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# On the Difference Between Wiener index and Graovac–Pisanski Index

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**Abstract**

Let $G$ be a connected graph. The Wiener index of $G$ is the sum of all distances in $G$, that is, $W(G) = \sum_{u,v\in V(G)} \mathrm{dist}(u,v)$. On the other hand, the Graovac-Pisanski index of $G$ is $\mathrm{GP}(G) = \frac{|V(G)|}{2|\mathrm{Aut}(G)|} \sum_{u\in V(G)} \sum_{\alpha\in \mathrm{Aut}(G)} \mathrm{dist}(u,\alpha(u))$, where $\mathrm{Aut}(G)$ is the group of automorphisms of $G$. In this paper we study the difference $\Delta_W(G) = W(G) - \mathrm{GP}(G)$. We show that this difference is nonnegative for trees, but there are graphs $G$ for which $\Delta_W(G)$ is negative. We also find infinitely many graphs $G$ which are not vertex-transitive and yet $\Delta_W(G) = 0$. For trees we completely determine the set of values of $\Delta_W(G)$.

## 1 Introduction

All graphs considered in this paper are connected. Let $G$ be a graph. We denote the vertex and edge sets of $G$ by $V(G)$ and $E(G)$, respectively. For two vertices $u$ and $v$, by $\mathrm{dist}(u,v)$ we denote the distance between $u$ and $v$. Wiener index of a graph, $W(G)$, was introduced by Wiener in [24]. It is the sum of distances between all (unordered) pairs of vertices in a graph, i.e.,

$$W(G) = \sum_{u,v \in V(G)} \mathrm{dist}(u,v).$$

A great deal of knowledge on this index is accumulated in several survey papers [7, 8, 13, 14, 25].

Let $G$ be a graph and let $U \subseteq V(G)$. The *Wiener index of* $U$ in $G$ is defined as the sum of distances between all vertex pairs in $U$

$$W_G(U) = \sum_{u,v \in U} \mathrm{dist}(u,v).$$

Obviously, $W_G(V(G)) = W(G)$. Let $v \in V(G)$. By $w_U(v)$ we denote the sum of distances from $v$ to all vertices of $U$. Hence,

$$w_U(v) = \sum_{u \in U} \mathrm{dist}(u,v).$$

If $U = V(G)$, we write $w_G(v)$ instead of $w_{V(G)}(v)$. Using this notation one can rewrite the Wiener index as follows

$$W(G) = \tfrac{1}{2} \sum_{v \in V(G)} w_G(v).$$

Graovac-Pisanski index, $\mathrm{GP}(G)$, of a graph was introduced in [10]. It is defined as

$$\mathrm{GP}(G) = \frac{|V(G)|}{2|\mathrm{Aut}(G)|} \sum_{u \in V(G)} \sum_{\alpha \in \mathrm{Aut}(G)} \mathrm{dist}(u, \alpha(u)),$$

where $\mathrm{Aut}(G)$ is the group of automorphisms of $G$.

Graovac-Pisanski index (originally known as the modified Wiener index) presents an algebraic approach for generalizing the Wiener index. The definition of Graovac-Pisanski index is based on distances but its advantage is in considering also the symmetries of a graph. It is known that symmetries of a molecule have an influence on its properties [20]. Črepnjak et al. showed that the Graovac-Pisanski index of some hydrocarbon molecules is correlated with their melting points [5]. Graovac-Pisanski index was considered for nanostructures, linear polymers, and some classes of fullerenes and fullerene-like molecules [1–3, 17–19, 21–23].

Upper and lower bounds for Graovac-Pisanski index were considered in [18]. Graovac-Pisanski index was further considered from computational point of view in [9, 22]. Exact formulae for the Graovac-Pisanski index for some graph operations are present in [4]. Recent studies were devoted to identifying graphs with maximal values of this index among $n$-vertex trees and unicyclic graphs [15, 16].

