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# On Spectral Radius and Energy of Arithmetic–Geometric Matrix of Graphs

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#### Abstract

Let G = (V, E) be a simple graph of order n with vertex set  $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E. Denote the sequence of its vertex degrees by  $d_1, d_2, \ldots, d_n$ . The arithmetic–geometric matrix  $A_{AG}(G) = (a_{i,j})$  of G is the square matrix of order n, where  $a_{i,j} = \frac{1}{2} \left( \sqrt{\frac{d_i}{d_j}} + \sqrt{\frac{d_j}{d_i}} \right)$  if  $v_i v_j \in E(G)$  and 0 otherwise. We give some bounds for the arithmetic–geometric spectral radius in terms of the maximum degree and minimum degree of G, the Randić index  $R_{-1}$ , and the first Zagreb index  $M_1$ . We also obtain some bounds for the arithmetic–geometric energy in terms of ordinary energy, the sum of 2-degrees of G, symmetric division deg index, the forgotten index, the second Zagreb index, and so on. Finally, some families of arithmetic–geometric equienergetic graphs are constructed by graph operations.

#### 1. Introduction

Throughout this paper, let G = (V, E) be a simple graph of order n and size m, with vertex set  $V(G) = \{v_1, v_2, \ldots, v_n\}$  and edge set E(G). An edge  $e \in E(G)$  with end vertices  $v_i$  and  $v_j$  is denoted by  $v_i v_j$ . Let  $d_i$  be the degree of a vertex  $v_i$ . The maximum and minimum degree of G are denoted by  $\Delta$  and  $\delta$ , respectively. The 2-degree of  $v_i$  is denoted by  $t_i$ , which is defined as the sum of degrees of all vertices adjacent to  $v_i$ , i.e.,  $t_i = \sum_{v_i v_i \in E(G)} d_j$ .

The adjacency matrix  $A(G) = (a_{i,j})$  of G is the matrix of order n, where  $a_{i,j} = 1$  if  $v_i v_j \in E(G)$  and 0 otherwise. The eigenvalues of A(G) are denoted by  $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ .

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The energy [11, 15, 16] of G is defined as  $\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|$ . In 1975, a so-called Randić index was proposed, associated with molecular structure [19]. It is defined by

$$R_{-1}(G) = \sum_{v_i v_j \in E(G)} \frac{1}{d_i d_j}$$

In [9], Gutman et al. introduced the Randić matrix  $R(G) = (r_{i,j})$  of G, where  $r_{i,j} = \frac{1}{\sqrt{d_i d_j}}$ if  $v_i v_j \in E(G)$  and 0 otherwise. The eigenvalues of R(G) are denoted by  $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$ . Moreover, Gutman et al. [9] also offered some bounds for the Randić spectral radius and Randić energy. In addition, Cavers et al. [3] obtained a bound on Randić index in terms of the normalized Laplacian energy of graphs.

In 1994, Yang et al. [28] proposed the extended adjacency matrix of G, denoted by  $A_{ex}(G) = (c_{i,j})$ , where  $c_{i,j} = \frac{1}{2}(\frac{d_i}{d_j} + \frac{d_j}{d_i})$  if  $v_i v_j \in E(G)$  and 0 otherwise. The energy of extended adjacent matrix  $A_{ex}(G)$  was first studied by Yang et al. [28]. The corresponding topological index is the symmetric division deg index [25], which is written as

$$SDD(G) = \sum_{v_i v_j \in E(G)} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right).$$

Recently, the spectral radius and energy of the extended adjacency matrix were also studied, see [2, 7, 27] and the references cited therein.

In 2009, Vukičević and Furtula [26] proposed the geometric–arithmetic matrix of G, denoted by  $A_{GA}(G) = (g_{i,j})$ , where  $g_{i,j} = \frac{2\sqrt{d_i d_j}}{d_i + d_j}$  if  $v_i v_j \in E(G)$  and 0 otherwise. The corresponding topological index is

$$GA_1 = \sum_{v_i v_j \in E(G)} \frac{2\sqrt{d_i d_j}}{d_i + d_j}$$

The spectral properties of the geometric–arithmetic index were considered in [20]. For more information about the geometric–arithmetic index, see [6, 26].

Recently, Shegehall and Kanabur [23] introduced the arithmetic–geometric index of G. In particular, they studied the arithmetic–geometric indices of path graph with pendent vertices attached to the middle vertices of path  $P_n$  [24]. Motivated by these papers, we consider the arithmetic–geometric matrix of G, which is defined as  $A_{AG} = A_{AG}(G) = (h_{i,j})$ , where  $h_{i,j} = \frac{d_i+d_j}{2\sqrt{d_id_j}} = \frac{1}{2} \left( \sqrt{\frac{d_i}{d_j}} + \sqrt{\frac{d_j}{d_i}} \right)$  if  $v_i v_j \in E(G)$  and 0 otherwise. Denote the eigenvalues of  $A_{AG}(G)$  by  $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_n$ , where  $\eta_1$  is called the arithmetic–geometric spectral radius of G. The arithmetic–geometric energy is defined as

$$\mathcal{E}_{AG} = \mathcal{E}_{AG}(G) = \sum_{i=1}^{n} |\eta_i|.$$

It is worth noting that some bounds of the arithmetic–geometric energy of graphs have been offered by Das et al. in [8].

