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Energy, Matching Number and Rank of Graphs

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Abstract

The energy $\mathcal{E}(G)$ of a graph G is the sum of the absolute values of all eigenvalues of the adjacency matrix $A(G)$ of G. For graphs with vertex-disjoint cycles, we prove that $\mathcal{E}(G) \geq 2\mu(G) + \frac{3}{5}c_1(G)$, where $\mu(G)$ is the matching number of G and $c_1(G)$ denotes the number of odd cycles in G. This result improves the bound $2\mu(G) + \frac{\sqrt{5}}{5}c_1(G)$ obtained by Wong et al. [Lower bounds of graph energy in terms of matching number. Linear Algebra Appl. 549 (2018) 276-286]. Moreover, we give a new lower bound $\sqrt{41}$ for the energy of connected graphs, which improves the result obtained by Zhou et al.

1 Introduction

Graphs considered in this paper are finite and simple. Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The *order* and the *size* of G are denoted by $\nu(G)$ and $\varepsilon(G)$, respectively. Let $\Delta(G)$ denote the *maximum degree* of G. An edge set M of G is called a *matching* if any two edges in M have no common vertices, moreover, if each vertex of G is incident with exactly one edge of M, then M is called a *perfect matching* of G. The *matching number* of a graph G, denoted by $\mu(G)$, is the number of edges in a maximum matching. For a non-empty subset $V' \subseteq V(G)$, let $G[V']$

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denote the *induced subgraph* of G, which is the subgraph whose vertex set is V' and whose edge set is the set of those edges of G that have both ends in V'. For an induced subgraph $H = (V(H), E(H))$ of G, let $G - V(H)$ denote the induced subgraph whose vertex set is $V(G) \setminus V(H)$. For the induced subgraphs H and $G - V(H)$, the *edge cut* F of G is a subset of $E(G)$ such that the edge of which has one end in $V(H)$ and the other in $V(G) \setminus V(H)$. Let $G - F$ denote the spanning subgraph of G with edge set $E(G) \setminus F$. Let $H_1 \oplus H_2$ denote the *symmetric difference* of the subgraphs H_1 and H_2 of G. Obviously, for the induced subgraphs H and $G - V(H)$ of G, $G - F = H \oplus (G - V(H))$. For a positive integer n, let P_n (resp. C_n; K_n; $K_{p,q}$; $K_{t,l,s}$) denote the path (resp. the cycle; the complete graph; a complete bipartite graph; a complete tripartite graph) of order n, where $p + q = t + l + s = n$. The *rank* of a graph G, denoted by $r(G)$, is that of its adjacency matrix $A(G)$.

The *graph energy* $\mathcal{E}(G)$ of G, proposed by Gutman [11], is the sum of the absolute values of all the eigenvalues of the adjacency matrix $A(G)$. In the last two decades, some lower bounds of graph energy have been obtained for graphs with given parameters, such as the order, size, rank, chromatic number, matching number and so on. Caporossi et al. [5] proved that $\mathcal{E}(G) \geq 2\sqrt{m}$ for all graphs G of size m. Gutman [12] characterized some n-vertex graphs G with $\mathcal{E}(G) \geq n$, and Zhou et al. [20] proved that $\mathcal{E}(G) \geq n$ if G is a Hamilton graph of order n. For all the graphs whose rank is r, Akbari et al. [1] proved that $\mathcal{E}(G) \geq r$. For quadrangle-free graphs G with order n and size m, Zhou [19] proved that $\mathcal{E}(G) \geq \frac{2\sqrt{2\delta\Delta}}{2(\delta+\Delta)-1}\sqrt{2mn}$, where δ and Δ denote the minimum degree and maximum degree of G, respectively. For a graph G with vertex cover number τ, Wang and Ma [17] proved that $\mathcal{E}(G) \geq 2\tau(G) - 2c(G)$, where $c(G)$ is the number of odd cycles in G. Das and Gutman [9] gave a lower bound of graph energy in terms of the order, the size and the determinant of the adjacency matrix. For more detailed results on graph energy, we refer the reader to the book [14].

