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Solution to a Problem on the Complexity of Connective Eccentric Index of Graphs*

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Abstract

The connective eccentricity index of a graph G is defined as $\xi^{ce}(G) = \sum\limits_{v \in V} \frac{\deg(v)}{\mathrm{ecc}(v)}$, where $\deg(v)$ and $\mathrm{ecc}(v)$ are the degree and the eccentricity of the vertex v, respectively. The complexity of the connective eccentricity index of a graph G, denoted by $C_{\xi^{ce}}(G)$, is the number of different $\frac{\deg(v)}{\mathrm{ecc}(v)}$ for any vertex v. Alizadeh and Klavžar [MATCH Commun. Math. Comput. Chem. 76(2016) 659-667] studied this invariant and left a problem "construct infinite families of graphs $\{G_n\}_{n\to\infty}$ such that $C_{\xi^{ce}}(G_n) = |V(G_n)| = n$". In this paper, we solve this problem by giving a construction of G_n for all $n \geq 7$.

1 Introduction

Let $G = (V(G), E(G))$ be a simple graph on n vertices and m edges. The distance $d_G(u, v)$ between two vertices u and v, is the length of a shortest $u - v$ path in G. The eccentricity $\mathrm{ecc}_G(v)$ of a vertex v is the largest distance between v and any other vertex u of G. There are several important topological indices based on the eccentricity of vertices. One of these topological indices is the connective eccentricity index of a graph G which is defined as $\xi^{ce}(G) = \sum\limits_{v \in V} \frac{\deg(v)}{\mathrm{ecc}(v)}$, introduced in [8] as a novel topological descriptor for predicting biological activity. Afterwards, many researchers began to study this topic and many results were obtained in [10–12]. We note that there is another topological index

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eccentric connectivity index $\xi^c(G)$ of a graph G, which was introduced in [9] and is defined as $\xi^c(G) = \sum_{v \in V} \deg(v)\mathrm{ecc}(v)$. For more information we can refer to [5–7].

Many topological indices are summation-type topological indices such as the Wiener index, the connective eccentricity index and so on. Suppose that $I(G)$ is a topological index and $I(G) = \sum_{v \in V} f(v)$. We can think of the value $f(v)$ as the contribution to the index $I(G)$ from the vertex v. It is interesting to study the different contributions of vertices of G. The initial study is about the Wiener index and named the Wiener dimension of a graph [1]. Because the word "dimension" is also an important geometrical concept, Y. Alizadeh and S. Klavžar [3] used the word "complexity" instead of "dimension", and introduced the complexity of the connectivity eccentricity index, for brevity, ξ^{ce}-complexity, denoted as $C_{\xi^{ce}}(G)$. Formally, ξ^{ce}-complexity of a graph G is defined as the number of different $\frac{\deg(v)}{\mathrm{ecc}(v)}$ for any vertex v of G. For more results about complexity, see [2, 4].

In [3], Y. Alizadeh and S. Klavžar studied the graph with its ξ^{ce}-complexity equal to some special values. For any $d \geq 2$ and $k \geq 1$, they made a construction of graph G with $diam(G) = d$ and $C_{\xi^{ce}}(G) = k$. It is obvious that $1 \leq C_{\xi^{ce}}(G) \leq n$. Note that the ξ^{ce}-complexity of a vertex transitive graph equals 1. For the graph G with $C_{\xi^{ce}}(G) = 1$, they constructed an infinite family of graphs that are not vertex-transitive. When they talked about the graph G with $C_{\xi^{ce}}(G) = |V(G_n)| = n$, they constructed a graph G for $n = 7$, and left a problem: it would be interesting to construct infinite families of graphs $\{G_n\}_{n\to\infty}$ such that $C_{\xi^{ce}}(G) = |V(G_n)| = n$. In this note, we will give a construction of G_n with $C_{\xi^{ce}}(G) = |V(G_n)| = n$ for all $n \geq 7$.

2 Graphs with ξ^{ce}-complexity equal to n

In this section, we will construct an infinite family of graphs G_n with ξ^{ce}-complexity equal to n. Before that, we sketch out our method.