The group of automorphisms of $G$ partitions $V(G)$ into orbits. We say that $u, v \in V(G)$ belong to the same *orbit* if there is an automorphism $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(u) = v$. Let $V_1, V_2, \ldots, V_t$ be all the orbits of $\mathrm{Aut}(G)$ in $G$. In [10] it was shown that for every graph $G$

$$\mathrm{GP}(G) = |V(G)| \cdot \sum_{i=1}^{t} \frac{W_G(V_i)}{|V_i|}, \tag{1}$$

where $V_1, V_2, \ldots, V_t$ are the orbits of $\mathrm{Aut}(G)$. Since all vertices $u$ in an orbit $U$ have the same value of $w_U(u)$, we can rewrite (1) in the following way (see [6])

$$\mathrm{GP}(G) = \frac{|V(G)|}{2} \cdot \sum_{i=1}^{t} w_{V_i}(v_i), \tag{2}$$

where $V_1, V_2, \ldots, V_t$ are the orbits of $\mathrm{Aut}(G)$ and $v_1, v_2, \ldots, v_t$, respectively, are their representatives. Using (2) one can see that the Graovac-Pisanski index of every graph is either an integer or half of an integer number. Moreover, in [6] it was shown that for bipartite graphs the following statement holds.

**Proposition 1.** *If $G$ is a bipartite graph then* $\mathrm{GP}(G)$ *is an integer number.*

Since trees are bipartite graphs, Proposition 1 implies that $\mathrm{GP}(T)$ is an integer number if $T$ is a tree. If a graph has no nontrivial automorphisms, that is if all its orbits consist of single vertices, then its Graovac-Pisanski index is 0.

Some researchers have shown interest in the difference between Wiener index and Graovac-Pisanski index. Denote

$$\Delta_W(G) = W(G) - \mathrm{GP}(G).$$

This difference was first considered in [11], and in [12] it was computed for some families of polyhedral graphs. In both these papers one can find an erroneous statement that a graph $G$ is vertex-transitive if and only $\Delta_W(G) = 0$. It was observed already in [10] that for a vertex-transitive graph $\Delta_W(G) = 0$, however we find infinitely many graphs $G$ which are not vertex-transitive and yet $\Delta_W(G) = 0$, see Section 2. In the same section we show that there exist graphs $G$ for which $\Delta_W(G)$ is negative. In the last section we prove that $\Delta_W(G)$ is nonnegative for trees. Moreover, we completely determine the set of values of $\Delta_W(G)$ if $G$ is a tree.

# 2 General graphs

Let $G$ be a graph. As already mentioned, $\mathrm{GP}(G) = W(G)$ if $G$ is vertex-transitive, i.e., if there is just one orbit of vertices of $\mathrm{Aut}(G)$. On the other hand, if $G$ has $|V(G)|$

orbits of $\mathrm{Aut}(G)$ then $\mathrm{GP}(G) = 0$. Regarding the number of orbits these two cases are extremal, and so maybe one could expect that $0 \le \mathrm{GP}(G) \le W(G)$. While the inequality $0 \le \mathrm{GP}(G)$ follows from the definition and it is obvious that $\mathrm{GP}(G) = 0$ if and only if $G$ has $|V(G)|$ orbits, the inequality $\mathrm{GP}(G) \le W(G)$ is not so obvious although one can object that all the distaces contribute to $W(G)$ while only some of them contribute to $\mathrm{GP}(G)$. The problem is caused by the normalizing factors $|V(G)|$ and $\frac{1}{|V_i|}$ in (1). From the terms in (1) one can deduce that $\Delta_W(G) = W(G) - \mathrm{GP}(G)$ can be negative if there are "big distances" inside orbits and "small distances" between them. A typical example of such a situation is a complete bipartite graph $K_{a,b}$, where $a \ne b$. We have the following statement.

