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Note on the Girth of a Borderenergetic Graph*

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Abstract

The energy $\mathcal{E}(G)$ of a graph G is defined as the sum of the absolute values of the eigenvalues of its adjacency matrix. If a graph G of order n has the same energy as the complete graph K_n, i.e., if $\mathcal{E}(G) = 2(n-1)$, then G is said to be borderenergetic. In this note, we investigate the girth of a borderenergetic graph G in the case that G is a dense graph, and get the result that the girth is 3.

1 Introduction

All graphs appeared in this note are simple and undirected. Let G be a graph with order n and size m. The complete graph of order n is denoted by K_n. The degree of vertex v_i in a graph G is denoted by $d_i(G)$, $i = 1, 2, \cdots, n$. The minimum degree of graph G is denoted by $\delta(G)$. For terminology and notation not given here, we refer to [1, 2].

Let $A(G)$ be the adjacency matrix of G and set $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of the adjacency matrix $A(G)$. If $D(G)$ is the diagonal matrix of the vertex degrees of G, then $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are the Laplacian matrix and signless Laplacian matrix of G, respectively. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ and

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$q_1 \geq q_2 \geq \cdots \geq q_n$ be the eigenvalues of $L(G)$ and $Q(G)$, respectively. The energy of a graph G [12,13], denoted by $\mathcal{E}(G)$, is defined as

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i| .$$

For additional information on the graph energy and its applications in chemistry, we refer to [4,11,13,14,16].

Gong et al. in [10] proposed the concept of *borderenergetic* graphs, namely, graphs of order n satisfying $\mathcal{E}(G) = 2(n-1)$. The corresponding results on borderenergetic graphs can be seen in [5,15,17,18,20].

Similarly, the concept of *Laplacian borderenergetic*, i.e., *L-borderenergetic* graphs was proposed in [23], that is, a graph G of order n is L-borderenergetic if $LE(G) = LE(K_n)$, where $LE(G) = \sum_{i=1}^{n} |\mu_i - \overline{d}(G)|$ and $\overline{d}(G)$ is the average degree of G. More results on L-borderenergetic graphs can be referred to [6–9, 19, 21, 23, 24]. Analogously, Tao and Hou [22] extended this concept to signless Laplacian energy $QE(G)$ of a graph G. If a graph G of order n has the same signless Laplacian energy as the complete graph K_n, i.e., $QE(G) = QE(K_n) = 2(n-1)$, then it is called *Q-borderenergetic.*

Although the borderenergetic graphs of any order $n \geq 7$ were constructed in [10], bearing in mind that the smallest borderenergetic graph has an order 7, there are very few results on the properties of the structures of borderenergetic graphs. Depending on the searching by computer, the borderenergetic graphs with order $7 \leq n \leq 11$ were depicted in [10, 17, 20], which are all not bipartite. Then there exits at least one odd cycle contained in each one borderenergetic graph found ahead. Especially, by observing these borderenergetic graphs, we can find that the girths of them are coincidentally the same and equal to 3, which means that these graphs are not bipartite. In [6], the authors proved that a borderenergetic graph G is not bipartite when G is a sparse graph.

Theorem 1. *[6] Let G be a borderenergetic graph of order n and size m. If $m < \frac{2(n-1)^2}{n}$, then G is not bipartite.*

In this note, we will consider borderenergetic graphs in the case that they are dense graphs, and prove that they are not bipartite and their girths are all equal to 3.

2 Main results

An asymptotically tight lower bound on the size m of a borderenergetic graph of order n is presented as follows.

Lemma 2. *[5] Let G be a borderenergetic graph of order n and size m. Then*

$$m \geq \left\lceil \frac{\left[2(n-1) - \sqrt{\frac{1}{n}\sum_{i=1}^{n} d_i^2}\right]^2}{2(n-1)} + \frac{\sum_{i=1}^{n} d_i^2}{2n} \right\rceil . \tag{1}$$

If G is $(n-3)$-regular, then the bound in (1) is asymptotically tight.

The following result(Lemma 3) is famous in extremal graph theory., originally due to Turán [25] and can be easilly found in [3].

Lemma 3. *A graph G with n vertices and more than $\left[\frac{n^2}{4}\right]$ edges contains at least one triangle. The only graphs without triangles having n vertices and $\left[\frac{n^2}{4}\right]$ edges are $K_{l,l}$ for $n = 2l$ and $K_{l+1,l}$ for $n = 2l+1$.*

Indeed, Lemma 3 shows that the girth of graph G is 3 if it satisfies $m > \left[\frac{n^2}{4}\right]$, which means that the graph G is rather dense. By using above lemmas, we now study the girth of borderenergetic graphs and obtain the following main result.

Theorem 4. *Let G be a 2-connected noncomplete borderenergetic graph. If it satisfies*

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n} d_i^2(G) \ &\geq \ \frac{1}{2n^2}\Big(4\sqrt{2}\sqrt{n^6 - 11n^5 + 44n^4 - 84n^3 + 83n^2 - 41n + 8} \\ &+ \ n^4 - 9n^3 + 33n^2 - 41n + 16\Big) , \end{aligned} \tag{2}$$

then the girth of the borderenergetic graph G is 3.

