MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **81** (2019) 785-800

ISSN 0340 - 6253

An Efficient Algorithm for Solving Coupled Lane–Emden Boundary Value Problems in Catalytic Diffusion Reactions: The Homotopy Analysis Method

Randhir Singh[a,*], **Abdul–Majid Wazwaz**[b]

[a] *Department of Mathematics, Birla Institute of Technology Mesra, Ranchi, India.*

[b] *Department of Mathematics, Saint Xavier University, Chicago, IL 60655, U.S.A.*

(Received July 6, 2018)

Abstract

In this paper, we use the homotopy analysis method (HAM) with Green's function [1] for solving the coupled Lane–Emden boundary value problems which appear in catalytic diffusion reactions. Due to the presence of singularity, these problems pose difficulties in obtaining their solutions. To overcome the singular behavior at the origin, the coupled Lane–Emden boundary value problems are transformed into an equivalent Fredholm integral equations. The integral forms of the Lane-Emden equations are then solved by the HAM. Unlike, Adomian decomposition method (ADM), the HAM contains an adjustable parameters to control the convergence of solution. For speed up the calculations, the discrete averaged residual error is used to obtain optimal value of the adjustable parameter c_0 to control the convergence of solution. The numerical results show that the HAM gives reliable algorithm for analytic approximate solutions of these systems. The error analysis of the sequence of the analytic approximate solutions has been performed by computing the residual error functions and the maximal residual error parameters, which demonstrate an approximate exponential rate of convergence.

*Corresponding author. E-mail addresses: randhir.math@gmail.com (R. Singh), wazwaz@sxu.edu (A.-M. Wazwaz)

1 Introduction

The astrophysicists Jonathan Homer Lane and Robert Emden studied the Lane–Emden equations, where they considered the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics [2]. The Lane–Emden equation has been used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas spheres and the theory of thermionic currents [3]. This equation also describes the temperature or concentration variation in many fields of physics, chemistry, biology, biochemistry, and many others such as isothermal and non-isothermal reaction diffusion process inside a porus cylindrical or spherical catalysts [3], solidification of cylindrical and spherical objects [4] and thermal explosion in rectangular slab [5]. A substantial amount of work has been done on these types of problems for various structures [6–23] and the references cited therein. The singular behavior that occurs at $x = 0$ gives the main difficulty for solving the Lane-Emden equations.

Systems of Lane-Emden equations arise in the modelling of several physical and chemical phenomena, such as pattern formation, population evolution, chemical reactions, and so on [24–28]. In [26, 27], the Adomian decomposition method was used to solve the Volterra integral form of the Lane–Emden equation with initial values and boundary conditions.

We consider the coupled of Lane-Emden boundary value problems:

$$\begin{cases} w_i''(x) + \dfrac{k_i}{x} w_i'(x) + f_i(w_1(x), w_2(x)) = 0, \quad k_i \geq 1 \quad i = 1,2 \quad x \in (0,1) \\ w_i'(0) = 0, \quad w_i(1) = c_i, \end{cases} \tag{1.1}$$

where c_i are real constants.

In this work we extend the HAM combined with the Green's function strategy [1] to solve the coupled Lane–Emden boundary value problems in catalytic diffusion reactions of the form (1.1). We will show that using the integral form facilitates the computational work and overcomes the singularity behavior at $x = 0$. The error analysis will be performed by using the residual error functions and the maximal error residual parameters, which demonstrate an approximate exponential rate of convergence.

2 The HAM for integral form of Lane–Emden equations

To convert the coupled of Lane-Emden boundary (1.1) into integral equation, it is written in the following form

$$(x^{k_i} w_i'(x))' = x^{k_i} f_i(w_1(x), w_2(x)), \qquad i = 1, 2. \tag{2.1}$$

Equation (2.1) is integrated twice first from 0 to x and then from x to 1, then by changing the order of integration, and applying the boundary conditions $w_i'(0) = 0,\ w_i(1) = c_i,\ \ i = 1, 2$, we obtain for $k_i = 1$ as

$$w_i(x) = c_i + \int_0^x \ln x\ s^{k_i} f_i(w_1(s), w_2(s))ds + \int_x^1 \ln s\ s^{k_i}\ f_i(w_1(s), w_2(s))ds, \quad i = 1, 2,$$

and for $k_i \neq 1$, we have

$$w_i(x) = c_i + \int_0^x \frac{x^{1-k_i} - 1}{1 - k_i} s^{k_i} f_i(w_1(s), w_2(s))ds + \int_x^1 \frac{s^{1-k_i} - 1}{1 - k_i} s^{k_i} f_i(w_1(s), w_2(s))ds, \quad i = 1, 2.$$