**Proposition 2.** *Let $a > b \ge 1$. Then*

$$\Delta_W(K_{a,b}) = a + b - ab.$$

*Proof.* Let $a > b \ge 1$. In $K_{a,b}$, denote by $V_1$ and $V_2$ the partite sets with $a$ and $b$ vertices, respectively. Further, let $v_1 \in V_1$ and $v_2 \in V_2$. Then

$$\begin{aligned} w_G(v_1) &= 2(a-1) + b, & w_{V_1}(v_1) &= 2(a-1), \\ w_G(v_2) &= 2(b-1) + a & w_{V_2}(v_2) &= 2(b-1). \end{aligned}$$

Hence we have

$$\begin{aligned} W(K_{a,b}) &= \tfrac{1}{2} \cdot a(2a + b - 2) + \tfrac{1}{2} \cdot b(a + 2b - 2) = a^2 + b^2 + ab - a - b, \\ \mathrm{GP}(G) &= (a+b)\big(\tfrac{1}{2} \cdot 2(a-1) + \tfrac{1}{2} \cdot 2(b-1)\big) = a^2 + b^2 + 2ab - 2a - 2b, \end{aligned}$$

and so $\Delta_W(K_{a,b}) = a + b - ab$. ■

Observe that $\Delta_W(K_{a,1}) = 1$. However, if $b \ge 2$ then

$$a + b - ab = b - (b-1)a \le b - a < 0.$$

Hence, we obtain the following corollary.

**Corollary 3.** *For every $n \ge 5$ there is a graph $G$ on $n$ vertices such that*

$$\Delta_W(G) < 0.$$

It is easy to check that all connected graphs $G$ on $n \le 4$ vertices satisfy $\Delta_W(G) \ge 0$. But Corollary 3 naturally opens the following problem.

**Problem 4.** *Let $G$ be a graph on $n$ vertices. What is the minimum value of $\Delta_W(G)$?*

Denote by $m(n)$ the minimum value of $\Delta_W(G)$ from Problem 4. For $K_{a,b}$ we cannot have $a = b$. Therefore, the minimum value of $\Delta_W(K_{a,b})$ occurs when $a - (n-a) = 1$ or $a - (n-a) = 2$ depending on the parity of $n$. Hence,

$$\begin{array}{rcll} m(n) & \leq & n + \frac{1}{4} - \frac{n^2}{4} & \quad \text{if } n \text{ is odd,} \\ m(n) & \leq & n + 1 - \frac{n^2}{4} & \quad \text{if } n \text{ is even.} \end{array}$$

However, it is not known whether these bounds are tight.

Now we turn our attention to graphs $G$ for which $\Delta_W(G) = 0$. Observe that $a+b-ab = 0$ is equivalent to $a = \frac{b}{b-1}$ for $b \geq 2$. Since $a$ is an integer we get $b = 2$, implying $a = b$, which is false. Hence, there are no $a > b \geq 1$ such that $\Delta_W(K_{a,b}) = 0$. But there are graphs $G$ which are not vertex-transitive and yet $\Delta_W(G) = 0$.

Let $A$, $B$ and $C$ be disjoint sets having $a$, $b$ and $c$ vertices, respectively. We define a graph $G_{a,b,c}$ as follows. $V(G_{a,b,c}) = A \cup B \cup C$ and $e = uv$ is an edge of $G_{a,b,c}$ if and only if either $u \in A$ and $v \in B$, or $u \in B$ and $v \in C$, or $u, v \in C$, $u \neq v$. In fact, $G_{a,b,c}$ is $(D_a \oplus K_c) + D_b$ where $D_n$ is an empty graph on $n$ vertices, $K_n$ is a complete graph on $n$ vertices, $\oplus$ denotes the disjoint union and $+$ denotes the join of graphs. Since $G_{a,b,1}$ is $K_{a+1,b}$, we require $c \geq 2$. We have the following statement.