Two non-isomorphic graphs of the same order without identical spectra are said to be equienergetic if they have the same energy. Similarly, two graphs are said to be arithmetic–geometric equienergric if they have the same arithmetic–geometric graph energy. More details on equienergetic graphs can be found in [1, 2, 14].

As usual, the isolated vertex, the complete bipartite graph, complete k-partite graph, the complete graph, and the star on n vertices are denoted by  $K_1$ ,  $K_{p,q}$ ,  $K_{n_1,n_2,\ldots,n_k}$ ,  $K_n$ and  $K_{1,n-1}$ , respectively.

In this paper, we also need the following three degree–based topological indices

- The first Zagreb index [11],  $M_1(G) = \sum_{i=1}^n d_i^2 = \sum_{v_i v_j \in E(G)} (d_i + d_j),$
- The second Zagreb index [10],  $M_2(G) = \sum_{v_i v_j \in E(G)} d_i d_j$ ,
- The forgotten index [11],  $F(G) = \sum_{i=1}^{n} d_i^3 = \sum_{v_i v_j \in E(G)} \left( d_i^2 + d_j^2 \right).$

In the rest of the paper, we obtain some upper and lower bounds for the spectral radius and energy of the arithmetic–geometric matrix and characterize the extremal graphs. This paper is organized as follows. In Section 2, we recall some earlier results which will be used in the later parts of the paper. Section 3 gives some upper and lower bounds for the spectral radius of the arithmetic–geometric matrix and characterizes the extremal graphs. Some bounds on the arithmetic–geometric energy are obtained in Section 4. In Section 5, we construct some pairs of arithmetic–geometric equienergetic graphs on nvertices for all  $n \geq 9$ .

## 2. Preliminaries

We recall some known results, which will be used in the next sections. Lemma 1 [12]. If G is a connected graph of order n with size m, then

$$\lambda_1 \le \sqrt{2m - n + 1} \,,$$

with equality holding if and only if G is isomorphic to  $K_n$  or  $K_{1,n-1}$ . Lemma 2 [8]. If G is a graph of order n, then

$$\gamma_1 \le \frac{1}{n} \sqrt{2n(n-1)R_{-1}}$$
.

**Lemma 3** (Rayleigh-Ritz) [29]. If *B* is a real symmetric matrix of order *n* with eigenvalues  $\rho_1 \ge \rho_2 \ge \cdots \ge \rho_n$ , then, for a nonzero vector *x*,

$$\rho_1 \ge \frac{x^T B x}{x^T x} \,,$$

with equality holding if and only if x is an eigenvector of B corresponding to  $\rho_1$ .

**Lemma 4** [30]. Let G be a graph of order n with degree sequence  $d_1, d_2, \ldots, d_n$ . Then

$$\lambda_1 \ge \sqrt{\frac{\sum\limits_{i=1}^n d_i^2}{n}}.$$

with equality holding if and only if G is regular or semiregular.

**Lemma 5** (Interlacing Lemma) [22]. Let B be a symmetric matrix of order n, and  $B_k$  be its  $k \times k$  submatrix. Then, for any integer i where  $1 \le i \le k$ ,

$$\rho_{n-k+i}(B) \le \rho_i(B_k) \le \rho_i(B) ,$$

where  $\rho_i(B)$ ,  $\rho_i(B_k)$  are the *i*-th largest eigenvalue of B and  $B_k$ , respectively.

**Lemma 6** [13]. Suppose that  $B = (b_{i,j})$  and  $C = (c_{i,j})$  are two nonnegative symmetric matrices of order n. If  $B \ge C$ , i.e.,  $b_{i,j} \ge c_{i,j}$  for all i, j, then  $\rho_1(B) \ge \rho_1(C)$ .

**Lemma 7** [17]. Let B and C be two  $n \times n$  complex matrices. For any integer k, where  $1 \le k \le n$ ,

$$\sum_{i=1}^{k} s_i(B+C) \le \sum_{i=1}^{k} s_i(B) + \sum_{i=1}^{k} s_i(C) ,$$

where  $s_i$  denotes the *i*-th largest singular value of matrices.

**Lemma 8** [17]. Let  $B_1, B_2, \ldots, B_m$  be  $n \times n$  complex matrices. For any integer k where  $1 \le k \le n$ ,

$$\sum_{i=1}^{k} s_i(B_1 B_2 \cdots B_m) \le \sum_{i=1}^{k} s_i(B_1) s_i(B_2) \cdots s_i(B_m).$$

In the next lemmas, we state some inequalities, that are needed in seeking bounds for the arithmetic–geometric energy.

**Lemma 9** [18]. For positive numbers  $x_1, x_2, \ldots, x_n$ ,

$$\frac{n}{\frac{1}{x_1} + \dots + \frac{1}{x_n}} \le \sqrt[n]{x_1 x_2 \cdots x_n}.$$

**Lemma 10** (Chebyshev's inequality) [5]. For real numbers  $a_1 \leq a_2 \leq \cdots \leq a_n$  and  $b_1 \leq b_2 \leq \cdots \leq b_n$ ,

$$\left(\sum_{i=1}^{n} a_i\right) \left(\sum_{i=1}^{n} b_i\right) \le n \sum_{i=1}^{n} a_i b_i$$

with equality holding if and only if  $a_1 = a_2 = \cdots = a_n$  or  $b_1 = b_2 = \cdots = b_n$ . Lemma 11 [18]. For non-negative numbers  $x_1, x_2, \ldots, x_n$  and  $k \ge 2$ ,

$$\sum_{i=1}^{n} (x_i)^k \le \left(\sum_{i=1}^{n} x_i^2\right)^{k/2}$$

**Lemma 12** [4]. Let G be a graph of order  $n \ge 2$ . Then the multiplicity of the eigenvalue zero in its adjacency spectrum equals to n-2 if and only if G is isomorphic to  $K_{p,q} \cup (n-p-q) K_1$ , where  $p+q \le n$ .

