section, we review theorem about L-borderenergetic graphs.

Theorem 3.1. [9] *For any integer $n \geq 4$, there is an L-borderenergetic graph.*

In [52], L-Bordereneregetic graphs was reported that there are exactly 65 non- isomorphic non-complete connected L-borderenergetic graphs upto 9 vertices. In [9], at the same period of time, it was reported that there exits exactly 75 such connected graphs, respectively. In [52], the number of connected L-borderenergetic graphs on 10 vertices have been presented. For $n = 10$, the number of connected L-borderenergetic graphs is reported 120, but the correct number is 232, see [18]. Also in [37], the authors introduced all non-isomorphic non-complete Laplacian borderenergetic disconnected graphs up to 9 vertices. The correct numbers for non-complete and non-isomorphic L-borderenergetic graphs of order at most 10 are reported in Table 1.

Table 1. The numbers of non-complete and non-isomorphic L-borderenergetic graphs of order at most 10.

n	4	5	6	7	8	9	10
# Connected	2	1	11	5	33	23	232
# Disconnected	2	2	5	5	27	26	?

Theorem 3.2. [9] *Let G be an L-borderenergetic graph. Suppose that G is obtained from the tensor product of two L-integral graphs G_1 and G_2, where G_1 and G_2 are k_1-regular and k_2-regular, respectively. Then both $|V(G_1)|$ and $|V(G_2)|$ are odd.*

Theorem 3.3. [37] *Let G be a graph on n vertices. Then $G \cong K_a \cup K_b$ is L-borderenergetic if only if $b = a + 2$, where $a \leq b$, $n = a + b$ and a, b are positive integers.*

Corollary 3.4. [37] *Let G be a graph on n vertices. Then the graph $G \cong K_a \cup K_b \cup K_1$ is L-borderenergetic if only if $b = a + 2$, where $a \leq b$ and $n = a + b + 1$.*

Theorem 3.5. [37] *Let G_1 and G_2 be two (n, m)-Laplacian borderenergetic graphs. Then $G_1 + G_2$ is L-borderenergetic graph if only if G_1 and G_2 are complete L-borderenergetic.*

Theorem 3.6. [57] *Let G be a L-borderenergetic graph of order n with average vertex degree $\overline{d}$. Then for $p \neq 0$, $G + \overline{K}_p$ is L-borderenergetic if $p = n - \overline{d}$.*

A graph G is self-complementary (sc), if it is isomorphic to its complement.

Theorem 3.7. [37] *Let G be regular sc-graph on $n \geq 9$ vertices. Then G is L-borderenergetic, if G is strongly regular graph with parameters $(9, 4, 1, 2)$.*

Theorem 3.8. [37] *Let G be a non-complete strongly regular graph (n, k, s, t) with three distinct eigenvalues k, s and t, where $m_t = m_s$ $(r \geq 2)$ and $n \geq 5$. If G is an integral L-borderenergetic connected graph, then G has parameters $(16, 5, 0, 2)$.*

For each integer $n \geq 3$, we define the graph G in $K_{n-1} \odot K_n$ to be the following join $G = (K_{n-1} \cup K_{n-2}) + K_1$ of order $2n - 2$, see Figure 4.

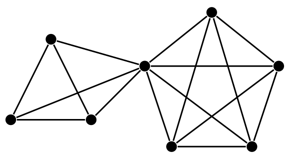


Figure 4. Graph $K_4 \odot K_5$.

Theorem 3.9. [54] *For each $n \geq 3$, $G = K_{n-1} \odot K_n$ is L-borderenergetic and non-co-L-spectral graph with K_{2n-2}.*

For each integer $n \geq 3$, we define the graph $G = K_n \cdot K_n$ of order $2n$; obtained from two copies of the complete graph by adding n edges between one vertex of a copy of K_n and n vertices of the other copy, see Figure 5.

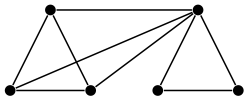


Figure 5. Graph $K_3 \cdot K_3$.

Theorem 3.10. [54] *For each $n \geq 3$, $G = K_n \cdot K_n$ is L-borderenergetic and non-co-L-spectra graph with K_{2n}.*

Let M_p be a set of p independent edges in a complete graph K_n, where $0 \leq b \leq \lfloor n/2 \rfloor$. Let E_q be a set of q edges possessing a common vertex in a complete graph K_n, where $0 \leq q \leq n-1$.

