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Arithmetic–Geometric Spectral Radius and Energy of Graphs

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Abstract

Let G be a graph of order n with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, and let d_i be the degree of the vertex v_i of G for $i = 1, 2, \ldots, n$. The arithmetic-geometric adjacency matrix $A_{ag}(G)$ of G is defined so that its (i, j)-entry is equal to $\frac{d_i+d_j}{2\sqrt{d_i d_j}}$ if the vertices v_i and v_j are adjacent, and 0 otherwise. The arithmetic-geometric spectral radius and arithmetic-geometric energy of G are the radius and energy of its arithmetic-geometric adjacency matrix, respectively. In this paper, some sharp lower and upper bounds on arithmetic-geometric radius and arithmetic-geometric energy are obtained, and the respective extremal graphs are characterized.

1 Introduction

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, and let $|E(G)| = m$, where n is the order and m is the size of G. Let d_i be the degree of the vertex v_i of G for $i = 1, 2, \ldots, n$. The maximum and minimum degrees of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively.

The adjacency matrix $A = A(G)$ of a graph G is the matrix of order n, where $a_{ij} = 1$ if $v_i v_j \in E(G)$ and 0 otherwise. The eigenvalues of $A(G)$ are denoted by $\rho^{(1)}(G) \geq \rho^{(2)}(G) \geq \cdots \geq \rho^{(n)}(G)$. The greatest eigenvalue $\rho^{(1)}(G)$ is called the spectral radius of G. The energy of G is

$$\mathcal{E}(G) = \sum_{i=1}^{n} \left|\rho^{(i)}(G)\right| \ .$$

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In 2015, Sehgehalli et al. [11] proposed the arithmetic-geometric index of a graph G, and defined the arithmetic-geometric adjacency matrix (AG matrix) of G, denoted by $A_{ag}(G) = (g_{ij})$, where $g_{ij} = \frac{d_i+d_j}{2\sqrt{d_i d_j}}$ if $v_i v_j \in E(G)$ and 0 otherwise. Note that $A_{ag}(G)$ is a real symmetric matrix of order n. All eigenvalues of $A_{ag}(G)$ are real, which can be denoted by $\rho_{ag}^{(1)}(G) \geq \rho_{ag}^{(2)}(G) \geq \cdots \geq \rho_{ag}^{(n)}(G)$. The greatest eigenvalue $\rho_{ag}^{(1)}(G)$ of $A_{ag}(G)$ is called the arithmetic-geometric radius (AG spectral radius) of G. The arithmetic-geometric energy (AG energy) of G is defined in an analogue way as

$$\mathcal{E}_{ag}(G) = \sum_{i=1}^{n} \left|\rho_{ag}^{(i)}(G)\right| \ .$$

In [15], some bounds for the AG spectral radius and AG energy were obtained.

It is usual and useful to define modified energies as inverse sum indeg energy [12], distance energy [14], ABC energy [3,6], matching energy [1,16], and Randić energy [5,9].

In this paper, we consider the AG spectral radius and AG energy of graphs. In Section 2, we give some useful lemmas. In Section 3, we give some lower and upper bounds on the AG spectral radius and characterize the extremal graphs. In Section 4, we obtain some lower and upper bounds on the AG energy and characterize the extremal graphs.

We shall need three graph invariants, namely the forgotten topological index F, the second Zagreb index M_2, and the modified second Zagreb index M_2^* of a graph G:

$$F = F(G) = \sum_{v_i v_j \in E(G)} \left(d_i^2 + d_j^2\right) \ ,$$
$$M_2 = M_2(G) = \sum_{v_i v_j \in E(G)} d_i d_j \ ,$$
$$M_2^* = M_2^*(G) = \sum_{v_i v_j \in E(G)} \frac{1}{d_i d_j} \ .$$

Throughout the paper, we use K_n and $K_{p,q}(p+q=n)$ to denote the complete graph and the complete bipartite graph of order n, respectively. For other undefined notions and terminology from graph theory, the readers are referred to [2,10].