**Lemma 13** [21]. A connected graph G of order n has only one positive eigenvalue in its adjacency spectrum if and only if G is a complete k-partite graph  $K_{n_1,n_2,\ldots,n_k}$ .

### 3. On arithmetic–geometric spectral radius

In this section, we obtain some lower and upper bounds on the spectral radius  $\eta_1$  of the arithmetic–geometric matrix. First, we give two lower bounds in terms of the first Zagreb index  $M_1$ .

**Theorem 1.** Let G be a graph of order n with maximum degree  $\Delta$ . Then

$$\eta_1 \ge \frac{M_1}{n\Delta} \,, \tag{1}$$

where the equality holds if and only if G is a regular graph.

*Proof.* Let  $x = (x_1, x_2, \dots, x_n)^T$  be a unit vector in  $\mathbb{R}^n$ . Then,

$$x^{T}A_{AG}x = \sum_{v_{i}v_{j} \in E(G)} \left( \sqrt{\frac{d_{i}}{d_{j}}} + \sqrt{\frac{d_{j}}{d_{i}}} \right) x_{i} x_{j} \ge \sum_{v_{i}v_{j} \in E(G)} \left( \frac{d_{i} + d_{j}}{\Delta} \right) x_{i} x_{j} .$$
(2)

Set  $x = \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}\right)^T$ . Then from Lemma 3, we have

$$\eta_1 \ge x^T A_{AG} x \ge \frac{\sum\limits_{v_i v_j \in E(G)} (d_i + d_j)}{n\Delta} = \frac{\sum\limits_{i=1}^n d_i^2}{n\Delta} = \frac{M_1}{n\Delta}$$

Now suppose that the equality holds in (1). Then all the inequalities in the proof must be equalities. From (2), we have  $d_1 = d_2 = \cdots = d_n = \Delta$ . Furthermore, from  $\eta_1 = x^T A_{AG}x$ , we have that the vector  $x = \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right)^T$  is an eigenvector corresponding to  $\eta_1$ . Hence, G is a regular graph.

Conversely, if G is a regular graph, it is easy to check that the equality holds in (1).

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**Theorem 2.** Let G be a graph of order n. Then

$$\eta_1 \ge \sqrt{\frac{M_1}{n}},\tag{3}$$

where the equality holds if and only if G is a regular graph.

*Proof.* Since  $\frac{1}{2}\left(\sqrt{\frac{d_i}{d_j}} + \sqrt{\frac{d_j}{d_i}}\right) \ge 1$ , then  $A_{AG} \ge A(G)$ . Furthermore, by Lemmas 4 and 6,

$$\eta_1 \ge \lambda_1 \ge \sqrt{\frac{\sum\limits_{i=1}^n d_i^2}{n}} = \sqrt{\frac{M_1}{n}}.$$

Finally, by the equality condition of Lemma 4 and  $\frac{1}{2}\left(\sqrt{\frac{d_i}{d_j}} + \sqrt{\frac{d_j}{d_i}}\right) = 1$ , we can get that the equality holds if and only if G is regular graph.

Note that the lower bound of  $\eta_1$  in (3) is better than that in (1) as

$$\frac{M_1^2}{n^2 \Delta^2} - \frac{M_1}{n} = \frac{M_1^2 - nM_1 \Delta^2}{n^2 \Delta^2} = \frac{M_1}{n^2 \Delta^2} \left( M_1 - n\Delta^2 \right) = \frac{M_1}{n^2 \Delta^2} \left( \sum_{i=1}^n d_i^2 - n\Delta^2 \right) \le 0.$$

Next we give an upper bound on  $\eta_1$  in terms of the number of vertices and edges. **Theorem 3.** Let G be a graph of order n and size m. Then

$$\eta_1 \le \frac{1}{2} \left( \sqrt{n-1} + \frac{1}{\sqrt{n-1}} \right) \sqrt{2m-n+1} \,, \tag{4}$$

where the equality holds if and only if G is isomorphic to  $K_{1,n-1}$ .

*Proof.* Since  $f(x) = x + \frac{1}{x}$  is an increasing function in the variable  $x \in [1, +\infty)$ , for any edge  $v_i v_j \in E(G)$ ,

$$\sqrt{\frac{d_i}{d_j}} + \sqrt{\frac{d_j}{d_i}} \le \sqrt{\frac{\Delta}{\delta}} + \sqrt{\frac{\delta}{\Delta}} \le \sqrt{n-1} + \frac{1}{\sqrt{n-1}} \,. \tag{5}$$

Let  $\rho_1$  be the spectral radius of the matrix  $\frac{1}{2}\left(\sqrt{\frac{\Delta}{\delta}} + \sqrt{\frac{\delta}{\Delta}}\right)A(G)$ . Then, from Lemmas 6 and 1, we have

$$\eta_1 \le \rho_1 = \frac{1}{2} \left( \sqrt{\frac{\Delta}{\delta}} + \sqrt{\frac{\delta}{\Delta}} \right) \lambda_1 \le \frac{1}{2} \left( \sqrt{n-1} + \frac{1}{\sqrt{n-1}} \right) \sqrt{2m-n+1}.$$
(6)

We now consider the sharpness of (4). Suppose that the equality holds in (4). Then all the inequalities in the proof must be equalities. From the equality in (5), we have  $d_i = 1, d_j = n - 1$  or  $d_i = 1, d_j = n - 1$  for any edge  $v_i v_j \in E(G)$ . From the equality in (6), we get  $\lambda_1 = \sqrt{2m - n + 1}$ . Then by Lemma 1, G is isomorphic to  $K_n$  or  $K_{1,n-1}$ . Hence, G must be isomorphic to  $K_{1,n-1}$ .