**Lemma 5.** *Let $a, b \geq 1$ and $c \geq 2$. Then*

$$\Delta_W(G_{a,b,c}) = \tfrac{1}{2}(-2ab + ac - bc + 3a + 3b + 4c).$$

*Proof.* Observe that $A$, $B$ and $C$ are the orbits of $\mathrm{Aut}(G_{a,b,c})$. Let $u \in A$, $v \in B$ and $z \in C$. We have

$$\begin{array}{ll} w_G(u) = 2(a-1) + b + 2c, & \qquad w_A(u) = 2(a-1), \\ w_G(v) = a + 2(b-1) + c, & \qquad w_B(v) = 2(b-1), \\ w_G(z) = 2a + b + (c-1), & \qquad w_C(z) = (c-1). \end{array}$$

Hence

$$\begin{array}{rcl} W(G_{a,b,c}) & = & \frac{1}{2}\big(a(2a+b+2c-2) + b(a+2b+c-2) + c(2a+b+c-1)\big) \\ & = & \frac{1}{2}(2a^2 + 2b^2 + c^2 + 2ab + 4ac + 2bc - 2a - 2b - c) \\ \mathrm{GP}(G_{a,b,c}) & = & \frac{1}{2}(a+b+c)\big((2a-2) + (2b-2) + (c-1)\big) \\ & = & \frac{1}{2}(2a^2 + 2b^2 + c^2 + 4ab + 3ac + 3bc - 5a - 5b - 5c) \end{array}$$

which gives $\Delta_W(G_{a,b,c})$ as required. ∎

Observe that if $a = b$ then

$$\Delta_W(G_{a,b,c}) = 3a - a^2 + 2c.$$

Hence if $c = \frac{a^2-3a}{2}$ we get $\Delta_W(G_{a,a,c}) = 0$. The expression $a^2 - 3a$ is even for every value of $a$ but $c \geq 2$ implies $a \geq 4$. So we have the following corollary.

**Corollary 6.** *Let $a \geq 4$ and let $c = \frac{a^2-3a}{2}$. Then $\Delta_W(G_{a,a,c}) = 0$.*

Corollary 6 gives an infinite class of graphs $G$ which are not vertex-transitive and yet $W(G) = \mathrm{GP}(G)$. The problem, for which values $n$ there are non-vertex-transitive graphs $G$ on $n$ vertices such that $\Delta_W(G) = 0$, remains open.

# 3 Trees

If $e \in E(G)$, then $G - e$ denotes the graph with vertex set $V(G)$ and edge set $E(G) \setminus \{e\}$. As shown in [24], if $G$ is a tree, then its Wiener index can be computed using the following lemma.

**Lemma 7.** *Let $T$ be a tree. For every edge $e$ in $T$, denote by $n(e)$ and $n'(e)$ orders of the two trees of $T - e$. Then*

$$W(T) = \sum_{e \in E(T)} n(e) \cdot n'(e).$$

In Lemma 7, every summand counts how many times an edge occurs on a (shortest, and unique) path connecting two vertices of $V(T)$. Therefore, we have the following analogue of Lemma 7 for $W_G(U)$.

**Lemma 8.** *Let $T$ be a tree and let $U \subseteq V(T)$. For every edge $e$ in $T$, denote by $n_U(e)$ and $n'_U(e)$, respectively, the number of vertices of $U$ in the two trees of $T - e$. Then*

$$W_T(U) = \sum_{e \in E(T)} n_U(e) \cdot n'_U(e).$$

By $P_n$ we denote a path on $n$ vertices. Since every tree has at most two central vertices and since vertices which are central cannot be in the same orbit as non-central ones, $P_1$ and $P_2$ are the only trees which have just one orbit of the group of automorphisms, i.e., which are vertex-transitive. Hence, $W(T) = \mathrm{GP}(T)$ if $T$ is $P_1$ or $P_2$. For all other trees the following statement is true.

