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A Short Proof for Graph Energy is at Least Twice of Minimum Degree

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Abstract

The energy $\mathcal{E}(G)$ of a graph G is the sum of the absolute values of all eigenvalues of G. Zhou in (*MATCH Commun. Math. Comput. Chem.* **55** (2006) 91–94) studied the problem of bounding the graph energy in terms of the minimum degree together with other parameters. He proved his result for quadrangle-free graphs. Recently, in (*MATCH Commun. Math. Comput. Chem.* **81** (2019) 393–404) it is shown that for every graph G, $\mathcal{E}(G) \geq 2\delta(G)$, where $\delta(G)$ is the minimum degree of G, and the equality holds if and only if G is a complete multipartite graph with equal size of parts. Here, we provide a short proof for this result. Also, we give an affirmative answer to a problem proposed in (*MATCH Commun. Math. Comput. Chem.* **81** (2019) 393–404).

Let G be a graph with the vertex set and the edge set $V(G)$ and $E(G)$, respectively. The *adjacency matrix* of a graph G of order n, $A(G) = [a_{ij}]$, is an $n \times n$ matrix, where $a_{ij} = 1$ if $v_i v_j \in E(G)$, and $a_{ij} = 0$, otherwise. The *energy* of a graph G, $\mathcal{E}(G)$, is defined as the sum of absolute values of eigenvalues of $A(G)$. The concept of graph energy was first introduced by Gutman in 1978, see [3]. For more properties of the energy of graphs we refer to [4]. There are many lower bounds on the energy of graphs. Zhou studied the problem of bounding the graph energy in terms of the minimum degree together with other parameters [7]. In the next theorem, we provide a short elementary proof for a problem given in [5].

Theorem 1. *Let G be a connected graph with average degree $\overline{d}$. Then $\mathcal{E}(G) \geq 2\overline{d}$ and the equality holds if and only if G is a complete multipartite graph with the equal size of parts.*

Proof. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of $A(G)$. Since $\sum_{i=1}^{n} \lambda_i = 0$, by the triangle inequality and [2, Theorem 3.8], we note that $\lambda_1 \geq \overline{d}$. Now, we have,

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i| = \lambda_1 + \sum_{i=2}^{n} |\lambda_i| \geq \lambda_1 + \left| \sum_{i=2}^{n} \lambda_i \right| = 2\lambda_1 \geq 2\overline{d}.$$

Also, $\mathcal{E}(G) = 2\overline{d}$, if and only if all of $\lambda_2, \ldots, \lambda_n$ have the same signs and $\lambda_1 = \overline{d}$. Note that $\lambda_2, \ldots, \lambda_n$ have the same signs if and only if λ_1 is the only positive eigenvalue of G which by [6] (see also [2, Theorem 6.7]) is equivalent to G being a complete multipartite graph. Moreover, by [1] (see also [2, Theorem 3.8]) $\lambda_1 = \overline{d}$ if and only if G is regular. Therefore, $\mathcal{E}(G) = 2\overline{d}$ if and only if G is a multipartite graph with equal size of parts. ■

As a consequence of Theorem 1, we state the next corollary which was proved in [5].

Corollary 1. *Let G be a connected graph with minimum degree $\delta(G)$. Then $\mathcal{E}(G) \geq 2\delta(G)$ and the equality holds if and only if G is a complete multipartite graph with equal size of parts.*

Now, we propose the following conjecture.

Conjecture. *For every graph with maximum degree $\Delta(G)$ whose adjacency matrix is non-singular, $\mathcal{E}(G) \geq \Delta(G) + \delta(G)$ and the equality holds if and only if G is a complete graph.*

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