Conversely, it is easy to check that the equality holds in (4) if G is isomorphic to  $K_{1,n-1}$ .

In a similar way as in Theorem 3, we can obtain an upper bound on  $\eta_1$  in terms of Randić index  $R_{-1}$ .

**Theorem 4.** Let G be a graph of order n and size m with maximum degree  $\Delta$ . Then

$$\eta_1 \le \frac{\Delta}{n} \sqrt{2n(n-1)R_{-1}} \,. \tag{7}$$

Equality in (7) holds if and only if G is isomorphic to  $K_n$ . *Proof.* For any edge  $v_i v_j \in E(G)$ ,

$$\frac{1}{2}\left(\sqrt{\frac{d_i}{d_j}} + \sqrt{\frac{d_j}{d_i}}\right) \le \Delta \frac{1}{\sqrt{d_i d_j}}.$$
(8)

Then, by Lemmas 6 and 2, we have

$$\eta_1 \le \Delta \gamma_1 \le \frac{\Delta}{n} \sqrt{n(n-1)tr(R^2)} = \frac{\Delta}{n} \sqrt{2n(n-1)R_{-1}} \,. \tag{9}$$

We next consider the sharpness of (7). Suppose that the equality holds in (7). Then all the inequalities in the proof must be equalities. From the equality in (8), we have  $d_1 = d_2 = \cdots = d_n = \Delta$ , i.e., *G* is a regular graph. Since *G* is a regular graph, then  $\eta_1 = \lambda_1 = r$ ,  $\gamma_1 = \frac{1}{r}\lambda_1 = 1$ , and  $R_{-1} = \sum_{\substack{v_i v_j \in E(G) \\ v_i v_j \in E(G)}} \left(\frac{1}{d_i d_j}\right) = \frac{m}{r^2} = \frac{n}{2r}$ , where *r* is the regular degree of *G*. From the equality in (9), we get

$$r = \frac{\Delta}{n}\sqrt{2n(n-1)R_{-1}} = \sqrt{\frac{n^2 r^2 (n-1)}{n^2 r}} = \sqrt{r(n-1)},$$

which implies that n - 1 = r. Hence G is a complete graph  $K_n$ .

Conversely, it is easy to check that the equality holds in (7) for  $K_n$ .

**Theorem 5.** Let G be a connected graph of order n with maximum degree  $\Delta$  and minimum degree  $\delta$ . Then

$$\eta_1 \le \left[1 + \frac{\left(\sqrt{\Delta} - \sqrt{\delta}\right)^2}{2\delta}\right] \lambda_1 \,, \tag{10}$$

with the equality holding if and only if G is a regular graph.

*Proof.* Let  $x = (x_1, x_2, ..., x_n)^T$  be the unit eigenvector corresponding to the eigenvalue  $\eta_1$ . Then  $\eta_1 = x^T A_{AG} x$ . By Lemma 3,

$$\lambda_1 \ge x^T A x = 2 \sum_{v_i v_j \in E(G)} x_i x_j \,. \tag{11}$$

By (2), we get

$$x^{T}A_{AG}x = \sum_{v_{i}v_{j}\in E(G)} \left(\frac{d_{i}+d_{j}}{\sqrt{d_{i}d_{j}}}\right)x_{i}x_{j} = \sum_{v_{i}v_{j}\in E(G)} \left(\frac{\left(\sqrt{d_{i}}-\sqrt{d_{j}}\right)^{2}+2\sqrt{d_{i}d_{j}}}{\sqrt{d_{i}d_{j}}}\right)x_{i}x_{j}$$

$$\leq 2\sum_{v_{i}v_{j}\in E(G)}x_{i}x_{j} + \frac{\left(\sqrt{\Delta}-\sqrt{\delta}\right)^{2}}{\delta}\sum_{v_{i}v_{j}\in E(G)}x_{i}x_{j} \qquad (12)$$

$$= 2\left[1 + \frac{\left(\sqrt{\Delta}-\sqrt{\delta}\right)^{2}}{2\delta}\right]\sum_{v_{i}v_{j}\in E(G)}x_{i}x_{j} \leq \left[1 + \frac{\left(\sqrt{\Delta}-\sqrt{\delta}\right)^{2}}{2\delta}\right]\lambda_{1}.$$

Suppose now that the equality holds in (10). Then all the inequalities in the proof must be equalities. From (11) and (12), we get  $d_1 = d_2 = \cdots = d_n = \delta$  and  $\lambda_1 = \eta_1$ . Hence G is a regular graph.

Conversely, it is easy to check that the equality holds in (10) for a regular graph G.