Theorem 3.11. [14] *For any even integer n, the graph $K_n - M_{\frac{n}{2}-1}$ is L-borderenergetic.*

Theorem 3.12. [47] *Let $\mathcal{G}_n = \{G_n(a,b) | a, b \geq 0, 2a+b+1 = n\}$, where $n \geq 4$. If n is odd, then $G_n(1, n-3) = (K_2 \cup (n-3)K_1) + K_1$ is the only L-borderenergetic graph in $\mathcal{G}_n$. If n is even, then $G_n(1, n-3) = (K_2 \cup (n-3)K_1) + K_1$ and $G_n(\frac{n-2}{2}, 1) = \left(\frac{n-2}{2} K_2 \cup K_1\right) + K_1$ are the only L-borderenergetic graphs in $\mathcal{G}_n$.*

Theorem 3.13. [47] *Let $\mathcal{H}_n = \left\{H_n(a,b) | a, b \geq 0, a+b = \frac{n}{2}\right\}$, where $n \geq 4$ is even. Then all graphs but $H_n(\frac{n}{2}, b)$ in $\mathcal{H}_n$ are non-complete L-borderenergetic graphs.*

Corollary 3.14. [47] *For an even integer $n \geq 6$, there exists at least $\frac{n}{2} + 4$ connected non-complete L-borderenergetic graphs which are $H_n(a,b)$, for $b = 1, 2, \ldots, \frac{n}{2}$ and $J_{n,i}$, for $i = 1, 2, 3, 4$.*

Theorem 3.15. [37] *Suppose that $p \geq 4$ is even and $G = K_{p,q} + p\,e$ is obtained from $K_{p,q}$ by adding p independent edges. Then G is L-borderenergetic.*

Theorem 3.16. [37] *Suppose that $r \geq 3$ and $0 \leq p \leq r-2$. Then $G = K_{r,r} + (r-1)\,e$ (see Figure 6) is L-borderenergetic.*

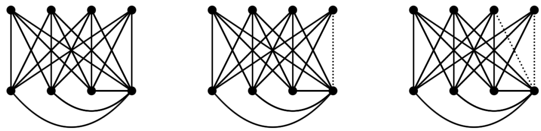


Figure 6. The graph $G = (K_{4,4} - pe) + 3e$ for $0 \leq p \leq 2$.

The $K_{a_n}(k)$-graph is obtained from K_n by removing k edges which have a common vertex. The $K_{b_n}(k)$-graph is obtained from K_n by removing k independent edges. The $K_{E_n}(k)$-graph is one obtained from K_n by deleting the edges of k independent paths P_3.

Theorem 3.17. [37] *The $K_{a_n}(k)$-graph is Laplacian borderenergetic, where $n = 4$. In addition, for even number n $(n \geq 4)$ all $K_{b_n}(k)$-graph, are L-borderenergetic, where $k = \frac{n-2}{2}$. Also, $K_{E_n}(k)$-graph is Laplacian borderenergetic, where $n = 6$.*

Let $S_n + e$ be the graph with $n+1$ edges obtained from a star S_n by connecting two pendant edges which have a vertex in common. Obviously, $S_n + e$ is a unicyclic threshold graph.

Theorem 3.18. [9,13] *The graph $K_1 + (K_2 \cup pK_1)(\cong S_n + e)$ is L-borderenergetic.*

Theorem 3.19. [37] *For each integer $p \geq 2$, the threshold graphs $0^p1^20^{p-1}$ are L- borderenergetic.*

Theorem 3.20. [37] *Let G be graph on $n \geq 4$ vertices. Then the threshold graph $0^{p+1}1^10^q1^10^p$ is L-borderenergetic, where $0 \leq p \leq \lfloor \frac{n-4}{2} \rfloor$ and $q = n - 2p - 3$.*

When a graph is bipartite, the Laplacian and signless Laplacian spectrum are the same. In next theorem, some asymptotically tight bounds of the first Zagreb index of a L- borderenergetic bipartite graph, in terms of the order and size are given.