**Theorem 9.** *Let $T$ be a tree on at least three vertices. Then*

$$W(T) > \mathrm{GP}(T).$$

*Proof.* By (1) we have to prove

$$W(T) > |V(T)| \cdot \sum_{i=1}^{t} \frac{W_T(V_i)}{|V_i|}, \tag{3}$$

where $V_1, V_2, \ldots, V_t$ are the orbits of $\mathrm{Aut}(T)$. We prove this statement using Lemma 8. Fix an endvertex of $T$ and denote it by $q$. For every $U \subseteq V(T)$, by $n_U(e)$ we denote the number of vertices of $U$ in the component of $T - e$ containing $q$, while $n'_U(e)$ is the number of vertices of $U$ in the other component of $T-e$. For every orbit $V_i$ of $\mathrm{Aut}(T)$, let $n_i = |V_i|$ and let $a_i(e) = n_{V_i}(e)$. Then $n'_{V_i}(e) = n_i - a_i(e)$. Since $n_{V(G)}(e) = \sum_{i=1}^{t} a_i(e)$, $n'_{V(G)}(e) = \sum_{i=1}^{t} \big(n_i - a_i(e)\big)$ and $|V(T)| = \sum_{i=1}^{t} n_i$, after reordering the sums, (3) is equivalent to

$$\sum_{e\in E(T)} \left( \sum_{i=1}^{t} a_i(e) \cdot \sum_{i=1}^{t} \big(n_i - a_i(e)\big) \right) > \sum_{e\in E(T)} \sum_{i=1}^{t} n_i \cdot \left( \sum_{i=1}^{t} \frac{a_i(e)\cdot \big(n_i - a_i(e)\big)}{n_i} \right)$$

Hence, (3) is true if for every edge $e$ we have

$$\sum_{i=1}^{t} a_i(e) \cdot \sum_{i=1}^{t} \big(n_i - a_i(e)\big) \geq \sum_{i=1}^{t} n_i \cdot \sum_{i=1}^{t} \frac{a_i(e)\cdot \big(n_i - a_i(e)\big)}{n_i}$$

and for at least one edge the inequality is strict.

So let $e \in E(T)$. Denote $p_i = a_i(e)/n_i$, $1 \leq i \leq t$. Then $0 \leq p_i \leq 1$. Observe that $0 \leq (p_i - p_j)^2$ and equality holds if and only if $p_i = p_j$. Hence,

$$0 \leq \sum_{1\leq i<j\leq t} n_i n_j (p_i - p_j)^2 \tag{4}$$

and equality holds if and only if $p_1 = p_2 = \cdots = p_t$. However, (4) is equivalent to the following

$$\begin{aligned}
\sum_{i=1}^{t} n_i^2 p_i^2 + \sum_{1\leq i<j\leq t} 2 n_i n_j p_i p_j &\leq \sum_{i=1}^{t} n_i^2 p_i^2 + \sum_{1\leq i<j\leq t} (n_i n_j p_i^2 + n_i n_j p_j^2) \\
\sum_{i=1}^{t} n_i p_i \cdot \sum_{i=1}^{t} n_i p_i &\leq \sum_{i=1}^{t} n_i \cdot \sum_{i=1}^{t} n_i p_i^2 \\
\sum_{i=1}^{t} n_i p_i \cdot \sum_{i=1}^{t} n_i - \sum_{i=1}^{t} n_i p_i \cdot \sum_{i=1}^{t} n_i p_i &\geq \sum_{i=1}^{t} n_i \cdot \sum_{i=1}^{t} n_i p_i - \sum_{i=1}^{t} n_i \cdot \sum_{i=1}^{t} n_i p_i^2 \\
\sum_{i=1}^{t} n_i p_i \cdot \sum_{i=1}^{t} \big(n_i - n_i p_i\big) &\geq \sum_{i=1}^{t} n_i \cdot \sum_{i=1}^{t} n_i p_i (1 - p_i) \\
\sum_{i=1}^{t} a_i(e) \cdot \sum_{i=1}^{t} \big(n_i - a_i(e)\big) &\geq \sum_{i=1}^{t} n_i \cdot \sum_{i=1}^{t} \frac{a_i(e)\cdot \big(n_i - a_i(e)\big)}{n_i}
\end{aligned} \tag{5}$$

and equality holds in (5) if and only if $\frac{a_1(e)}{n_1} = \frac{a_2(e)}{n_2} = \cdots = \frac{a_t(e)}{n_t}$.