# 4. On arithmetic–geometric energy

In this section, we mainly use some fundamental inequalities to obtain upper and lower bounds of the arithmetic–geometric energy. First, we give some upper bounds in terms of the maximum degree and the minimum degree and some topological indices. **Theorem 6.** Let G be a graph of order n and size m, with maximum degree  $\Delta$  and minimum degree  $\delta$  ( $\delta \geq 1$ ). Then

$$\mathcal{E}_{AG} \leq \sqrt{2n\Delta^2 R_{-1}} \tag{13}$$

$$\mathcal{E}_{AG} \leq \frac{1}{2}\sqrt{2mn}\left(\sqrt{\frac{\Delta}{\delta}} + \sqrt{\frac{\delta}{\Delta}}\right) \tag{14}$$

$$\mathcal{E}_{AG} \leq \sqrt{\frac{n}{2\delta^2} \left(F + 2M_2\right)} \tag{15}$$

$$\mathcal{E}_{AG} \leq \sqrt{n\left(\frac{F}{2\delta^2} + m\right)} \tag{16}$$

$$\mathcal{E}_{AG} \leq \sqrt{n\left(\frac{1}{2\delta}\sum_{i=1}^{n}t_{i}+m\right)}$$
(17)

where the equalities hold in (13)–(17) if and only if G is isomorphic to  $\frac{n}{2}K_2$ .

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*Proof.* Remark that these bounds are direct consequences of Corollary 2 in [8], in order to obtain the equalities' condition, we here give a new proof. By the Cauchy–Schwarz inequality,

$$\mathcal{E}_{AG} = \sum_{i=1}^{n} |\eta_i| \le \sqrt{n \sum_{i=1}^{n} \eta_i^2} = \sqrt{\frac{n}{2} \sum_{v_i v_j \in E(G)} \left(\sqrt{\frac{d_i}{d_j}} + \sqrt{\frac{d_j}{d_i}}\right)^2}.$$
 (18)

Since

$$\sqrt{\frac{d_i}{d_j}} + \sqrt{\frac{d_j}{d_i}} \le \sqrt{\frac{\Delta}{\delta}} + \sqrt{\frac{\delta}{\Delta}}, \quad \sqrt{\frac{d_i}{d_j}} + \sqrt{\frac{d_j}{d_i}} \le \frac{2\Delta}{\sqrt{d_i d_j}}, \quad \sqrt{\frac{d_i}{d_j}} + \sqrt{\frac{d_j}{d_i}} \le \frac{d_i + d_j}{\delta}, \quad (19)$$

from (18) and (19), we arrive at (13)-(15) directly.

On the other hand, from (18) it follows

$$\mathcal{E}_{AG} \leq \sqrt{n \sum_{i=1}^{n} \eta_i^2} = \sqrt{\frac{n}{2} \sum_{v_i v_j \in E(G)} \left(\sqrt{\frac{d_i}{d_j}} + \sqrt{\frac{d_j}{d_i}}\right)^2}$$
$$= \sqrt{\frac{n}{2} \sum_{v_i v_j \in E(G)} \left(\frac{d_i}{d_j} + \frac{d_j}{d_i}\right) + mn}$$
$$(20)$$

$$\leq \sqrt{\frac{n}{2}} \sum_{v_i v_j \in E(G)} \left(\frac{d_i^2 + d_j^2}{\delta^2}\right) + mn = \sqrt{\left(\frac{F}{2\delta^2} + m\right)n}.$$
(21)

Then, we arrive at (16).

From (20), we have

$$\mathcal{E}_{AG} \leq \sqrt{\frac{n}{2} \left[ \sum_{v_i \in V(G)} \left( \frac{1}{d_i} \right) \sum_{v_i v_j \in E(G)} (d_j) \right] + mn} \\ \leq \sqrt{\frac{n}{2\delta} \left[ \sum_{v_i \in V(G)} \left( \sum_{v_i v_j \in E(G)} (d_j) \right) \right] + mn} = \sqrt{\frac{n}{2\delta} \sum_{i=1}^n t_i + mn}, \quad (22)$$

implying that the (17) follows.

We now examine the sharpness of (13)–(17). Suppose that the equalities hold in (13)–(17). Then all the inequalities in the proof must be equalities. From the equalities in (19), (21), and (22), we conclude that G is a regular graph. From the equality in (18), we get  $|\eta_1| = |\eta_2| = \cdots = |\eta_n|$ . If  $d_i = 1$  for any  $1 \le i \le n$ , then G is isomorphic to  $\frac{n}{2}K_2$ . Otherwise,  $d_i \ge 2$  for any  $1 \le i \le n$ . Then G contains a connected component H with at

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least 3 vertices. If *H* is the complete graph of order p ( $p \ge 3$ ), then  $|\eta_1| = p-1 > 1 = |\eta_2|$ , a contradiction. So *H* is not a complete graph. On the other hand, since the arithmetic– geometric matrix is a nonnegative irreducible matrix, then  $\eta_1(H) > \eta_2(H)$ . By Lemma 5,  $\eta_2(H) \ge 0$ , which leads to a contradiction. Hence,  $|\eta_1| = |\eta_2| = \cdots = |\eta_n|$  holds only if *G* is isomorphic to  $\frac{n}{2}K_2$ .

Conversely, it is easy to check that the equalities hold in (13)–(17) if G is isomorphic to  $\frac{n}{2}K_2$ .

We now offer some other upper bound on the arithmetic–geometric energy in terms of the energy  $\mathcal{E}(G)$  of graphs.