Let G_1 and G_2 be two graphs with the same order. Suppose that G_1 is the graph with vertex set $V(G_1) = \{u_1, u_2, \ldots, u_n\}$ and edge set $E(G_1)$, and G_2 with vertex set $V(G_2) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G_2)$. Let $P_l = t_1t_2\ldots t_l$ be a path and P_l^i be the path P_l with a designated vertex t_i for $1 \leq i \leq l$.

We now define an operation of G_1, G_2 and P_l^i, denoted by $R(G_1, G_2, P_l^i)$, as follows. The resultant graph $G = R(G_1, G_2, P_l^i)$ has vertex set $V(G)$ and edge set $E(G)$, where $V(G) = V(G_1) \cup V(G_2) \cup V(P_l)$ and $E(G) = E(G_1) \cup E(G_2) \cup E(P_l) \cup E_1 \cup E_2$ where E_1 is the set of all edges u_iv_j satisfying $i+j \leq n+1$, i.e., $E_1 = \{u_1v_1, u_1v_2, \ldots, u_1v_n, u_2v_1, \ldots, u_2v_{n-1}, u_3v_1, \ldots, u_3v_{n-2}, \ldots, u_nv_1\}$ and E_2 is the set of edges connecting the i-th vertex t_i of P_l to all vertices of G_2.

We illustrate an example $R(P_3, K_3, P_5^2)$ in Figure 2.1.

In [3], Y. Alizadeh and S. Klavžar gave a graph of order 7 with ξ^{ce}-complexity equal to 7, see Figure 2.2.

For any number $n \geq 8$, we will demonstrate our constructions. Let C_n and K_n be the cycle and the complete graph of order n, respectively.

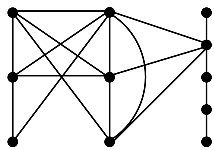

Figure 2.1. $R(P_3, K_3, P_5^2)$

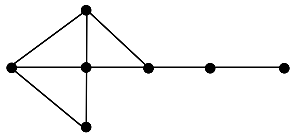

Figure 2.2. $\xi^{ce}(G) = 7$

Theorem 2.1 *If $n = 4k+4$ with $k \geq 3$ and $k \neq 5$, then $G = R(G_1, K_{2k-1}, P_6^2)$ satisfies that its ξ^{ce}-complexity is n, where G_1 is a 4-regular graph of order $2k-1$.*

Proof. First of all, we claim that there exists a 4-regular graph of order $2k-1$ for any $k \geq 3$. Note that the Cayley graph $\mathrm{Cay}(G, S)$ is the graph whose vertex set is G and two vertices $a, b \in G$ are adjacent whenever $ab^{-1} \in S$. It can be verified that the Cayley graph $\mathrm{Cay}(Z_{2k-1}, \{1, 2, 2k-2, 2k-3\})$ is 4-regular.

Suppose that $V_1 = V(G_1)$, $V_2 = V(K_{2k-1})$ and $V_3 = V(P_6^2)$. By computation, we find that the contributions (i.e. $\frac{\deg(v)}{\mathrm{ecc}(v)}$) of all vertices of V_1, which is called the contributions of V_1 for short, constitute a set $S_1 = \{\frac{5}{6}, \frac{6}{6}, \ldots, \frac{2k+3}{6}\}$. The contributions of V_2 constitute a set $S_2 = \{\frac{2k}{5}, \frac{2k+1}{5}, \ldots, \frac{4k-2}{5}\}$. The contributions of V_3 form a set $S_3 = \{\frac{1}{5}, \frac{2k+1}{4}, \frac{2}{3}, \frac{2}{4}, \frac{2}{5}, \frac{1}{6}\}$. Observe that S_1, S_2 and S_3 are all simple sets. To prove that ξ^{ce}-complexity of G is n, it is suffice to show that S_1, S_2 and S_3 are disjoint.

Note that $6 \cdot \frac{2k+1}{4}$ and $5 \cdot \frac{2k+1}{4}$ are not integer, so $\frac{2k+1}{4}$ is not in S_1 or S_2. The maximal value of $S_3 \backslash \{\frac{2k+1}{4}\}$ is $\frac{2}{3}$ which is less than the minimum $\frac{5}{6}$ of S_1. It implies that $S_1 \cap S_3 = \oslash$. By similar method, we deduce that $S_2 \cap S_3 = \oslash$. It remains to show that $S_1 \cap S_2 = \oslash$.