Wong et al. [16] investigated the lower bound of graph energy by matching number and proved that $\mathcal{E}(G) \geq 2\mu(G)$, where $\mu(G)$ is the matching number of G. Moreover, Wong et al. [16] proved that for graphs with pairwise vertex-disjoint cycles, $\mathcal{E}(G) \geq 2\mu(G) + \frac{\sqrt{5}}{5}c_1(G)$, where $c_1(G)$ denotes the number of odd cycles of G. Ashraf [3] pointed out that the result cannot be improved to $\mathcal{E}(G) \geq 2\mu(G) + c_1(G)$ by giving a counterexample, moreover, obtained that $\mathcal{E}(G) \geq 2\mu(G) + c_0(G)$ for triangle-free graphs, where $c_0(G)$

denotes the number of odd cycles of G with length at least 5. In Section 2 of this paper, we prove that if G is a graph whose cycles (if exist) are pairwise vertex-disjoint, then $\mathcal{E}(G) \geq 2\mu(G) + \frac{3}{5}c_1(G)$, which improve the result of [16].

For a connected graph G, Wong et al. [16] proved that $\mathcal{E}(G) \geq 2\sqrt{5}$ apart from K_1, K_3 and $K_{p,q}$ with $1 \leq pq \leq 4$. Later, Zhou et al. [20] improved the bound from $2\sqrt{5}$ to 6. In this paper, we give a new lower bound $\sqrt{41}$ for the energy of connected graphs in Section 3, which improves the results obtained in [16, 20].

2 Energy and matching number

We now list some preliminary results which will be used in this section.

Lemma 1. *[10] If F is an edge cut of a simple graph G, then $\mathcal{E}(G-F) \leq \mathcal{E}(G)$.*

Lemma 2. *[10] Let H be an induced subgraph of a simple graph G. Then $\mathcal{E}(H) \leq \mathcal{E}(G)$ and equality holds if and only if $E(H) = E(G)$.*

Lemma 3. *[3] Let G be a graph whose cycles are vertex-disjoint. If G is not a cycle and any cut edge of G lies on every maximum matching of G, then G has a perfect matching.*

Lemma 4. *[3] Let G be a graph whose cycles have odd lengths and are vertex-disjoint. If we remove one edge from each cycle of G to obtain a tree T, then $\mathcal{E}(G) \geq \mathcal{E}(T)$.*

Let Ψ_n denote the class of n-vertex trees which have a perfect matching and whose vertex degrees do not exceed 3. Gutman [13] conjectured that for any tree $T \in \Psi_n$, $\mathcal{E}(T) \geq \mathcal{E}(\widehat{P}_{n/2})$, where $\widehat{P}_{n/2}$ is obtained by adding a pendant edge to each vertex of the path $P_{n/2}$. Zhang and Li confirmed this conjecture in [18]. We now give a new lower bound for the energy of these trees.

Lemma 5. *For any tree $T \in \Psi_n$, if $n \notin \{4, 6, 10, 12, 18, 24\}$, then $\mathcal{E}(T) \geq n + \frac{3}{5}\lfloor\frac{n}{3}\rfloor$ and equality holds if and only if $T = P_2$.*

Proof. Since T has a perfect matching, then the order n of T is even. If $n = 2$, then $T = P_2$ and thus $\mathcal{E}(T) = \mathcal{E}(P_2) = 2 = 2 + \frac{3}{5}\lfloor\frac{2}{3}\rfloor$. Assume now that $n \geq 4$. Note that $\mathcal{E}(T) \geq \mathcal{E}(\widehat{P}_{n/2})$ for any tree $T \in \Psi_n$. Ashraf [2] proved that

$$\mathcal{E}(\widehat{P}_{n/2}) = \sum_{i=1}^{n/2} 2\sqrt{1 + \cos^2\left(\frac{\pi i}{\frac{n}{2}+1}\right)} > 1.21n - 3.23.$$