Theorem 3.21. [15] *Let G be a L-borderenergetic bipartite graph. Then*

1) $M_1(G) \leq \frac{(2(n-1)-2\frac{m}{n})^2}{2} + \frac{6m^2}{n^2} + \frac{4m^2}{n} - 2m.$

2) $M_1(G) \geq \frac{(2(n-1)-4\frac{m}{n})^2}{n-2} + \frac{8m^2}{n^2} + \frac{4m^2}{n} - 2m.$ *Also, if G is $\left(\sqrt{4k_1^2 - 2k_1 + 4}\right)$-regular and $m = k_1 n + k_2$, where $k_1 > 0$ and $k_2 \geq 0$, then the lower bound above is asymptotically tight.*

In Appindex (Table 4), some classes of L-borderenergetic graph are given in [37,47,54].

3.1 Bounds on the order and size of L-borderenergetic graphs

In this sections, two bounds for the size of L-borderenergetic graphs with maximum degree 4 are given. Also, a lower bound for the order of borderenergetic graphs is given.

Theorem 3.22. [10] *If G is an L-borderenergetic graph with $\Delta = 4$, then*

$$m \leq \frac{1}{16} M_1(G) + \frac{5n}{4} - \frac{(n-3)^2}{4(n-1)} - 1. \tag{7}$$

When G is 4-regular, the bound in inequality (7) is asymptotically tight.

Theorem 3.23. [10] *If G is an L-borderenergetic graph with $\Delta = 4$, then*

$$m \leq \frac{1}{16} M_1(G) + \frac{5n}{4} - \frac{(n-1)^2}{4n}. \tag{8}$$

When G is 4-regular, the bound in inequality (8) is asymptotically tight.

Theorem 3.24. [14] *If G is an L-borderenergetic graph of order n and size m, then*

$$m \leq \frac{1}{2(2\bar{d}-1)} \left[M_1(G) + (n-1)\bar{d}^2 - \frac{(2n-2-\bar{d})^2}{n-1} \right]. \tag{9}$$

When G is 4-regular, the bound in inequality (9) is asymptotically tight.

Corollary 3.25. [14] *If G is an L-borderenergetic k-regular graph of order n and size m, then*

$$m \leq \frac{1}{2(2k-1)} \left[(2n-1)k^2 - \frac{(2n-2-k)^2}{n-1} \right]. \tag{10}$$

Due to regularity, the bound in inequality (10) is also fit for borderenergetic graphs.

Theorem 3.26. [14] *If G is an L-borderenergetic graph of order n, then*

$$n \geq 2\bar{d} - \delta + 1.$$

Corollary 3.27. [14] *If G is an L-borderenergetic graph of order n and size m with $\delta = 1$, then $n \geq 2\sqrt{m}$.*

3.2 Non-L-borderenergetic graphs

Here, we show that all trees, cycles, the complete bipartite graphs, and many 2-connected graphs are not L-borderenergetic.

Theorem 3.28. [14] *The complete bipartite graph $K_{a,b}$ $(1 \leq a \leq b)$, is not L- borderenergetic.*

Theorem 3.29. [14,37] *There is no Laplacian borderenergetic tree with $n \geq 3$ vertices.*

Theorem 3.30. [14] *If G is a 2-connected graph with $\Delta = 3$ and $t(G) \geq 7$, then G is not L-borderenergetic, where $t(G)$ the number of vertices of degree 3 in G.*

In [10], the authors obtained a better result, namely Theorem 3.31, which improves Theorem 3.30.

Theorem 3.31. [10] *If G is a 2-connected graph of order $n \geq 5$ with $\Delta = 3$, then G is not L-borderenergetic.*

Theorem 3.30 only considers 2-connected graphs with maximum degree 3 and $t(G) \geq 7$. But, the other cases, such as the 2-connected graphs with $1 \leq t \leq 6$ and the graphs with $\Delta \leq 4$, need to be further studied.

4 Signless Laplacian borderenergetic graphs

Tao et al. in [53] generalized the concept of borderenegetic graphs for the signless Laplacian matrices of graphs. That is, a graph G of order n is signless Laplacian borderenergetic or Q-borderenergetic for short, if $\mathcal{E}_Q(G) = \mathcal{E}_Q(K_n) = 2n - 2$. In [37], it was shown that there exits Q-borderenergetic graphs on small order n with $4 \leq n \leq 9$. At the same period of time, Tao et al. in [53] obtained some bounds on the order and size of Q-borderenergetic graphs and by using a computer search they explore all Q-borderenergetic connected graphs on no more than 10 vertices. In [13,53] two infinite family of these graphs were constructed.