It is straightforward to verify that the assertion holds for $3 \leq k \leq 7$. For $k \geq 8$, we have that the maximum $\frac{2k+3}{6}$ of S_1 is less than the minimum $\frac{2k}{5}$ of S_2. Thus, $S_1 \cap S_2 = \oslash$.

We conclude that S_1, S_2 and S_3 are disjoint. It follows that $\xi^{ce}(G) = n$. ■

Theorem 2.2 *If $n = 4k + 3$ with $k \geq 2$, then $G = R(C_{2k}, K_{2k}, P_3^1)$ satisfies that its ξ^{ce}-complexity is n.*

Proof. Let $V_1 = V(C_{2k})$, $V_2 = V(K_{2k})$ and $V_3 = V(P_3^1)$. The contributions of V_1, V_2 and V_3 are $S_1 = \{\frac{3}{4}, \frac{4}{4}, \ldots, \frac{2k+2}{4}\}$, $S_2 = \{\frac{2k+1}{3}, \frac{2k+2}{3}, \ldots, \frac{4k}{3}\}$, $S_3 = \{\frac{2k+1}{2}, \frac{2}{3}, \frac{1}{4}\}$, respectively. Note that S_1, S_2 and S_3 are all simple sets. It suffices to prove that S_1, S_2 and S_3 are disjoint.

Since $\frac{3}{4} > \frac{2}{3}$ and $\frac{2k+1}{2} > \frac{2k+2}{4}$, we get $S_1 \bigcap S_3 = \oslash$. Because $3 \cdot \frac{2k+1}{2}$ is not an integer and $\frac{2}{3} < \frac{2k+1}{3}$, we deduce that $S_2 \bigcap S_3 = \oslash$.

For $k \geq 2$, we get that $\frac{2k+2}{4} < \frac{2k+1}{3}$. Then $S_1 \bigcap S_2 = \oslash$.

We conclude that S_1, S_2 and S_3 are disjoint. It follows that $\xi^{ce}(G) = n$. ■

Theorem 2.3 *If $n = 4k + 6$ with $k \geq 1$, then $G = R(C_{2k+1}, K_{2k+1}, P_4^2)$ satisfies that its ξ^{ce}-complexity is n.*

Proof. Let $V_1 = V(C_{2k+1})$, $V_2 = V(K_{2k+1})$ and $V_3 = V(P_4^2)$. The contributions of V_1, V_2 and V_3 are $S_1 = \{\frac{3}{4}, \frac{4}{4}, \ldots, \frac{2k+3}{4}\}$, $S_2 = \{\frac{2k+2}{3}, \frac{2k+3}{3}, \ldots, \frac{4k+2}{3}\}$, $S_3 = \{\frac{1}{3}, \frac{2k+3}{2}, \frac{2}{3}, \frac{1}{4}\}$, respectively. Note that S_1, S_2 and S_3 are all simple sets.

By simple calculations, we deduce that $S_1 \bigcap S_3 = \oslash$ and $S_2 \bigcap S_3 = \oslash$.

For $k \geq 1$, we get that $\frac{2k+3}{4} < \frac{2k+2}{3}$. Then $S_1 \bigcap S_2 = \oslash$. Thus, S_1, S_2 and S_3 are disjoint and the proof is complete. ■

Theorem 2.4 *If $n = 4k + 5$ with $k \geq 3$ and $k \neq 4, 7$, then $G = R(G_1, K_{2k}, P_5^1)$ satisfies that its ξ^{ce}-complexity is n, where G_1 is a 4-regular graph of order $2k$.*

Proof. Let $V_1 = V(G_1)$, $V_2 = V(K_{2k})$ and $V_3 = V(P_5^1)$. The contributions of V_1, V_2 and V_3 are $S_1 = \{\frac{5}{6}, \frac{6}{6}, \ldots, \frac{2k+4}{6}\}$, $S_2 = \{\frac{2k+1}{5}, \frac{2k+2}{5}, \ldots, \frac{4k}{5}\}$, $S_3 = \{\frac{2k+1}{4}, \frac{2}{3}, \frac{2}{4}, \frac{2}{5}, \frac{1}{6}\}$, respectively. Note that S_1, S_2 and S_3 are all simple sets.

By simple calculations, we deduce that $S_1 \bigcap S_3 = \oslash$ and $S_2 \bigcap S_3 = \oslash$ for $k \geq 3$.