For $n \geq 323$, since $1.21n - 3.23 \geq 1.2n \geq n + \frac{3}{5}\lfloor\frac{n}{3}\rfloor$, thus $\mathcal{E}(T) > n + \frac{3}{5}\lfloor\frac{n}{3}\rfloor$. For $n \leq 322$, with the help of Matlab, we obtain that if $n \notin \{4, 6, 10, 12, 18, 24\}$,

$$\mathcal{E}(T) \geq \mathcal{E}(\widehat{P}_{n/2}) = \sum_{i=1}^{n/2} 2\sqrt{1 + \cos^2\left(\frac{\pi i}{\frac{n}{2}+1}\right)} > n + \frac{3}{5}\left\lfloor\frac{n}{3}\right\rfloor.$$

Moreover, we have $\mathcal{E}(T) \geq \mathcal{E}(\widehat{P}_{n/2}) > n + \frac{3}{5}\lfloor\frac{n}{3}\rfloor - \frac{3}{5}$ for $n = 4, 6, 10, 12, 18, 24$. Thus for any tree $T \in \Psi_n$, if $n \notin \{4, 6, 10, 12, 18, 24\}$, $\mathcal{E}(T) \geq n + \frac{3}{5}\lfloor\frac{n}{3}\rfloor$ and equality holds if and only if $T = P_2$. ■

Lemma 6. *[3] For any odd cycle C of length n, $\mathcal{E}(C) \geq n + 1$.*

Lemma 7. *Let G be an n-vertex connected graph with a perfect matching and $\Delta(G) \leq 3$. If all cycles of G are of odd length and vertex-disjoint, then $\mathcal{E}(G) \geq n + \frac{3}{5}c_1(G)$ and equality holds if and only if $G = P_2$, where $c_1(G)$ denotes the number of odd cycles in G.*

Proof. We can obtain a spanning tree T with a perfect matching by removing one edge from each cycle of G. It is obvious that $\Delta(T) \leq 3$. Since all cycles of G are of odd length and pairwise vertex-disjoint, thus $c_1(G) \leq \lfloor\frac{n}{3}\rfloor$. If $n = 2$, then $G = P_2$ and thus $\mathcal{E}(G) = \mathcal{E}(P_2) = 2 = 2 + \frac{3}{5}c_1(P_2) = 2 + \frac{3}{5}c_1(G)$. Assume now that $n \geq 4$. If $n \notin \{4, 6, 10, 12, 18, 24\}$, together with Lemma 4 and Lemma 5, we have

$$\mathcal{E}(G) \geq \mathcal{E}(T) > n + \frac{3}{5}\left\lfloor\frac{n}{3}\right\rfloor \geq n + \frac{3}{5}c_1(G).$$

For $n = 4, 6, 10, 12, 18, 24$, by Lemma 5, $\mathcal{E}(T) > n + \frac{3}{5}\lfloor\frac{n}{3}\rfloor - \frac{3}{5}$. By Lemma 4, $\mathcal{E}(G) \geq \mathcal{E}(T)$. If $c_1(G) < \lfloor\frac{n}{3}\rfloor$, then

$$\mathcal{E}(G) \geq \mathcal{E}(T) > n + \frac{3}{5}\left\lfloor\frac{n}{3}\right\rfloor - \frac{3}{5} \geq n + \frac{3}{5}c_1(G).$$