Here, we present some basic theorem used to study Q-borderenergetic graphs.

Theorem 4.1. [16] *If G is a connected Q-borderenergetic graph, then G is not a tree.*

Theorem 4.2. •

1) [37] *There is no non-complete Q-borderenergetic graph of order, $n \leq 5$ and 7.*

2) [37] *There are exactly two non-complete Q-borderenergetic graphs of order 6.*

3) [37] *There exist exactly 14 non-complete Q-borderenergetic graphs of order 8.*

4) [37] *There exist exactly* 16 *non-complete Q-borderenergetic graphs of order* 9.

5) [53] *There are totally* 39 *non-complete Q-borderenergetic connected graphs of order* $n \leq 10$.

If G is a regular graph of degree k, noting that $D = kI$, $Q = kI + A$, and $L = kI - A$, it follows that $\mathcal{E}(G) = \mathcal{E}_L(G) = \mathcal{E}_Q(G)$, also note that the graph $\overline{pC_4 \cup qC_6 \cup rC_3}$ is regular of degree $n-3$, from [22] we can easily get the following theorem.

Theorem 4.3. [53] *Let p, $q$and r are non-negative integers with $p + q = 2$, then $\overline{pC_4 \cup qC_6 \cup rC_3}$ is Q-borderenergetic.*

Theorem 4.4. *Suppose that $G = K_1 + (K_t \cup pK_{t-1})$. For each integer $p \geq 1$,*

1) [13] *If $t = 2$ or $t > 3$, then G is not Q-borderenergetic.*

2) [53] *If $t = 3$, then G is L-borderenergetic and Q-borderenergetic.*

Note that graph $K_1 + (K_3 \cup pK_2)$ can be seen as constructed by connecting one vertex of K_4 with both ends of each of p copies of K_2. If we do the same thing on two or three vertices of K_4, which has the form as graph H_1 and H_2 in Figure 7, respectively, we obtain another families of Q-borderenergetic graphs.

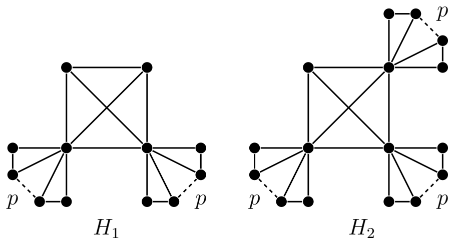


Figure 7. Two families of Q-borderenergetic graphs.

Theorem 4.5. [53]

1) *For each integer $p \geq 1$, let H_1 be a graph constructed by connecting two vertices of K_4 with both ends of each of p copies of K_2, respectively. Then H_1 is a Q-borderenergetic graph of order $4p + 4$.*

2) *For each integer $p \geq 1$, let H_2 be a graph constructed by connecting three vertices of K_4 with both ends of each of p copies of K_2, respectively. Then H_2 is a Q-borderenergetic graph of order $6p + 4$.*

Theorem 4.6. [53] *If G is a Q-borderenergetic graph of order n with m edges, then*

$$m > \frac{1}{4}\left(n - n^2 + \sqrt{n^2(n-1)^2 + 8nM_1(G)n}\right).$$

Theorem 4.7. [53] *If G is a Q-borderenergetic graph of order n with m edges, then*

$$\frac{1}{2}\left((2m+2-\Delta) - \sqrt{(2m+2-\Delta)^2 - 8m}\right) < n < \frac{1}{2}\left((2m+2-\Delta) + \sqrt{(2m+2-\Delta)^2 - 8m}\right).$$

In [16] Deng et al. presented some upper bounds on the order of a Q-borderenergetic graph in items of its size, graph energy, degrees and the first Zagreb index.

Theorem 4.8. [16] *Let G be a Q-borderenergetic graph with order n and size m. Then*

$$n \leq (\mathcal{E}(G) + \sum_{i=1}^{n} |d_i - \overline{d}|)/2 + 1.$$

If the graph G is connected, then the equality holds if and only if G is regular.

Theorem 4.9. [16] *Let G be a Q-borderenergetic graph with order n and size m. Then*

$$n \leq (\mathcal{E}(G) + \sqrt{nM_1(G) - 4m^2})/2 + 1.$$

If the graph G is connected, then the equality holds if and only if G is regular.