2 Lemmas

Lemma 2.1 *[13] If B is an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, then for any $0 \neq \boldsymbol{x} \in \boldsymbol{R}^n$, $\boldsymbol{x}^T B \boldsymbol{x} \leq \lambda_1 \boldsymbol{x}^T \boldsymbol{x}$. Equality holds if and only if $\boldsymbol{x}$ is an eigenvector of B corresponding to λ_1.*

Lemma 2.2 *[8] Let $B = (b_{ij})$ and $D = (d_{ij})$ be real symmetric, nonnegative matrices of order n. If $B \geq D$, i.e., $b_{ij} \geq d_{ij}$ for all i, j, then $\lambda_1(B) \geq \lambda_1(D)$, where λ_1 is the largest eigenvalue.*

Lemma 2.3 *[7] Let G be a connected graph of order n and size m. Then $\rho^{(1)}(G) \leq \sqrt{2m-n+1}$, with equality if and only if $G \cong K_{1,n-1}$ or $G \cong K_n$.*

The nullity $n_0(G)$ of a graph G is the multiplicity of the eigenvalue zero of $A(G)$.

Lemma 2.4 *[4] Let G be a graph of order $n \geq 2$. Then $n_0(G) = n-2$ if and only if $G \cong K_{p,q} \cup (n-p-q)K_1$, where $p+q \leq n$.*

The following Lemma 2.5 is clear from the Perron-Frobenius theorem.

Lemma 2.5 *Let G be a connected graph of order $n \geq 2$. Then $\rho_{ag}^{(1)}(G) > \rho_{ag}^{(2)}(G)$.*

Lemma 2.6 *Let G be a graph of order n. Then $|\rho_{ag}^{(1)}(G)| = |\rho_{ag}^{(2)}(G)| = \cdots = |\rho_{ag}^{(n)}(G)|$ if and only if $G \cong nK_1$ or $G \cong \frac{n}{2}K_2$.*

Proof. Let $|\rho_{ag}^{(1)}(G)| = |\rho_{ag}^{(2)}(G)| = \cdots = |\rho_{ag}^{(n)}(G)|$, and k be the number of isolated vertices in G. If $k \geq 1$, then $\rho_{ag}^{(1)}(G) = \rho_{ag}^{(2)}(G) = \cdots = \rho_{ag}^{(n)}(G) = 0$, and so $G \cong nK_1$. Otherwise, $k = 0$. If $\Delta(G) = 1$, then $G \cong \frac{n}{2}K_2$. If $\Delta(G) \geq 2$, then G contains a connected component H with at least 3 vertices, and so $\rho_{ag}^{(1)}(H) > \rho_{ag}^{(2)}(H)$ by Lemma 2.5, a contradiction. ■

3 On AG spectral radius of a graph

In this section we give some sharp lower and upper bounds on AG spectral radius.

Theorem 3.1 *[15] Let G be a connected graph of order n and size m. Then*

$$\rho_{ag}^{(1)}(G) \leq \frac{1}{2}\left(\sqrt{n-1} + \frac{1}{\sqrt{n-1}}\right)\sqrt{2m-n+1}\,, \tag{1}$$

with equality holding if and only if $G \cong K_{1,n-1}$.

Theorem 3.2 *Let G be a graph of order n and size m with the maximum degree Δ and minimum degree δ. Then*

$$\rho_{ag}^{(1)}(G) \geq \frac{2m\delta}{n\Delta}\,, \tag{2}$$

with equality holding if and only if G is a regular graph.

Proof. Let $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$ be any unit vector in $\mathbf{R}^n$. By Lemma 2.1,

$$\rho_{ag}^{(1)}(G) \geq \mathbf{x}^T A_{ag}(G)\mathbf{x} = \sum_{v_iv_j \in E(G)} \frac{d_i + d_j}{\sqrt{d_i d_j}} x_i x_j \geq \frac{2\delta}{\Delta} \sum_{v_iv_j \in E(G)} x_i x_j \ . \tag{3}$$

Taking $\mathbf{x} = (\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \cdots, \frac{1}{\sqrt{n}})^T$ in (3), we have

$$\rho_{ag}^{(1)}(G) \geq \mathbf{x}^T A_{ag}(G)\mathbf{x} \geq \frac{2m\delta}{n\Delta} \ .$$

Then (2) holds.

If the equality in (2) holds, then all the above inequalities must be equalities. From (3), we have $d_1 = d_2 = \cdots = d_n = \delta = \Delta$. Then G is a regular graph.