Assume now that $c_1(G) = \lfloor\frac{n}{3}\rfloor$. If $n = 4$, then $c_1(G) = 1$ and G is the graph obtained from a C_3 by attaching a pendant vertex to one of its vertices. It is easy to calculate that $\mathcal{E}(G) \approx 4.96 > 4 + \frac{3}{5}c_1(G)$. If $n = 10$, then $c_1(G) = 3$ and we can obtain three C_3 after removing three cut edges of G. By Lemma 1, $\mathcal{E}(G) \geq 3\mathcal{E}(C_3) = 12 > 10 + \frac{3}{5}c_1(G)$. If $n = 6, 12, 18, 24$, then $c_1(G) = \frac{n}{3}$ and removing $\frac{n-3}{3}$ cut edges of G yields $\frac{n}{3}$ C_3. By Lemma 1, $\mathcal{E}(G) \geq \frac{n}{3}\mathcal{E}(C_3) = \frac{4}{3}n > n + \frac{3}{5}c_1(G)$. Thus, $\mathcal{E}(G) \geq n + \frac{3}{5}c_1(G)$ and equality holds if and only if $G = P_2$. ■

We now give a new lower bound for the energy of graphs whose cycles are pairwise vertex-disjoint in terms of matching number and the number of its odd cycles.

Theorem 8. *Let G be a graph and let $c_1(G)$ denote the number of the odd cycles of G. If all cycles (if exist) of G are pairwise vertex-disjoint, then $\mathcal{E}(G) \geq 2\mu(G) + \frac{3}{5}c_1(G)$ and equality holds if and only if G is the disjoint union of some copies of K_2, some copies of C_4 and some isolated vertices.*

Proof. If $c_1(G) = 0$, the result holds by Theorem 1.2 in [16]. We now prove that $\mathcal{E}(G) > 2\mu(G) + \frac{3}{5}c_1(G)$ for $c_1(G) \geq 1$. Let M be a maximum matching of G. Let $k := c_1(G)$. Then $k \geq 1$. Let $C_1, C_2, \ldots, C_k$ denote the odd cycles of G and let $C = C_1 \cup C_2 \cup \cdots \cup C_k$. Consider the subgraph H of G induced by the edges of M and C. Since M is a maximum matching of G, then $\mu(H) = \mu(G)$ and $\Delta(H) \leq 3$. Let H_1 be the resulted graph by removing any cut edges from H such that the removal does not decrease the matching number. Then $\mu(H_1) = \mu(H) = \mu(G)$ and $c_1(H_1) = c_1(H) = c_1(G)$. Moreover, each connected component of H_1 is either a complete graph K_2, an odd cycle, or a graph satisfying Lemma 3. Let $H_1 = H_{11} \cup H_{12} \cup H_{13}$, where H_{11} is the union of the components which are K_2, H_{12} is the union of the odd cycles of H_1, and H_{13} is the union of components satisfying Lemma 3. Thus, $\mu(H_1) = \mu(H_{11}) + \mu(H_{12}) + \mu(H_{13})$ and $c_1(H_1) = c_1(H_{12}) + c_1(H_{13})$. It is obvious that $\mathcal{E}(H_{11}) = 2\mu(H_{11})$. Note that H_{12} is the union of odd cycles of H_1. Assume that $H_{12} = C_{i_1} \cup C_{i_2} \cup \cdots \cup C_{i_t}$, where $t = c_1(H_{12})$. By Lemma 6,

$$\begin{aligned}
\mathcal{E}(H_{12}) &= \mathcal{E}(C_{i_1}) + \mathcal{E}(C_{i_2}) + \cdots + \mathcal{E}(C_{i_t}) \\
&\geq (\nu(C_{i_1}) + 1) + (\nu(C_{i_2}) + 1) + \cdots + (\nu(C_{i_t}) + 1) \\
&= (\nu(C_{i_1}) + \nu(C_{i_2}) + \cdots + \nu(H_{i_t})) + c_1(H_{12}) \\
&= 2\mu(H_{12}) + 2c_1(H_{12}) \\
&\geq 2\mu(H_{12}) + \frac{3}{5}c_1(H_{12}). \qquad (2.1)
\end{aligned}$$