**Theorem 7.** Let G be a graph of order n and size m, with maximum degree  $\Delta$  and minimum degree  $\delta$ . Then

$$\mathcal{E}_{AG} \leq \sqrt{\frac{\Delta}{\delta}} \mathcal{E}(G)$$
.

*Proof.* Let A, B, C be  $n \times n$  matrices, where A is the adjacency matrix of G and  $C = \text{diag}(\frac{1}{\sqrt{d_1}}, \frac{1}{\sqrt{d_2}}, \dots, \frac{1}{\sqrt{d_n}})$ . Then by the definition of the arithmetic-geometric matrix, we have

$$A_{AG} = \frac{B + B^T}{2}, \ B = C^{-1}AC.$$

From Lemmas 7 and 8, we obtain

$$\begin{aligned} \mathcal{E}_{AG} &= \sum_{i=1}^{n} s_i \left( A_{AG} \right) = \sum_{i=1}^{n} s_i \left( \frac{B + B^T}{2} \right) \\ &\leq \sum_{i=1}^{n} s_i \left( \frac{B}{2} \right) + \sum_{i=1}^{n} s_i \left( \frac{B^T}{2} \right) = \sum_{i=1}^{n} s_i \left( B \right) \\ &= \sum_{i=1}^{n} s_i \left( C^{-1}AC \right) \leq \sum_{i=1}^{n} s_i \left( C^{-1} \right) s_i \left( A \right) s_i \left( C \right) \\ &= \sum_{i=1}^{n} \sqrt{\frac{d_i}{d_{n-i+1}}} s_i \left( A \right) \leq \sqrt{\frac{\Delta}{\delta}} \sum_{i=1}^{n} s_i \left( A \right) = \sqrt{\frac{\Delta}{\delta}} \mathcal{E} \left( G \right) \end{aligned}$$

implying the required result.

In what follows, we obtain two lower bounds for  $\mathcal{E}_{AG}$  in terms of the first Zagreb index, the number of vertices, and the number of edges.

**Theorem 8.** Let G be a connected graph of order n. Then

$$\mathcal{E}_{AG} \ge 2\sqrt{\frac{M_1}{n}},$$

with the equality holding if and only if G is isomorphic to the complete k-partite graph  $G_{r_1,r_2,\ldots,r_k}$ , where  $|r_1| = |r_2| = \cdots = |r_k|$ .

Proof. Applying Theorem 2, we have

$$\mathcal{E}_{AG} = \sum_{i=1}^{n} |\eta_i| = 2 \sum_{i=1, \eta_i \ge 0}^{n} \eta_i \ge 2\eta_1 \ge 2\sqrt{\frac{M_1}{n}},$$

where the second equality holds if and only if G is a regular graph. So  $\eta_i = \lambda_i$  for any  $1 \le i \le n$ . From Lemma 13, the equality holds if and only if G is isomorphic to  $G_{r_1,r_2,\ldots,r_k}$ , where  $|r_1| = |r_2| = \cdots = |r_k|$ .

**Theorem 9.** If G is a graph of order n with m edges, then

$$\mathcal{E}_{AG} \ge 2\sqrt{m}\,,\tag{23}$$

with the equality holding if and only if G is isomorphic to  $K_{p,p} \cup (n-2p) K_1$ , where  $p = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$ .

*Proof.* For m = 0, the equality holds obviously. For  $m \ge 1$ , let p be the number of isolated vertices and k be the number of connected components. In addition, let  $G_i$  be the *i*-th connected component of G with order  $n_i \ge 2$  and  $m_i \ge 1$  edges. Then we have  $n = p + \sum_{i=1}^{k} n_i, \ m = \sum_{i=1}^{k} m_i.$ 

Consider first the connected component  $G_i$  with vertex set  $V(G_i) = \{v_{i1}, \ldots, v_{in_i}\}$ and edge set  $E(G_i)$ . Let  $d_{i1}, \ldots, d_{in_i}$  be the degree sequence of  $G_i$ . And let  $\eta_{i1}, \ldots, \eta_{in_i}$ be the eigenvalues of  $A_{AG}(G_i)$ . Then, by Theorem 4 in [8], we have

$$\mathcal{E}_{AG}(G_i) \ge \sqrt{2\sum_{j=1}^{n_i} \eta_{ij}^2} = \sqrt{\sum_{v_{ij}v_{ik} \in E(G_i)} \left(\sqrt{\frac{d_{ij}}{d_{ik}}} + \sqrt{\frac{d_{ik}}{d_{ij}}}\right)^2} \ge 2\sqrt{m_i}.$$
 (24)

Notice that, for real numbers  $a \ge b > 0$ ,

$$\sqrt{a} + \sqrt{b} \ge \sqrt{a+b} \,, \tag{25}$$

with equality holding if and only if b = 0. From this inequality, we have

$$\mathcal{E}_{AG}(G) = \sum_{i=1}^{k} \mathcal{E}_{AG}(G_i) \ge 2\sqrt{m_1} + 2\sqrt{m_2} + \dots + 2\sqrt{m_k}$$
  
 
$$\ge 2\sqrt{m_1 + m_2} + \dots + 2\sqrt{m_k} \ge \dots \ge 2\sqrt{m_1 + \dots + m_k} = 2\sqrt{m} .$$

The first part of the proof is done.