Let $e^* = qu$, where $q$ is the fixed endvertex. Since $T$ has at least three vertices, $q$ and $u$ have different degrees, and so they belong to different orbits of $\mathrm{Aut}(T)$. Assume that $q \in V_1$ and $u \in V_2$. Then $n_{V_1}(e^*) = a_1(e^*) = 1$ while $n_{V_2}(e^*) = a_2(e^*) = 0$. Hence $p_1 = a_1(e^*)/n_1 > 0$ while $p_2 = a_2(e^*)/n_2 = 0$. So in this case we have a strict inequality in (5) which proves (3). ■

Now we find the values of $\Delta_W(T)$, when $T$ is a tree. For this we need to calculate $\Delta_W$ for four types of trees, namely $T_a^1$, $T_{a,b}^1$, $T_a^2$ and $T_{a,b}^2$. The graphs $T_a^1$ and $T_{a,b}^1$ have one central vertex, say $c$. $T_a^1$ is obtained when $a$ pendant vertices are attached to $c$, where $a \geq 2$. Hence, $T_a^1$ is the star $K_{a,1}$. $T_{a,b}^1$ is obtained from $T_a^1$ by attaching $b$ pendant vertices to every endvertex of $T_a^1$, where $b \geq 1$. Observe that $T_{a,b}^1$ has $1 + a + ab$ vertices. The graphs $T_a^2$ and $T_{a,b}^2$ have two adjacent central vertices, say $c_1$ and $c_2$. $T_a^2$ is obtained from the edge $c_1c_2$ by attaching $a$ pendant vertices to each endvertex of $c_1c_2$, where $a \geq 1$. So $T_a^2$ has $2 + 2a$ vertices. $T_{a,b}^2$ is obtained from $T_a^2$ by attaching $b$ pendant vertices to every endvertex of $T_a^2$, where $b \geq 1$. Observe that $T_{a,b}^2$ has $2 + 2a + 2ab$ vertices. We have the following lemma.

**Lemma 10.** *We have*

$$\Delta_W(T_a^1) = 1, \qquad \Delta_W(T_{a,b}^1) = 2 + a + b,$$
$$\Delta_W(T_a^2) = 2, \qquad \Delta_W(T_{a,b}^2) = 4 + 2a + 2b.$$

*Proof.* By Proposition 2, $\Delta_W(T_a^1) = 1$, so we consider only $T_{a,b}^1$, $T_a^2$ and $T_{a,b}^2$ here. Let $T$ be a tree with orbits $V_1, V_2, \ldots V_t$, where $n_i = |V_i|$ for $1 \leq i \leq t$, and for $e \in E(T)$ let

$$\Delta_W(e) = \sum_{i=1}^{t} a_i(e) \cdot \sum_{i=1}^{t} \big(n_i - a_i(e)\big) - \sum_{i=1}^{t} n_i \cdot \sum_{i=1}^{t} \frac{a_i(e) \cdot \big(n_i - a_i(e)\big)}{n_i},$$

where $a_1(e), a_2(e), \ldots, a_t(e)$ are numbers of vertices of $V_1, V_2, \ldots, V_t$, respectively, in the component of $T - e$ containing a specific vertex $q$. As shown in the proof of Theorem 9,

$$\Delta_W(T) = \sum_{e \in E(T)} \Delta_W(e).$$

Observe that this specific vertex $q$ can be chosen for every edge $e$ separately, important is only that all $a_1(e), a_2(e), \ldots, a_t(e)$ relate to the same component of $T - e$.