Obviously, since all cycles of G are vertex-disjoint, the components of H_{13} also satisfy Lemma 7, then $\mathcal{E}(H_{13}) \geq \nu(H_{13}) + \frac{3}{5}c_1(H_{13}) = 2\mu(H_{13}) + \frac{3}{5}c_1(H_{13})$. Since H_1 is obtained from G by removing some cuts, we have

$$\begin{aligned}
\mathcal{E}(G) &\geq \mathcal{E}(H_1) \\
&= \mathcal{E}(H_{11}) + \mathcal{E}(H_{12}) + \mathcal{E}(H_{13}) \\
&\geq 2\mu(H_{11}) + [2\mu(H_{12}) + \frac{3}{5}c_1(H_{12})] + [2\mu(H_{13}) + \frac{3}{5}c_1(H_{13})] \\
&= 2[\mu(H_{11}) + \mu(H_{12}) + \mu(H_{12})] + \frac{3}{5}[c_1(H_{12}) + c_1(H_{13})] \\
&= 2\mu(G) + \frac{3}{5}c_1(G).
\end{aligned}$$

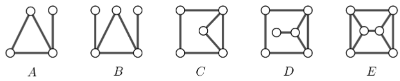

Figure 1. Connected graphs with rank 4.

If the equality holds, then by (2.1), $c_1(H_{12}) = 0$, and moreover by Lemma 7, $c_1(H_{13}) = 0$, a contradiction to $c_1(G) = c_1(H_1) = c_1(H_{12}) + c_1(H_{13}) \geq 1$. Thus, if $c_1(G) \geq 1$, then $\mathcal{E}(G) > 2\mu(G) + \frac{3}{5}c_1(G)$. Therefore, $\mathcal{E}(G) \geq 2\mu(G) + \frac{3}{5}c_1(G)$ and equality holds if and only if $c_1(G) = 0$, that is, G is the disjoint union of some copies of K_2, some copies of C_4 and some isolated vertices which is obtained in Theorem 1.2 in [16]. ■

3 Energy and rank

Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, and let $\mathbf{m} = (m_1, m_2, \ldots, m_n)$ be a vector of positive integers. Denote by $G \circ \mathbf{m}$ the graph obtained from G by replacing each vertex v_i of G with m_i isolated vertices $v_i^1, v_i^2, \ldots, v_i^{m_i}$ and joining v_i^s with v_j^t if and only if v_i and v_j are adjacent in G. $\{v_i^1, v_i^2, \ldots, v_i^{m_i}\}$ is the vertices of $G \circ \mathbf{m}$ corresponding to the vertex v_i of G, and the graph $G \circ \mathbf{m}$ is said to be obtained from G by *multiplication of vertices*. Graphs with small rank $r(r = 2, 3, 4, 5)$ have been characterized and completely determined as follows.

Lemma 9. *[4] A connected graph G is of rank 2 if and only if it is a complete bipartite graph, and G is of rank 3 if and only if G is a complete tripartite graph.*

Lemma 10. *[6] Let G be a connected graph. Then $r(G) = 4$ if and only if G can be obtained from K_4, P_4, P_5 or one of the graphs shown in Figure 1 by multiplication of vertices.*

Chang et al. [7] determined all connected graphs G with rank 5. By Theorem 3 in [7], we can get the following result.

Lemma 11. *Let G be a connected graph with rank 5. Then one of the graphs shown in Figure 2 is an induced subgraph of G.*

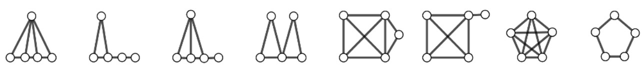

Figure 2. Induced subgraphs of a connected graph with rank 5.

Together with Lemma 2 and Lemma 11, we can give a lower bound on the energy of all connected graphs G with rank 5 as below.