Conversely, if G is regular, then $d_1 = d_2 = \cdots = d_n = \Delta$, and so $\mathbf{x} = (\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \cdots, \frac{1}{\sqrt{n}})^T$ is an eigenvector of $A_{ag}(G)$ corresponding to the eigenvalue $\rho_{ag}^{(1)}(G)$. Then the equality holds in (2). ■

In what follows, we obtain an upper bound on $\rho_{ag}^{(1)}(G)$ in terms of maximum degree $\Delta(G)$, minimum degree $\delta(G)$, and spectral radius $\rho^{(1)}(G)$.

Theorem 3.3 *Let G be a graph of order n with the maximum degree Δ and minimum degree δ. Then*

$$\rho_{ag}^{(1)}(G) \leq \frac{\Delta}{\delta}\, \rho^{(1)}(G)\,, \tag{4}$$

with equality holding if and only if G is regular.

Proof. Let $\mathbf{y} = (y_1, y_2, \cdots, y_n)^T$ be a unit eigenvector of $A_{ag}(G)$ corresponding to the eigenvalue $\rho_{ag}^{(1)}(G)$. Then

$$\rho_{ag}^{(1)}(G) = \mathbf{y}^T A_{ag}(G)\mathbf{y} = \sum_{v_iv_j \in E(G)} \frac{d_i + d_j}{\sqrt{d_i d_j}}\, y_i y_j \leq \frac{2\Delta}{\delta} \sum_{v_iv_j \in E(G)} y_i y_j \ . \tag{5}$$

By Lemma 2.1,

$$\rho^{(1)}(G) \geq \mathbf{y}^T A(G)\mathbf{y} = 2 \sum_{v_iv_j \in E(G)} y_i y_j \ . \tag{6}$$

Combining (5) and (6), the inequality (4) holds.

If the equality in (4) holds, then all the inequalities in (5) and (6) must be equalities. From (5), $d_1 = d_2 = \cdots = d_n = \delta = \Delta$. So G is regular.

Conversely, if G is regular, then $d_1 = d_2 = \cdots = d_n = \delta = \Delta$. Moreover, $A_{ag}(G) = A(G)$, and $\rho_{ag}^{(1)}(G) = \rho^{(1)}(G)$. Hence the equality in (4) holds. ■

Corollary 3.4 *Let G be a graph of order n with the maximum degree Δ and minimum degree δ. Then*

$$\rho_{ag}^{(1)}(G) \leq \frac{\Delta^2}{\delta}\,, \tag{7}$$

with equality holding if and only if G is regular.

4 On AG energy of a graph

In this section we establish some sharp lower and upper bounds on the AG energy.

Let G be a graph of order n. Note that $A_{ag}(G)$ is a real symmetric matrix with zero diagonal. Then

$$\sum_{i=1}^{n} \rho_{ag}^{(i)}(G) = 0\,, \qquad \sum_{i=1}^{n} \left(\rho_{ag}^{(i)}(G)\right)^2 = -2 \sum_{1\leq i\leq j\leq n} \rho_{ag}^{(i)}(G)\rho_{ag}^{(j)}(G)\,, \tag{8}$$

$$\sum_{i=1}^{n} \left(\rho_{ag}^{(i)}(G)\right)^2 = \frac{1}{2} \sum_{v_iv_j\in E(G)} \left(\frac{d_i+d_j}{\sqrt{d_jd_j}}\right)^2\,. \tag{9}$$

Theorem 4.1 *Let G be a graph of order n and size m with the maximum degree Δ and minimum degree δ. Then*

$$\mathcal{E}_{ag}(G) \leq \frac{\Delta}{\delta}\sqrt{2nm}\,, \tag{10}$$

$$\mathcal{E}_{ag}(G) \leq \Delta\sqrt{2nM_2^*(G)}\,, \tag{11}$$

$$\mathcal{E}_{ag}(G) \leq \frac{1}{\delta}\sqrt{\frac{2}{n}\left(F(G)+M_2(G)\right)}\,. \tag{12}$$

In all three relations equality holds if and only if $G \cong \frac{n}{2}K_2$.