Now suppose that the equality holds for  $m \ge 1$ . Then all the above inequalities must be equalities. By the equality condition in Theorem 4 of [8], the first equality in (24) holds if and only if  $\eta_{i1} = -\eta_{in_i}, \eta_{ij} = 0$  for  $2 \le j \le n_i - 1$ . The second equality in (24)

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holds if and only if  $G_i$  is a regular graph. Since  $G_i$  is a regular graph, then  $\eta_{ij} = \lambda_{ij}$ for  $1 \leq j \leq n_i$ , where  $\lambda_{ij}$  is the adjacency eigenvalue of  $G_i$ . Then by Lemma 12,  $G_i$  is isomorphic to  $K_{p,p}$  as  $G_i$  is regular. From the equality in (25), we have k = 1. Hence Gis isomorphic to  $K_{p,p} \cup (n-2p) K_1$ , where  $p = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$ .

Conversely, it is easy to check that the equality holds in (23) if G is isomorphic to  $K_{p,p} \cup (n-2p) K_1$ .

Given a graph G, if all eigenvalues in its adjacency spectrum are nonzero, then G is said to be nonsingular. Similarly, if all eigenvalues of the arithmetic–geometric matrix of G are nonzero, then G is called AG nonsingular. Next we give a lower bound on  $\mathcal{E}_{AG}$  for an AG nonsingular connected graph G.

**Theorem 10.** If G is an AG nonsingular connected graph of order n, then

$$\mathcal{E}_{AG} \ge \sqrt{\frac{M_1}{n}} - \ln \sqrt{\frac{M_1}{n}} + \ln \left| \det A_{AG} \right| + n - 1.$$

*Proof.* Since  $x \ge 1 + \ln x$  for any x > 0, we have

$$\begin{aligned} \mathcal{E}_{AG} &= \sum_{i=1}^{n} |\eta_i| = \eta_1 + \sum_{i=2}^{n} |\eta_i| \ge n - 1 + \sum_{i=2}^{n} \ln |\eta_i| + \eta_1 \\ &= n - 1 + \eta_1 + \ln \prod_{i=2}^{n} |\eta_i| = n - 1 + \eta_1 + \ln |\det A_{AG}| - \ln \eta_1 \end{aligned}$$

Since  $h(x) = n - 1 + x + \ln |\det A_{AG}| - \ln x$  is an increasing function in the variable  $x \in [1, +\infty)$ , by Theorem 2 we obtain

$$h(x) \ge h\left(\sqrt{\frac{M_1}{n}}\right) = \sqrt{\frac{M_1}{n}} - \ln\sqrt{\frac{M_1}{n}} + \ln\left|\det A_{AG}\right| + n - 1.$$

Thus the proof is completed.

**Theorem 11.** If G is a graph of order n with m edges, then

$$e^{-\sqrt{\frac{1}{2}SDD+m}} < \mathcal{E}_{AG} < e^{\sqrt{\frac{1}{2}SDD+m}}$$

*Proof.* First, we prove the right part of the result by using fundamental inequality and power series expansion method. Since  $x < e^x$  for any real number x, it follows

$$\mathcal{E}_{AG} = \sum_{i=1}^{n} |\eta_i| < \sum_{i=1}^{n} e^{|\eta_i|} = \sum_{i=1}^{n} \sum_{k \ge 0} \frac{(|\eta_i|)^k}{k!} = \sum_{k \ge 0} \frac{1}{k!} \sum_{i=1}^{n} (|\eta_i|)^k.$$

Note that  $\sum_{i=1}^n \eta_i^2 = \frac{1}{2} \sum_{v_i v_j \in E(G)} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right) + m = \frac{1}{2}SDD + m$ . From Lemma 11, we have

$$\mathcal{E}_{AG} < \sum_{k \ge 0} \frac{1}{k!} \sum_{i=1}^{n} (|\eta_i|)^k \le \sum_{k \ge 0} \frac{1}{k!} \left( \sum_{i=1}^{n} (|\eta_i|)^2 \right)^{k/2}$$

$$= \sum_{k \ge 0} \frac{1}{k!} \left( \sqrt{\frac{1}{2}SDD + m} \right)^k = e^{\sqrt{\frac{1}{2}SDD + m}}$$

The right part of the proof is done.

Let  $\sigma$  be the number of the nonzero eigenvalues of the matrix  $A_{AG}$ , and let  $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_{\sigma}$  be the absolute values of all these nonzero eigenvalues, given in a non-increasing order. By Lemma 5,

$$\eta_n \le \rho_2[A_{AG}] \le -\frac{1}{2} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right) \le -1 \,,$$

where  $[A_{AG}]$  is the leading  $2 \times 2$  submatrix of  $A_{AG}$ . Therefore,  $|\eta_n| \ge 1$ . Hence,

$$\sum_{i=1}^{\sigma} \xi_i = \sum_{i=1}^{n} |\eta_i| \ge |\eta_n| \ge 1.$$
(26)

Using the arithmetic–geometric mean inequality, we have

$$\mathcal{E}_{AG} = \sum_{i=1}^{n} |\eta_i| = \sum_{i=1}^{\sigma} \xi_i = \sigma \left( \sum_{i=1}^{\sigma} \frac{1}{\sigma} \xi_i \right) \ge \sigma \left( \sqrt[\sigma]{\xi_1 \xi_2 \cdots \xi_\sigma} \right) \,.$$