Proof. Observing the lower bound in Lemma 2, suppose $x = \frac{1}{n}\sum_{i=1}^{n} d_i^2$ and define a function $f(x)$ on x below.

$$f(x) = \frac{[2(n-1) - \sqrt{x}]^2}{2(n-1)} + \frac{x}{2}.$$

Then the derivative of the function $f(x)$ on x is

$$\frac{\partial f(x)}{\partial x} = \frac{n(\sqrt{x} - 2) + 2}{2(n-1)\sqrt{x}}.$$

Since G is a 2-connected graph, we have $\delta(G) \geq 2$. Then it arrives at

$$x = \frac{1}{n}\sum_{i=1}^{n} d_i^2(G) \geq 4 \ \ and \ \sqrt{x} \geq 2.$$

In this case, we have $\frac{\partial f(x)}{\partial x} \geq 0$, which shows that $f(x)$ is increasing for $x \geq 4$.

Let

$$f(x) = \frac{n^2+1}{4}. \tag{3}$$

By direct computation, we obtain an approximate solution x_0 for the equation (3). That is,

$$\begin{aligned} x_0 &= \frac{1}{2n^2}\Big(4\sqrt{2}\sqrt{n^6 - 11n^5 + 44n^4 - 84n^3 + 83n^2 - 41n + 8} \\ &+ n^4 - 9n^3 + 33n^2 - 41n + 16\Big). \end{aligned}$$

Thus, we have

$$f(x_0) \approx \frac{n^2+1}{4}. \tag{4}$$

Since $f(x)$ is increasing for $x \geq 4$, by $x \geq x_0$, i.e., the inequality (2), we obtain

$$f(x) \geq f(x_0).$$

Thus, by (1) and (4), we have

$$m \geq \lceil f(x) \rceil \geq f(x) \geq f(x_0) \approx \frac{n^2+1}{4} > \frac{n^2}{4} \geq \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Hence, from Lemma 3 we know that a borderenergetic graph has at least one triangle, i.e., its girth is 3. ■

A natural corollary from Theorem 4 is that such a 2-connected noncomplete borderenergetic graph satisfying the condition (2) is not bipartite. In fact, the rationality of the conditions in Theorem 4 can partly be checked from borderenergetic graphs with order $7 \leq n \leq 9$ depicted in [10], which are all 2-connected. Assume

$$x = \frac{1}{n}\sum_{i=1}^{n} d_i^2(G),$$

and

$$x_0 = \frac{1}{2n^2}(4\sqrt{2}\sqrt{n^6 - 11n^5 + 44n^4 - 84n^3 + 83n^2 - 41n + 8}$$

$$+n^4 - 9n^3 + 33n^2 - 41n + 16).$$

Let G_n^i be the i-th borderenergetic graph of order n, where $7 \le n \le 9$. The following Table 1 presents the values of x and x_0 for these borderenergetic graphs.

Graph	G_7^1	G_8^1	G_8^2	G_8^3	G_8^4	G_8^5	G_8^6	G_9^1
x	24.57	24.25	29	24.25	25	20.5	40	26.22
x_0	13.41	19.32	19.32	19.32	19.32	19.32	19.32	26.20
Graph	G_9^2	G_9^3	G_9^4	G_9^5	G_9^6	G_9^7	G_9^8	G_9^9
x	26.22	32.89	35.11	16	16	32.89	31.56	35.11
x_0	26.20	26.20	26.20	26.20	26.20	26.20	26.20	26.20
Graph	G_9^{10}	G_9^{11}	G_9^{12}	G_9^{13}	G_9^{14}	G_9^{15}	G_9^{16}	G_9^{17}
x	43.56	32.89	31.56	36.22	45.78	39.56	49.56	57.56
x_0	26.20	26.20	26.20	26.20	26.20	26.20	26.20	26.20

Table 1. The values of x and x_0 for the borderenergetic graph G_n^i, where $7 \le n \le 9$, $1 \le i \le 17$.

Furthermore, we consider the girth of a borderenergetic graph G of order n when the order n of G is large enough. Using Theorem 4, we derive a result below.

Theorem 5. *Let G be a 2-connected noncomplete borderenergetic graph of order n. If the order n of G is large enough and G satisfies*

$$d_i^2(G) \ge O(2\sqrt{2}n), 1 \le i \le n, \tag{5}$$

then the girth of the borderenergetic graph G is 3.

Proof. Since n is large enough, the inequality (2) for the graph G can be derived from $d_i^2(G) \ge O(2\sqrt{2}n), 1 \le i \le n$. From Theorem 4, we can see that the girth of the borderenergetic graph G is 3. ■

Obviously, the borderenergetic graph G appeared in Theorem 5 is not bipartite. By the inequality 5, we get

$$\begin{aligned} d_i(G) &\ge O(\sqrt{2\sqrt{2}n}), \\ 2m = \sum_{i=1}^{n} d_i(G) &\ge O(\sqrt{2\sqrt{2}}n^{3/2}), \\ m &\ge O(\frac{\sqrt{2\sqrt{2}}}{2}n^{3/2}). \end{aligned}$$

And what's more important, comparing with Lemma 3, in the case that G is borderenergetic and its order is large enough, G only satisfies $m \ge O(\frac{\sqrt{2\sqrt{2}}}{2}n^{3/2})$ and then it contains at least one triangle.

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