Proof. Applying the Cauchy-Schwarz inequality, we get

$$\sum_{i=1}^{n} |\rho_{ag}^{(i)}(G)| \leq \sqrt{n\sum_{i=1}^{n}\left(\rho_{ag}^{(i)}(G)\right)^2} = \sqrt{\frac{n}{2}\sum_{v_iv_j\in E(G)}\left(\frac{d_i+d_j}{\sqrt{d_jd_j}}\right)^2}\,, \tag{13}$$

with equality holding if and only if $|\rho_{ag}^{(1)}(G)| = |\rho_{ag}^{(2)}(G)| = \cdots = |\rho_{ag}^{(n)}(G)|$. Note that

$$\frac{d_i+d_j}{\sqrt{d_jd_j}} \leq \frac{2\Delta}{\delta}, \qquad \frac{d_i+d_j}{\sqrt{d_jd_j}} \leq \frac{2\Delta}{\sqrt{d_jd_j}}, \qquad \frac{d_i+d_j}{\sqrt{d_jd_j}} \leq \frac{d_i+d_j}{\delta}\,.$$

So inequalities (10)-(12) hold.

If the equality in (10) (or (11), or (12)) holds, then the equality in (13) holds, and so $|\rho_{ag}^{(1)}(G)| = |\rho_{ag}^{(2)}(G)| = \cdots = |\rho_{ag}^{(n)}(G)|$. By Lemma 2.6, we have $G \cong \overline{K}_n$, or $G \cong \frac{n}{2}K_2$. If $G \cong \overline{K}_n$, then $\delta = 0$, a contradiction. Thus $G \cong \frac{n}{2}K_2$.

If $G \cong \frac{n}{2}K_2$, then it is easy to see that the equalities in (10), (11), and (12) hold. ■

Theorem 4.2 *Let G be a graph of order n and size m with the maximum degree Δ and minimum degree δ. Then*

$$\mathcal{E}_{ag}(G) \geq \frac{4m\delta}{n\Delta} .$$

Proof. From (2) and (8),

$$\mathcal{E}_{ag}(G) = \sum_{i=1}^{n} |\rho_{ag}^{(i)}(G)| = 2 \sum_{\rho_{ag}^{(i)}(G) \geq 0} |\rho_{ag}^{(i)}(G)| \geq 2\rho_{ag}^{(1)}(G) \geq \frac{4m\delta}{n\Delta} .$$

The theorem follows. ■

Theorem 4.3 *Let G be a connected graph of order $n \geq 2$ and size m. Then*

$$\mathcal{E}_{ag}(G) \geq 2\sqrt{m}\,, \tag{14}$$

with equality holding if and only if $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

Proof. For $n = 2$, $G \cong K_2$ and hence the equality holds. Let $n \geq 3$. We have

$$\begin{aligned}
(\mathcal{E}_{ag}(G))^2 &= \sum_{i=1}^{n} \left(\rho_{ag}^{(i)}(G)\right)^2 + 2 \sum_{1\leq i\leq j\leq n} \left|\rho_{ag}^{(i)}(G)||\rho_{ag}^{(j)}(G)\right| \\
&\geq \sum_{i=1}^{n} \left(\rho_{ag}^{(i)}(G)\right)^2 + 2 \left| \sum_{1\leq i\leq j\leq n} \rho_{ag}^{(i)}(G)\rho_{ag}^{(j)}(G) \right| \qquad (15) \\
&= 2\sum_{i=1}^{n} \left(\rho_{ag}^{(i)}(G)\right)^2 = \sum_{v_iv_j\in E(G)} \left(\frac{d_i + d_j}{\sqrt{d_jd_j}}\right)^2 \geq 4m\,, \qquad (16)
\end{aligned}$$

and so inequality (14) follows.

The equality in (14) holds if and only if the inequalities in (15) and (16) must be equalities. The equality in (16) holding implies that $\sqrt{\frac{d_i}{d_j}} + \sqrt{\frac{d_j}{d_i}} = 2$ for each edge $v_iv_j \in E(G)$, that is, $d_i = d_j$ for each edge $v_iv_j \in E(G)$. This means G is regular. The equality in (15) holding implies that $A_{ag}(G)$ has two nonzero eigenvalues and all the remaining eigenvalues are zero, that is, $\rho_{ag}^{(1)}(G) = -\rho_{ag}^{(n)}(G)$, and $\rho_{ag}^{(i)}(G) = 0$ for $2 \leq i \leq n-1$. Since G is regular, $A_{ag}(G) = A(G)$, and $n_0(G) = n-2$. Note that G is connected. By Lemma 2.4, $G \cong K_{\frac{n}{2},\frac{n}{2}}$. ■

Note that the graph G in Theorem 4.3 is connected. For general graphs (not necessarily connected), the paper [15] gives the following result.