Theorem 12. *Let G be a connected graph with rank 5. Then $\mathcal{E}(G) > \sqrt{41}$.*

Proof. Let H denote any one of the eight graphs shown in Figure 2. It is easy to verify that $\mathcal{E}(H) > \sqrt{41}$. By Lemma 11, G has an induced subgraph as given in Figure 2. Thus, by Lemma 2, we have $\mathcal{E}(G) \geq \mathcal{E}(H) > \sqrt{41}$. ■

Some lower bounds on the graph energy with respect to the rank of a graph have obtained as follows.

Lemma 13. *[1] If G is a graph of rank r, then $\mathcal{E}(G) \geq r$. Further, if G is a connected bipartite graph of rank r, then $\mathcal{E}(G) \geq \sqrt{(r+1)^2 - 5}$.*

Lemma 14. *[16] Let G be a connected graph of rank r. If G has at least one odd cycle, then $\mathcal{E}(G) \geq \sqrt{r^2 + r - 1}$. Moreover, if G is not of full rank, then $\mathcal{E}(G) \geq r + \frac{1}{2}$.*

Combined Lemma 13 with Lemma 14, we investigate and give a lower bound of the energy for graphs which are of rank 6.

Theorem 15. *Let G be a connected graph of rank 6. Then $\mathcal{E}(G) > \sqrt{41}$.*

Proof. If G is a connected bipartite graph of rank 6, by Lemma 13, $\mathcal{E}(G) \geq \sqrt{(6+1)^2 - 5}$ $= \sqrt{44} > \sqrt{41}$. Assume now that G has at least one odd cycle. If G is not of full rank, by Lemma 14, $\mathcal{E}(G) \geq 6 + \frac{1}{2} = \frac{13}{2} > \sqrt{41}$. If G is of full rank, by Lemma 14, $\mathcal{E}(G) \geq \sqrt{6^2 + 6 - 1} = \sqrt{41}$, moreover, according to the results obtained in [8] about the connected graphs on six vertices, it is easy to check that $\mathcal{E}(G) > \sqrt{41}$. Thus, if G is a connected graph of rank 6, then $\mathcal{E}(G) > \sqrt{41}$. ■

Zhou et al. [20] proved that $\mathcal{E}(G) > 6$ for any connected graph G with $r(G) > 4$. We now improve this bound from 6 to $\sqrt{41}$.

Theorem 16. *If G is a connected graph with $r(G) > 4$, then $\mathcal{E}(G) > \sqrt{41}$.*

Proof. If $r(G) \geq 7$, it follows from Lemma 13 that $\mathcal{E}(G) \geq 7 > \sqrt{41}$. Thus, by Theorem 12 and Theorem 15, $\mathcal{E}(G) > \sqrt{41}$ for any connected graph G with $r(G) > 4$. ■

By Theorem 16, if G is a connected graph with $r(G) > 4$, then $\mathcal{E}(G) > \sqrt{41}$. Thus, the graph G with $\mathcal{E}(G) \leq \sqrt{41}$ must have rank 2, 3, or 4. In order to characterize the graph G with $\mathcal{E}(G) \leq \sqrt{41}$, we then give the following results.

Lemma 17. *[15] For a graph G of size m, $\mathcal{E}(G) \geq 2\sqrt{m}$ and equality holds if and only if G consists of a complete bipartite graph $K_{p,q}$ such that $pq = m$ and arbitrarily many isolated vertices.*

By Lemma 17, the lower bound of the energy for connected graphs can be characterized by its order as below.