By using Theorem 4.1, we have

Theorem 4.10. [16] *Let G be a connected Q-borderenergetic graph with the maximum degree 4 and $n \geq 31$. Then*

$$n \leq \sqrt{\frac{3}{2}(m-1)(m-6)} + 4.$$

Specially, Deng et al. [16] studied the order of a regular Q-borderenergetic graph and obtained an upper bound on the order of the graph.

Theorem 4.11. [16] *Let G be a k-regular Q-borderenergetic graph with order n and $2 \leq k \leq 4$. Then*

$$n \leq \frac{2(k^2 - k + 1)}{2k^2 - (k-1)^{3/2}k - 3k + 2},$$

with equality if and only if $G \cong K_3$.

Note that if G is a graph with maximum degree $\Delta \leq 4$, then G is called a chemical graph. Next results are the cases of $k = 3$ and $k = 4$ for Theorem 4.11.

Corollary 4.12. [16] *There is no* 3*-regular non-complete* Q*-borderenergetic graph.*

When $k = 4$, from Theorem 4.11, one can see $n \leq 21$.

Corollary 4.13. [16] *If* G *is a* 4*-regular* Q*-borderenergetic graph, then* $n \leq 21$.

In fact, by using a computer, we can find all the k-regular Q-borderenergetic graphs with order $8 \leq n \leq 11$ (see Table 2), and these graphs are shown in Figure 8.

Table 2. The k-regular Q-borderenergetic graphs with order $8 \leq n \leq 11$, $4 \leq k \leq 8$.

n	8	9	9	10	10	10	11
k	5	4	4	7	6	6	8
G	G_1	G_2	G_3	G_4	G_5	G_6	G_7

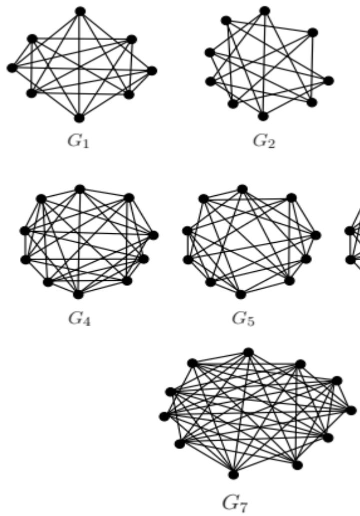


Figure 8. The regular Q-borderenergetic graphs G_i, $1 \leq i \leq 7$.

5 Seidel borderenergetic graphs

Hakimi-Nezhaad et al. [38] proposed the concept of a Seidel borderenergetic graph, which means $\mathcal{E}_S(G) = \mathcal{E}_S(K_n) = 2n - 2$ and they obtained several classes of Seidel borderenergetic graphs.

In this section, a characterization of Seidel borderenergetic graphs with at most three Seidel eigenvalues is given.

Theorem 5.1. [38] *Let G be a graph on n vertices with two distinct Seidel eigenvalues. Then G is Seidel borderenergetic if and only if $G \cong K_n$ or $\overline{K}_n$ or $K_i \cup K_j$ or $K_{i,j}$, where i+j=n.*

Corollary 5.2. [38] *Let G be a graph on n vertices with two distinct Seidel eigenvalues. Then graph $G - \{v\}$ is Seidel borderenergetic if and only if $G - \{v\} \cong K_{n-1}$ or $\overline{K}_{n-1}$ or $K_i \cup K_j$ or $K_{i,j}$, where $i + j = n - 1$.*

Theorem 5.3. [38] *Let G be connected r-regular graph on $n \geq 3$ vertices with three distinct eigenvalues. Then G is Seidel borderenergetic if and only if $G \cong K_{\frac{n}{2},\frac{n}{2}}$, where n is even.*

Corollary 5.4. [38] *If G is a r-regular graph with exactly distinct three Seidel eigenvalues, then G is not Seidel borderenergetic.*

All non-isomorphic Seidel borderenergetic graphs of order n, where $2 \leq n \leq 10$ together their Seidel eigenvalues are reported in Appendix (Table 5). These computations are done by the aid of nauty package developed by McKay (McKay 2006) [49] and the The GNU MPFR library [20].