Theorem 4.4 *[15] Let G be a graph of order n with m edges. Then*

$$\mathcal{E}_{ag}(G) \geq 2\sqrt{m}\,, \tag{17}$$

with equality holding if and only if $G \cong nK_1$, or $G \cong K_{p,p} \cup (n-2p)K_1$ with $1 \leq p \leq \lfloor \frac{n}{2} \rfloor$.

Theorem 4.5 *Let G be a connected graph of order n and size m with the maximum degree Δ and minimum degree δ. Then*

$$\mathcal{E}_{ag}(G) \geq \min\{\mathcal{E}_{ag}^{\alpha}, \mathcal{E}_{ag}^{\beta}\}\,, \tag{18}$$

where

$$\mathcal{E}_{ag}^{\alpha} = 1 + \sqrt{\sum_{v_iv_j \in E(G)} \left(\frac{d_i + d_j}{\sqrt{d_i d_j}} \right)^2 - 3}\,,$$

and

$$\mathcal{E}_{ag}^{\beta} = \frac{\Delta}{\delta}\rho^{(1)}(G) + \sqrt{\sum_{v_iv_j \in E(G)} \left(\frac{d_i + d_j}{\sqrt{d_i d_j}} \right)^2 - \frac{3\Delta^2}{\delta^2}\left(\rho^{(1)}(G)\right)^2}\,.$$

Equality in (18) holds if and only if $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

Proof. Note that

$$\left(\rho_{ag}^{(n)}(G)\right)^2 = \left(\sum_{i=1}^{n-1} \rho_{ag}^{(i)}(G)\right)^2 = \sum_{i=1}^{n-1}\left(\rho_{ag}^{(i)}(G)\right)^2 + 2\sum_{1 \leq i \leq j \leq n-1} \rho_{ag}^{(i)}(G)\rho_{ag}^{(j)}(G)\,,$$

and

$$\left(\sum_{i=1}^{n-1} \left|\rho_{ag}^{(i)}(G)\right|\right)^2 = \sum_{i=1}^{n-1}\left(\rho_{ag}^{(i)}(G)\right)^2 + 2\sum_{1 \leq i \leq j \leq n-1} \left|\rho_{ag}^{(i)}(G)\right|\left|\rho_{ag}^{(j)}(G)\right|\,.$$

Then

$$\begin{aligned}
\left(\mathcal{E}_{ag}(G) - \left|\rho_{ag}^{(n)}(G)\right|\right)^2 &= \sum_{i=1}^{n-1}\left(\rho_{ag}^{(i)}(G)\right)^2 + 2\sum_{1 \leq i \leq j \leq n-1} \left|\rho_{ag}^{(i)}(G)\right|\left|\rho_{ag}^{(j)}(G)\right| \\
&\geq \sum_{i=1}^{n-1}\left(\rho_{ag}^{(i)}(G)\right)^2 + 2\left|\sum_{1 \leq i \leq j \leq n-1} \rho_{ag}^{(i)}(G)\rho_{ag}^{(j)}(G)\right| \qquad (19) \\
&= \sum_{i=1}^{n-1}\left(\rho_{ag}^{(i)}(G)\right)^2 + \left|\left(\rho_{ag}^{(n)}(G)\right)^2 - \sum_{i=1}^{n-1}\left(\rho_{ag}^{(i)}(G)\right)^2\right|\,.
\end{aligned}$$

Since $\rho_{ag}^{(1)}(G) \geq |\rho_{ag}^{(n)}(G)|$, we have $\left(\rho_{ag}^{(n)}(G)\right)^2 \leq \sum\limits_{i=1}^{n-1} \left(\rho_{ag}^{(i)}(G)\right)^2$. Thus

$$\left(\mathcal{E}_{ag}(G) - \left|\rho_{ag}^{(n)}(G)\right|\right)^2 \geq 2\sum_{i=1}^{n-1} \left(\rho_{ag}^{(i)}(G)\right)^2 - \left(\rho_{ag}^{(n)}(G)\right)^2 = 2\sum_{i=1}^{n} \left(\rho_{ag}^{(i)}(G)\right)^2 - 3\left(\rho_{ag}^{(n)}(G)\right)^2,$$

that is,

$$\mathcal{E}_{ag}(G) \geq \left|\rho_{ag}^{(n)}(G)\right| + \sqrt{2\sum_{i=1}^{n} \left(\rho_{ag}^{(i)}(G)\right)^2 - 3\left(\rho_{ag}^{(n)}(G)\right)^2}\,.$$