Corollary 18. *Let G be a connected graph of order n. Then $\mathcal{E}(G) \geq 2\sqrt{n-1}$.*

Lemma 19. *Let G be a connected graph with $\mathcal{E}(G) \leq \sqrt{41}$. Then*

(i) *$\nu(G) \leq 7$ if G is not a star;*

(ii) *$\nu(G) \leq 11$ if G is a star;*

(iii) *$\varepsilon(G) \leq 10$ and equality holds if and only if $G = K_{p,q}$ with $pq = 10$.*

Proof. (i) In order to prove (i), by Lemma 2, we just need to prove that $\mathcal{E}(G) > \sqrt{41}$ for every connected graph of order 8 which is not a star. Let G be a connected graph of order 8 which is not a star. Then $\mu(G) \geq 2$. Let M be a maximum matching of G, and let $e := u_1u_2$ be an edge of M. Then for the induced subgraph $G - u_1 - u_2$, it is obvious that $\mu(G - u_1 - u_2) \geq 1$. Let H be a component of $G - u_1 - u_2$ which contains at least one edge, and let $K := G - V(H)$. Since G is a connected graph, then K is connected and $G - F = H \oplus K$, where F is the edge cut with respect to H and K. Moreover, since H is a subgraph with at least one edge and $e \in K$, $\nu(H) \geq 2$ and $\nu(K) \geq 2$. Since $\nu(G) = \nu(H) + \nu(K) = 8$, without loss of generality, assume that $\nu(H) = t$, where $t = 2, 3, 4$. Together with Lemma 1 and Corollary 18, we have

$$\mathcal{E}(G) \geq \mathcal{E}(G - F) = \mathcal{E}(H) + \mathcal{E}(K) \geq 2\sqrt{t-1} + 2\sqrt{7-t} \geq 2 + 2\sqrt{5} > \sqrt{41},$$

a contradiction. Hence, $\nu(G) \leq 7$ if G is not a star and with $\mathcal{E}(G) \leq \sqrt{41}$.

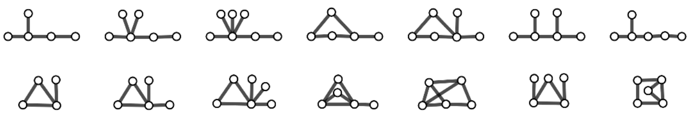

Figure 3. Graphs with energy not exceeding $\sqrt{41}$.

(ii) If G is a star of order $\nu(G)$, then $\mathcal{E}(G) = 2\sqrt{\nu(G) - 1} \leq \sqrt{41}$. Thus, $\nu(G) \leq 11$.

(iii) By Lemma 17, we have $2\sqrt{\varepsilon(G)} \leq \mathcal{E}(G) \leq \sqrt{41}$, which implies that $\varepsilon(G) \leq 10$ and equality holds if and only if $G = K_{p,q}$ with $pq = 10$. ■

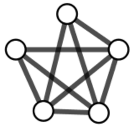

Figure 4. A graph K_4^1 obtained from K_4 by multiplication of vertices.

Theorem 20. *Let G be a connected graph. Then $\mathcal{E}(G) > \sqrt{41}$ if and only if G is not in $\{K_3, K_4, P_4, P_5, K_{1,1,2}, K_{1,1,3}, K_{1,1,4}, K_{p,q} : 1 \leq pq \leq 10\}$, and not a graph shown in Figure 3.*

Proof. Let G be a connected graph with $\mathcal{E}(G) \leq \sqrt{41}$. By Lemma 16, $r(G) \leq 4$. We now consider three cases as below.

Case 1. $r(G) = 2$.

In this case, by Lemma 9, $G = K_{p,q}$, where p, q are positive integers. Since $\mathcal{E}(K_{p,q}) = 2\sqrt{pq} \leq \sqrt{41}$, then $pq \leq 10$.

Case 2. $r(G) = 3$.

It follows from Lemma 9 that $G = K_{t,l,s}$ with $1 \leq t \leq l \leq s$. By Lemma 19, $\varepsilon(K_{t,l,s}) \leq 9$, and so $(t, l, s) \in \{(1,1,1), (1,1,2), (1,1,3), (1,1,4), (1,2,2)\}$. Note that $\mathcal{E}(K_{1,2,2}) \approx 6.4722 > \sqrt{41}$. Then, G is $K_{1,1,1}$, $K_{1,1,2}$, $K_{1,1,3}$ or $K_{1,1,4}$.