A Seidel switching of graph G can be constructed as follows. Let $V(G) = U_1 \cup U_2$ be a partition of vertices of G and G' be a graph obtained from G by removing all edges between U_1 and U_2 and adding all edges between them not presented in G. We say that G' is a Seidel switching of G with respect to U_1 and in this case G' and G are Seidel co-spectral, see [35]. Two graphs G and G' are called switching equivalent, if G' is constructed by a sequence of Seidel switching from G.

Finally, Haemers in [35] conjectured that the Seidel energy of any graph of order n is at least $2n - 2$ and, up to Seidel equivalence, the equality holds for K_n. Akbari et al. in [1] proved this conjecture and so we conclude the following result.

Theorem 5.5. *Let G be graph of order n. Then G is Seidel borderenergetic if and only if $G \cong K_n$ or $\overline{K}_n$ or $K_i \cup K_j$ or $K_{i,j}$, where $i + j = n$.*

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Appendix

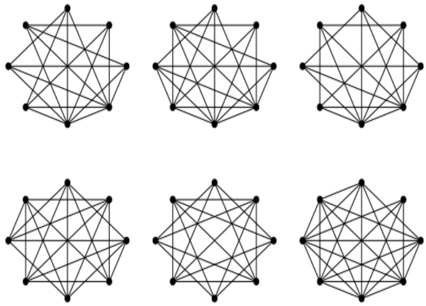

Figure 9. The six non-complete borderenergetic graphs of order 8. These have different number of edges (between 19 and 25).

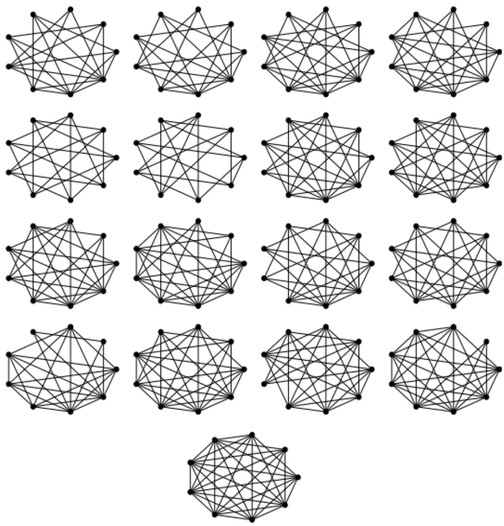

Figure 10. The seventeen non-complete borderenergetic graphs of order 9. These have different number of edges (between 22 and 34).

Table 3. The distribution of twelve-vertex borderenergetic graphs by the number of edges.

# Edges	# Graphs	# Edges	# Graphs
25	2	41	20
26	5	42	26
27	1	43	12
28	8	44	14
29	7	45	14
30	42	46	7
31	20	47	4
32	62	48	7
33	58	50	2
34	50	51	1
35	44	52	4
36	43	54	1
37	37	55	2
38	27	56	1
39	25	57	1
40	24	58	1

Table 4. Some classes of L-borderenergetic graphs.