For $k = 3, 5, 6$, we can check that $S_1 \bigcap S_2 = \oslash$. For $k \geq 8$, we get that $\frac{2k+4}{6} < \frac{2k+1}{5}$. Then $S_1 \bigcap S_2 = \oslash$. Thus, S_1, S_2 and S_3 are disjoint and the proof is complete. ■

To sum up, the above results illustrate the graphs with ξ^{ce}-complexity equal to n for all $n \geq 7$ except $\{8, 9, 12, 13, 21, 24, 33\}$. For these special graphs, we will give the concrete constructions in the following. For brevity, we omit their proofs.

(1) For $n = 8$, the graph $G = R(P_2, K_2, P_4^2)$ satisfies that its ξ^{ce}-complexity is 8. The contributions of vertices are $S_1 = \{\frac{2}{4}, \frac{3}{4}\}$, $S_2 = \{\frac{3}{3}, \frac{4}{3}\}$, $S_3 = \{\frac{1}{3}, \frac{4}{2}, \frac{2}{3}, \frac{1}{4}\}$.

(2) For $n = 9$, the graph $G = R(P_2, K_2, P_5^2)$ satisfies that its ξ^{ce}-complexity is 9. $S_1 = \{\frac{2}{5}, \frac{3}{5}\}$, $S_2 = \{\frac{3}{4}, \frac{4}{4}\}$, $S_3 = \{\frac{1}{4}, \frac{4}{3}, \frac{2}{3}, \frac{2}{4}, \frac{1}{5}\}$.

(3) For $n = 12$, the graph $G = R(2P_2, K_4, P_4^2)$ satisfies that its ξ^{ce}-complexity is 12. $S_1 = \{\frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \frac{5}{4}\}$, $S_2 = \{\frac{5}{3}, \frac{6}{3}, \frac{7}{3}, \frac{8}{3}\}$, $S_3 = \{\frac{1}{3}, \frac{6}{2}, \frac{2}{3}, \frac{1}{4}\}$.

(4) For $n = 13$, the graph $G = R(C_5, C_5, P_3^1)$ satisfies that its ξ^{ce}-complexity is 13. $S_1 = \{\frac{3}{4}, \frac{4}{4}, \frac{5}{4}, \frac{6}{4}, \frac{7}{4}\}$, $S_2 = \{\frac{4}{3}, \frac{5}{3}, \frac{6}{3}, \frac{7}{3}, \frac{8}{3}\}$, $S_3 = \{\frac{6}{2}, \frac{2}{3}, \frac{1}{4}\}$.

(5) For $n = 21$, the graph $G = R(4P_2, K_8, P_5^2)$ satisfies that its ξ^{ce}-complexity is 21. $S_1 = \{\frac{2}{5}, \frac{3}{5}, \ldots, \frac{9}{5}\}$, $S_2 = \{\frac{9}{4}, \frac{10}{4}, \ldots, \frac{16}{4}\}$, $S_3 = \{\frac{1}{4}, \frac{10}{3}, \frac{2}{3}, \frac{2}{4}, \frac{1}{5}\}$.

(6) For $n = 24$, the graph $G = R(C_{10}, K_{10}, P_4^1)$ satisfies that its ξ^{ce}-complexity is 24. $S_1 = \{\frac{3}{5}, \frac{4}{5}, \ldots, \frac{12}{5}\}$, $S_2 = \{\frac{11}{4}, \frac{12}{4}, \ldots, \frac{20}{4}\}$, $S_3 = \{\frac{11}{3}, \frac{2}{3}, \frac{2}{4}, \frac{1}{5}\}$.

(7) For $n = 33$, the graph $G = R(7P_2, K_{14}, P_5^2)$ satisfies that its ξ^{ce}-complexity is 33. $S_1 = \{\frac{2}{5}, \frac{3}{5}, \ldots, \frac{15}{5}\}$, $S_2 = \{\frac{15}{4}, \frac{16}{4}, \ldots, \frac{28}{4}\}$, $S_3 = \{\frac{1}{4}, \frac{16}{3}, \frac{2}{3}, \frac{2}{4}, \frac{1}{5}\}$.

Remark: By calculations, we can check that for $2 \le n \le 6$, there exists no graph on order n with its ξ^{ce}-complexity equal to n.

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