Case 3. $r(G) = 4$.

By Lemma 10, G is obtained from G' by multiplication of vertices, where G' is K_4, P_4, P_5 or one of the graphs shown in Figure 1. Moreover, by Lemma 19, $\nu(G) \leq 7$ and $\varepsilon(G) \leq 9$. By a direct calculation of the graph energy for graphs in Figure 1, $\mathcal{E}(D) \approx 7.1232 > \sqrt{41}$ and $\mathcal{E}(E) = 8 > \sqrt{41}$. By Lemma 3, G' can not be of type D or E. Thus, we now consider the following five subcases.

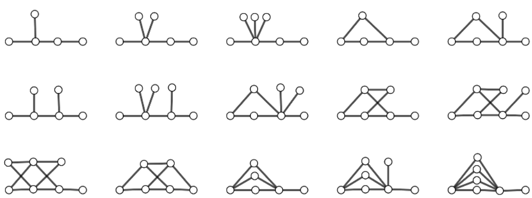

Figure 5. Graphs obtained from P_4 by multiplication of vertices.

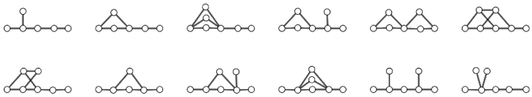

Figure 6. Graphs obtained from P_5 by multiplication of vertices.

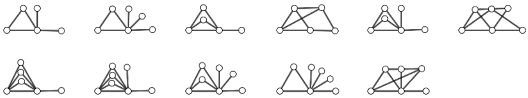

Figure 7. Graphs obtained from A by multiplication of vertices.

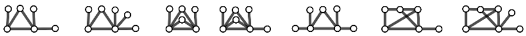

Figure 8. Graphs obtained from B by multiplication of vertices.

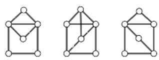

Figure 9. Graphs obtained from C by multiplication of vertices.

If $G' = K_4$, then G is possibly the graph K_4 or K_4^1 which is shown in Figure 4. Since $\mathcal{E}(K_4^1) \approx 7.2916 > \sqrt{41}$, then $G = K_4$.

If $G' = P_4$, G is P_4 or one of the following graphs shown in Figure 5. It is easy to verify that only graphs on the first row and the first graph on the second row have energy not exceeding $\sqrt{41}$. G is P_4 or possibly one of the first six graphs shown in Figure 5.

If $G' = P_5$, then G is P_5 or one of the following graphs shown in Figure 6. Denote by P_5^1 the first graph shown in Figure 6. It is easy to calculate that the graphs shown in Figure 6 other than P_5^1 all have an energy exceeding $\sqrt{41}$. Then if G is obtained from P_5 by multiplication of vertices, then G is either P_5 or P_5^1.

If G' is the graph A shown in Figure 1, then G is possibly the graph A or one of the following graphs shown in Figure 7. One can easily check that only the first four graphs in Figure 7 have an energy not exceeding $\sqrt{41}$. Hence, if G' is the graph A, then G is either the graph A or one of the first four graphs shown in Figure 7.

If G' is the graph B shown in Figure 1, then G is possibly the graph B or one of the following graphs shown in Figure 8. One can easily check that all the graphs in Figure 8 have an energy exceeding $\sqrt{41}$. Hence, if G' is the graph B, then $G = B$.

If G' is the graph C shown in Figure 1, then G is possibly C or one of the graphs shown in Figure 9. It is easy to check that the energy of the graphs shown in Figure 9 are all exceed $\sqrt{41}$. Thus, $G = C$.

Therefore, if G is a connected graph, then $\mathcal{E}(G) > \sqrt{41}$ if and only if G is not in $\{K_3, K_4, P_4, P_5, K_{1,1,2}, K_{1,1,3}, K_{1,1,4}, K_{p,q} : 1 \leq pq \leq 10\}$, and not a graph shown in Figure 3. ■

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