Graph	
$K_1 + (pK_2 \cup K_{1,2})$	$p \geq 0$
$K_1 + (pK_2 \cup K_1 \cup K_{1,2})$	$p \geq 0$
$(pK_2 \cup K_1) + (pK_2 \cup K_1)$	$p \geq 1$
$(pK_2 \cup K_1 \cup K_{1,2}) + (pK_2 \cup K_1 \cup K_{1,2})$	$p \geq 1$
$K_1 + (pK_2 \cup K_1)$	$p \geq 1$
$K_1 \cup (pK_2 + K_1)$	$p \geq 1$
$(pK_1 \cup (K_1 + (p+1)K_1)) + (pK_1 \cup (K_1 + (p+1)K_1))$	$p \geq 1,\, n = 4p+4$
$(p+1)K_2 + (p+1)K_2$	$p \geq 1,\, n = 4p+4$
$(K_2 \cup (2p+1)K_1) + (2r+1)K_1$	$p \geq 1,\, n = 4p+4$
$((2p+1)K_1) + (2p+2)K_1 + K_1$	$p \geq 1,\, n = 4p+4$
$(pK_1 \cup (K_1 + (p+1)K_1)) + (p+1)K_2$	$p \geq 1,\, n = 4p+4$
$(pK_1 \cup (K_1 + (p+1)K_1)) + (pK_2 \cup 2K_1)$	$p \geq 1,\, n = 4p+4$
$((p+1)K_2) + (pK_2 \cup 2K_1)$	$p \geq 1,\, n = 4p+4$
$((p+1)K_2) + ((2p+1)K_1 + K_1)$	$p \geq 1,\, n = 4p+4$
$(pK_2 \cup 2K_1) + ((2p+1)K_1 + K_1)$	$p \geq 1,\, n = 4p+4$
$((2p+1)K_1) + ((2p+1-q)K_1) \cup (K_1 + (q+1)K_1)$	$p \geq 1,\, n = 4p+4,\, 0 \leq q \leq p$
$K_1 + (K_q + pK_{q-1})$	$p \geq 1,\, q \geq 2$
$((2pK_1 + 2pK_1) + 2pK_1)$	$p \geq 1,\, n = 6p+1$
$K_1 + (K_{p-1} \cup K_{p,p})$	$p \geq 2$
$K_1 + (K_1 \cup ((\frac{p}{2}-1)K_1 + (\frac{p}{2}-1)K_1))$	$p \geq 6$
$K_1 + ((\frac{p}{2}-1)K_2 \cup K_1)$	$p \geq 6$
$(K_2 \cup (\frac{p}{2}-2)K_1) + (K_1 + (\frac{p}{2}-1)K_1)$	$p \geq 6$
$(K_2 \cup (\frac{p}{2}-2)K_1) + ((K_1 + (\frac{p}{2}-2)K_1) \cup K_1)$	$p \geq 6$
$pK_1 \cup (K_1 + (qK_1 \cup K_{1,p} \cup K_2)$	$1 \leq p \leq \lceil \frac{n-4}{2} \rceil,\, q = n-2p-4$

Table 5. Seidel borderenergetic graphs of order $2 \leq n \leq 10$ and their Seidel Spectra.

n	Graphs	S-Spectra
2	$K_2, \bar{K}_2$	$\{[1]^1, [-1]^1\}$
3	$K_3, K_1 \cup K_2$ $\bar{K}_3, K_{1,2}$	$\{[2]^1, [-1]^2\}$ $\{[1]^2, [-2]^1\}$
4	$K_4, K_2 \cup K_2, K_3 \cup K_1$ $\bar{K}_4, K_{2,2}, K_{1,3}$	$\{[3]^1, [-1]^3\}$ $\{[1]^3, [-3]^1\}$
5	$K_5, K_3 \cup K_2, K_4 \cup K_1$ $\bar{K}_5, K_{3,2}, K_{1,4}$	$\{[4]^1, [-1]^4\}$ $\{[1]^4, [-4]^1\}$
6	$K_6, K_3 \cup K_3, K_2 \cup K_4, K_3 \cup K_5$ $\bar{K}_6, K_{3,3}, K_{4,2}, K_{5,1}$	$\{[5]^1, [-1]^5\}$ $\{[1]^5, [-5]^1\}$
7	$K_7, K_4 \cup K_3, K_5 \cup K_2, K_6 \cup K_1$ $\bar{K}_7, K_{4,3}, K_{5,2}, K_{6,1}$	$\{[6]^1, [-1]^6\}$ $\{[1]^6, [-6]^1\}$
8	$K_8, K_4 \cup K_4, K_5 \cup K_3, K_6 \cup K_2, K_7 \cup K_1$ $\bar{K}_8, K_{4,4}, K_{5,3}, K_{6,2}, K_{7,1}$	$\{[7]^1, [-1]^7\}$ $\{[1]^7, [-7]^1\}$
9	$K_9, K_5 \cup K_4, K_6 \cup K_3, K_7 \cup K_2, K_8 \cup K_1$ $\bar{K}_9, K_{5,4}, K_{6,3}, K_{7,2}, K_{8,1}$	$\{[8]^1, [-1]^8\}$ $\{[1]^8, [-8]^1\}$
10	$K_{10}, K_5 \cup K_5, K_6 \cup K_4, K_7 \cup K_3, K_8 \cup K_2, K_9 \cup K_1$ $\bar{K}_{10}, K_{5,5}, K_{6,4}, K_{7,3}, K_{8,2}, K_{9,1}$	$\{[9]^1, [-1]^9\}$ $\{[1]^9, [-9]^1\}$