It follows from Lemmas 9, 10 and (26) that

$$\sigma\left(\sqrt[\sigma]{\xi_1\xi_2\cdots\xi_\sigma}\right) \ge \sigma\left(\frac{\sigma}{\sum\limits_{i=1}^{\sigma}\frac{1}{\xi_i}}\right) \ge \sigma\left(\frac{\sigma}{\sum\limits_{i=1}^{\sigma}\frac{1}{\xi_i}\sum\limits_{i=1}^{\sigma}\xi_i}\right) \ge \sigma\left(\frac{\sigma}{\sigma\sum\limits_{i=1}^{\sigma}\frac{1}{\xi_i}\xi_i}\right).$$

Applying power series expansion of  $e^x$ , we obtain

$$\sigma\left(\frac{\sigma}{\sigma\sum_{i=1}^{\sigma}\frac{1}{\xi_i}\xi_i}\right) \ge \sigma\left(\frac{\sigma}{\sigma^2\sum_{i=1}^{\sigma}\xi_i}\right) > \frac{1}{\sum_{i=1}^{\sigma}e^{\xi_i}} = \frac{1}{\sum_{i=1}^{\sigma}\sum_{k\ge 0}\frac{(\xi_i)^k}{k!}} = \frac{1}{\sum_{k\ge 0}\frac{1}{k!}\sum_{i=1}^{\sigma}(\xi_i)^k}.$$

It follows from Lemma 11 that

$$\frac{1}{\sum_{k\geq 0}\frac{1}{k!}\sum_{i=1}^{\sigma}\left(\xi_{i}\right)^{k}} \geq \frac{1}{\sum_{k\geq 0}\frac{1}{k!}\sum_{i=1}^{\sigma}\left(\xi_{i}^{2}\right)^{\frac{k}{2}}} = \frac{1}{\sum_{k\geq 0}\frac{1}{k!}\left(\sqrt{\frac{1}{2}SDD} + m\right)^{k}} = e^{-\sqrt{\frac{1}{2}SDD} + m},$$

implying the required lower bound.

# 5. Arithmetic-geometric equienergetic graphs

In this section, we consider constructions of arithmetic–geometric equienergetic nonregular graphs. Clearly, the energy is equal to arithmetic–geometric energy for a regular graph. Hence, if two regular graphs are equienergetic, then they are also arithmetic– geometric equienergetic.

We first recall the join operation on graphs. Let G and H be two graphs. The join  $G \vee H$  of G and H is the graph obtained by the vertex set V(G) of G and V(H) of H and then joining each of the vertices of V(G) to all the vertices of V(H).

**Theorem 12.** Let  $G_i$  be an  $r_i$  regular graph of order  $n_i$  for i = 1, 2. Then the arithmetic– geometric energy of  $G_1 \vee G_2$  is

$$\mathcal{E}_{AG} = \mathcal{E}(G_1) + \mathcal{E}(G_2) - (r_1 + r_2) + \sqrt{(r_1 + r_2)^2 - 4r_1 r_2 + n_1 n_2 \left(\sqrt{\frac{r_1 + n_2}{r_2 + n_1}} + \sqrt{\frac{r_2 + n_1}{r_1 + n_2}}\right)^2}.$$

*Proof.* Since  $G_i$  is an  $r_i$  regular graph of order  $n_i$  for i = 1, 2, it follows

$$A_{AG}(G_1 \vee G_2) = \begin{bmatrix} A_{AG}(G_1) & \frac{1}{2} \left( \sqrt{\frac{r_1 + n_2}{r_2 + n_1}} + \sqrt{\frac{r_2 + n_1}{r_1 + n_2}} \right) J_{n_1 \times n_2} \\ \frac{1}{2} \left( \sqrt{\frac{r_1 + n_2}{r_2 + n_1}} + \sqrt{\frac{r_2 + n_1}{r_1 + n_2}} \right) J_{n_2 \times n_1} & A_{AG}(G_2) \end{bmatrix}.$$

Then, the eigenvalues of the arithmetic–geometric matrix of  $G_1 \vee G_2$  are  $\lambda_j(G_i)$ , where  $j = 2, 3, \ldots n_i$  for i = 1, 2, and

$$\frac{1}{2} \left[ (r_1 + r_2) \pm \sqrt{(r_1 + r_2)^2 - 4r_1r_2 + n_1n_2\left(\sqrt{\frac{r_1 + n_2}{r_2 + n_1}} + \sqrt{\frac{r_2 + n_1}{r_1 + n_2}}\right)^2} \right],$$

which implies the required result.

**Theorem 13.** For all  $n \ge 9$ , there exists a pair of arithmetic–geometric equienergetic graphs of order n.

*Proof.* Let  $G_1$  and  $G_2$  be two graphs as shown in Figure 1. From the Example 4.1 in [16],  $G_1$  and  $G_2$  are 4-regular equienergetic graphs. Moreover,  $\mathcal{E}_{AG}(G_1) = \mathcal{E}_{AG}(G_2) = \mathcal{E}(G_1) = 16$ .



Figure 1: Two equienergetic 4-regular graphs used for constructing pairs of arithmetic– geometric equienergetic species.

Then, by Theorem 12, we have

 $\mathcal{E}_{AG}\left(G_1 \vee K_t\right) = \mathcal{E}_{AG}\left(G_2 \vee K_t\right)$ 

$$= t + 11 + \sqrt{(t+3)^2 - 16(t-1) + 9t\left(\sqrt{\frac{4+t}{8+t}} + \sqrt{\frac{8+t}{4+t}}\right)^2}$$

Hence,  $G_1 \vee K_t$  and  $G_2 \vee K_t$  are two arithmetic–geometric equienergetic graphs for all  $n \geq 9$ .

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