The graph $T_{a,b}^1$ has exactly three orbits of vertices. Let $V_1$ be the orbit of endvertices of $T_{a,b}^1$, let $V_2$ be the orbit of vertices adjacent to endvertices and let $V_3 = \{c\}$. Further, let $e_1$ be a pendant edge and let $e_2$ be an edge incident with $c$. For computing $\Delta_W(e_1)$, we choose $q$ to be the endvertex of $e_1$ of degree 1, and to obtain $\Delta_W(e_2)$ let $q$ be the endvertex of $e_2$ different from $c$. Then

$$\begin{aligned}\Delta_W(e_1) &= 1\cdot(a{+}ab)-(1{+}a{+}ab)\Big(\tfrac{1\cdot(ab-1)}{ab}+\tfrac{0\cdot a}{a}+\tfrac{0\cdot 1}{1}\Big)=\tfrac{1+a}{ab},\\ \Delta_W(e_2) &= (1{+}b)\cdot(a{-}b{+}ab)-(1{+}a{+}ab)\Big(\tfrac{b\cdot(ab-b)}{ab}+\tfrac{1\cdot(a-1)}{a}+\tfrac{0\cdot 1}{1}\Big)=\tfrac{1+b}{a}.\end{aligned}$$

Since $T^1_{a,b}$ has $ab$ edges like $e_1$ (that is, incident with a pendant vertex) and $a$ edges like $e_2$, we have $\Delta_W(T^1_{a,b})=2+a+b$.

Analogously, the graph $T^2_a$ has exactly two orbits of vertices. Let $V_1$ be the orbit of endvertices of $T^2_a$ and let $V_2=\{c_1,c_2\}$. Further, let $e$ be a pendant edge. Then

$$\Delta_W(e)=1\cdot(1+2a)-(2+2a)\Big(\tfrac{1\cdot(2a-1)}{2a}+\tfrac{0\cdot 2}{2}\Big)=\tfrac{1}{a},$$

while $\Delta_W(c_1c_2)=0$ since $a_i(c_1c_2)/n_i=\frac{1}{2}$ for $1\le i\le 2$ in this case. Since $T^2_a$ has $2a$ pendant edges, we get $\Delta_W(T^2_a)=2$.

The graph $T^2_{a,b}$ has three orbits of vertices. Let $V_1$ be the orbit of endvertices of $T^2_{a,b}$, let $V_2$ be the orbit of vertices adjacent to endvertices and let $V_3=\{c_1,c_2\}$. Further, let $e_1$ be a pendant edge and let $e_2$ be an edge incident with a central vertex, $e\neq c_1c_2$. Then

$$\begin{aligned}\Delta_W(e_1) &= 1\cdot(1{+}2a{+}2ab)-(2{+}2a{+}2ab)\Big(\tfrac{1\cdot(2ab-1)}{2ab}+\tfrac{0\cdot 2a}{2a}+\tfrac{0\cdot 2}{2}\Big)=\tfrac{1+a}{ab},\\ \Delta_W(e_2) &= (1{+}b)\cdot(1{+}2a{-}b{+}2ab)-(2{+}2a{+}2ab)\Big(\tfrac{b\cdot(2ab-b)}{2ab}+\tfrac{1\cdot(2a-1)}{2a}+\tfrac{0\cdot 2}{2}\Big)\\ &= \tfrac{1+b}{a},\end{aligned}$$

while $\Delta_W(c_1c_2)=0$ since $a_i(c_1c_2)/n_i=\frac{1}{2}$ for $1\le i\le 3$ in this case. Since $T^2_{a,b}$ has $2ab$ edges like $e_1$ and $2a$ edges like $e_2$, we have $\Delta_W(T^2_{a,b})=2+2a+2+2b=4+2a+2b$. ■

Using Lemma 10 we can prove the following theorem.

**Theorem 11.** *There exists a tree $T$ satisfying $\Delta_W(T)=\ell$ if and only if $\ell$ is integer, $\ell\ge 0$ and $\ell\notin\{3,4\}$.*

*Proof.* By Proposition 1 and Theorem 9, we have that $\ell$ is a nonnegative integer.