By (9),

$$\mathcal{E}_{ag}(G) \geq \left|\rho_{ag}^{(n)}(G)\right| + \sqrt{\sum_{v_iv_j \in E(G)} \left(\frac{d_i + d_j}{\sqrt{d_i d_j}}\right)^2 - 3\left(\rho_{ag}^{(n)}(G)\right)^2}\,.$$

Note that G is a connected graph. Then G has at least one edge. Without loss of generality, assume $v_iv_j \in E(G)$. Then

$$B = \begin{bmatrix} 0 & \frac{d_i+d_j}{2\sqrt{d_id_j}} \\ \frac{d_i+d_j}{2\sqrt{d_id_j}} & 0 \end{bmatrix}$$

is a 2×2 principal submatrix of $A_{ag}(G)$ based on indices i and j. It is easy to see that the eigenvalues of B are $\pm\frac{d_i+d_j}{2\sqrt{d_id_j}}$, and $\frac{d_i+d_j}{2\sqrt{d_id_j}} \leq 1$. By the interlacing theorem of eigenvalues of real symmetric matrices [8], we have

$$\rho_{ag}^{(n)}(G) \leq -\frac{d_i + d_j}{2\sqrt{d_id_j}} \leq -1\,.$$

It implies that $\left|\rho_{ag}^{(n)}(G)\right| \geq 1$. By Theorem 3.3, we get

$$\left|\rho_{ag}^{(n)}(G)\right| \leq \rho_{ag}^{(1)}(G) \leq \frac{\Delta}{\delta}\rho^{(1)}(G)\,. \tag{20}$$

We now consider the function

$$f(x) = x + \sqrt{\sum_{v_iv_j \in E(G)} \left(\frac{d_i + d_j}{\sqrt{d_id_j}}\right)^2 - 3x^2}\,,$$

where $1 \leq x \leq \frac{\Delta}{\delta}\rho^{(1)}(G)$. It is not difficult to see that $f(x)$ is increasing for $1 \leq x \leq \lambda$ and decreasing for $\lambda \leq x \leq \frac{\Delta}{\delta}\rho^{(1)}(G)$, where

$$\lambda = \sqrt{\frac{1}{12}\sum_{v_iv_j \in E(G)} \left(\frac{d_i + d_j}{\sqrt{d_id_j}}\right)^2}\,.$$

Thus

$$\mathcal{E}_{ag}(G) \geq \min\{f(1), f(\frac{\Delta}{\delta}\rho^{(1)}(G))\},$$

where $f(1) = \mathcal{E}_{ag}^{\alpha}$, and $f(\frac{\Delta}{\delta}\rho^{(1)}(G)) = \mathcal{E}_{ag}^{\beta}$. Then (18) holds.

Note that if the equality in (18) holds, then either $\mathcal{E}_{ag}(G) = \mathcal{E}_{ag}^{\alpha}$ or $\mathcal{E}_{ag}(G) = \mathcal{E}_{ag}^{\beta}$. If the equality in (18) holds, then all the above inequalities must be equalities. From (19), we conclude that $A_{ag}(G)$ has exactly two nonzero eigenvalues, that is,

$$\rho_{ag}^{(1)}(G) = -\rho_{ag}^{(n)}(G), \text{ and } \rho_{ag}^{(i)}(G) = 0, \;\; 2 \leq i \leq n-1 \,. \tag{21}$$

If $\mathcal{E}_{ag}(G) = \mathcal{E}_{ag}^{\alpha}$, then $\rho_{ag}^{(n)}(G) = -1$, and so $\rho_{ag}^{(1)}(G) = 1$. It implies that $G \cong K_2$. If $\mathcal{E}_{ag}(G) = \mathcal{E}_{ag}^{\beta}$, then $\rho_{ag}^{(n)}(G) = -\frac{\Delta}{\delta}\rho^{(1)}(G)$, and $\rho_{ag}^{(1)}(G) = \frac{\Delta}{\delta}\rho^{(1)}(G)$. By Theorem 3.3, G must be regular. Since G is connected, by Lemma 2.4, $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

Conversely, it is easy to check that the equality in (18) holds for $G \cong K_{\frac{n}{2},\frac{n}{2}}$. ■

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