Since the paths $P_1$ and $P_2$ are vertex-transitive, $\Delta_W(P_1)=\Delta_W(P_2)=0$. By Lemma 10, $\Delta_W(T^1_a)=1$ and $\Delta_W(T^2_a)=2$. Further, $\Delta_W(T^1_{a,b})=2+a+b$, where $a\ge 2$ and $b\ge 1$. Hence, if $\ell\ge 5$ then $\Delta_W(T^1_{\ell-3,1})=\ell$ since $\ell-3\ge 2$. In the rest of the proof we show that $\ell$ cannot be 3 or 4.

So let $T$ be a tree such that $\Delta_W(T)=3$ or $\Delta_W(T)=4$. Further, let $V_i$ be an orbit containing only pendant vertices of $T$. Observe that $V_i$ does not need to contain all pendant vertices of $T$. As in the previous proofs, denote $n_i = |V_i|$. Let $e$ be an edge containing a vertex of $V_i$. Then its contribution to $\Delta_W(T)$ is

$$\Delta_W(e) = 1 \cdot (n-1) - n \cdot \tfrac{1 \cdot (n_i - 1)}{n_i} = \tfrac{n - n_i}{n_i},$$

where $n = |V(T)|$. Since there are exactly $n_i$ edges with one endvertex in $V_i$, these edges contribute to $\Delta_W(T)$ by $n - n_i$.

Now suppose that there are two distinct orbits, say $V_1$ and $V_2$, containing (possibly not all) endvertices of $T$. Then pendant edges contribute to $\Delta_W(T)$ by at least $(n-n_1)+(n-n_2) = 2n - n_1 - n_2$. Since $\Delta_W(T) \neq 0$, $T$ has at least three vertices, and so the central vertex of $T$ cannot be pendant. Denote by $k$ the number of vertices of $T$ which are not in $V_1 \cup V_2$, i.e. $k = n - n_1 - n_2$. Then the pendant edges contribute to $\Delta_W(T)$ by at least $n + k$. Since $n \geq 3$ and $k \geq 1$, we have $n + k \geq 4$ and equality holds only if $T$ has exactly three vertices. But then $T$ is $P_3$ and all endvertices of $T$ belong to one orbit of $\mathrm{Aut}(T)$, a contradiction.

So all endvertices of $T$ belong to a single orbit of $\mathrm{Aut}(T)$, say $V_1$. Observe that if there are at least five vertices in $T$ which are not pendant, then $n_1 \leq n - 5$ and so the pendant edges contribute to $\Delta_W(T)$ by at least 5, a contradiction. Hence, deleting all endvertices of $T$ results in a tree $T'$ which has at most 4 vertices. To complete the proof we distinguish 5 cases according to the structure of $T'$.

**Case 1**: $|V(T')| = 1$. In this case $T$ is $T_a^1$ for some $a \geq 2$, and so $\Delta_W(T) = 1$, by Lemma 10.

**Case 2**: $|V(T')| = 2$. Then $T$ is $T_a^2$ for some $a \geq 1$, and so $\Delta_W(T) = 2$, by Lemma 10.

**Case 3**: $|V(T')| = 3$. Then $T'$ is the path $P_3$. The graph $T$ is obtained from $T'$ by attaching pendant vertices to both endvertices of $P_3$ and possibly also to the central vertex of $P_3$. But since there is only one orbit of pendant vertices in $T$, there are no pendant vertices adjacent to the central vertex of $P_3$. Hence, $T$ is $T_{2,b}^1$ for some $b \geq 1$. Consequently, $\Delta_W(T) = 2 + 2 + b \geq 5$, by Lemma 10.

**Case 4**: $|V(T')| = 4$ *and* $T'$ *is the path* $P_4$. Then analogously as above one can show that $T$ is $T_{1,b}^2$ for some $b \geq 1$. But then $\Delta_W(T) = 4 + 2 + 2b \geq 8$, by Lemma 10.

**Case 5**: $|V(T')| = 4$ *and* $T'$ *is the claw* $T_3^1$. Then $T$ is $T_{3,b}^1$ for some $b \geq 1$, which means that $\Delta_W(T) = 2 + 3 + b \geq 6$, by Lemma 10.

Since we obtained a contradiction in all cases, the proof is complete. ■

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