The equivalent the Fredholm integral form of Lane-Emden equation (1.1) (for details see [18]) is given by

$$w_i(x) = c_i + \int_0^1 K_i(x, s) s^{k_i} f_i(w_1, w_2)ds, \qquad i = 1, 2 \tag{2.2}$$

where $K_i(x, s)$ are given below. For $k_i = 1$, $i = 1, 2$

$$K_i(x, s) = \begin{cases} \ln s, & x \leq s, \\ \ln x, & s \leq x \end{cases} \tag{2.3}$$

and for $k_i > 1$, $i = 1, 2$

$$K_i(x, s) = \begin{cases} \dfrac{s^{1-k_i} - 1}{1 - k_i}, & x \leq s, \\ \dfrac{x^{1-k_i} - 1}{1 - k_i}, & s \leq x \end{cases} \tag{2.4}$$

As stated earlier, we will apply in this work the HAM combined with the Green's function strategy [1]. The HAM was developed and improved by S. Liao [29–34] for solving a wide class of functional equations. Various modifications of HAM have been also elaborated, for example, the optimal homotopy asymptotic method (OHAM) was proposed by Marinca and Herisanu [35–38], the optimal homotopy analysis method was proposed in [39–41], and the spectral homotopy analysis method [42]. Other works based on HAM can be found in [1, 43].

According to the HAM, the zero-order deformation equation may be written as

$$(1-q)[\phi_i(x,q) - w_{i0}] = q c_{i0} N_i[\phi_i(x,q)], \qquad i = 1,2, \tag{2.5}$$

where $q \in [0,1]$ is an embedding parameter, w_{i0} are initial guesses, $c_{i0} \neq 0$ are convergence control parameters, $\phi_i(x,q)$ are unknown functions and $N_i[\phi_i(x,q)]$ are defined as

$$N_i[\phi_i(x,q)] = \phi_i(x,q) - c_i - \int_0^1 K_i(x,s) s^{k_i} f_i(\phi_1(s,q), \phi_2(s,q)) ds = 0, \quad i = 1,2. \tag{2.6}$$

At $q = 0$, (2.5) reduces to $\phi_i(x,0) = w_{i,0}$ and at $q = 1$, it leads to $N_i[\phi_i(x,1)] = 0$ which is exactly the same as (2.2) provided that $\phi_i(x,1) = w_i(x)$. Thus, as q increasing form 0 to 1, $\phi_i(x,q)$ moves from w_{i0} to w_i.

We expand $\phi_i(x,q)$ in a Taylor series with respect to q to get

$$\phi_i(x,q) = w_{i0}(x) + \sum_{m=1}^{\infty} w_{im}(x) q^m, \qquad i = 1,2, \tag{2.7}$$

where

$$w_{im}(x) = \frac{1}{m!} \frac{\partial^m \phi_i(x,q)}{\partial q^m}\bigg|_{q=0}, \qquad i = 1,2. \tag{2.8}$$

If the convergence parameter $c_0 \neq 0$ is chosen properly, the series (2.7) converges for $q = 1$ and it becomes

$$\phi_i(x,1) \equiv w_i(x) = w_{i0}(x) + \sum_{m=1}^{\infty} w_{im}(x), \qquad i = 1,2, \tag{2.9}$$

which will be the solutions of the problem (2.2).

We now define the vector

$$\overrightarrow{w}_{im} = \{w_{i0}(x), w_{i1}(x), \ldots, w_{im}(x)\}, \qquad i = 1, 2$$

Differentiating (2.5) m-times with respect to q, dividing them by $m!$, setting subsequently $q = 0$, the mth-order deformation equations are obtained

$$w_{im}(x) - \chi_m \; w_{i(m-1)}(x) = c_{i0} \; R_{im}(\overrightarrow{w}_{i(m-1)}, x), \qquad i = 1, 2 \tag{2.10}$$

where χ_m is given by

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases} \tag{2.11}$$

and

$$\begin{aligned} R_{im}(\overrightarrow{w}_{i(m-1)}, x) &= \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} N_i[\phi_i(x,q)]\Big|_{q=0} = \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} N_i\Big(\sum_{k=0}^{\infty} w_{ik} q^k\Big)\Big|_{q=0} \\ &= w_{i(m-1)}(x) - (1-\chi_m) c_i - \int_0^1 K_i(x,s) \; s^{k_i} \; H_{i(m-1)} ds, \qquad i = 1, 2 \end{aligned} \tag{2.12}$$

where H_{im} are given by

$$H_{i(m)} = \frac{1}{(m)!} \frac{\partial^m}{\partial q^m} f\left(\sum_{k=0}^{\infty} w_{1k} q^k, \sum_{k=0}^{\infty} w_{2k} q^k \right)\Bigg|_{q=0}, \qquad i = 1, 2. \tag{2.13}$$

Using (2.10) and (2.12), the mth-order deformation equations are simplified as

$$w_{im} - \chi_m w_{i(m-1)} = c_{i0} \left(w_{i(m-1)} - (1-\chi_m) c_i - \int_0^1 K_i(x,s) \; s^{k_i} \; H_{i(m-1)} ds \right), \; i = 1, 2. \tag{2.14}$$

Taking $w_{i0} = c_i,\ i = 1, 2$, the solution components will be computed as:

$$\begin{cases} w_{i1}(x) = c_{i0}\left(w_{i0}(x) - c_{i0} - \int\limits_0^1 K_i(x,s)s^{k_i}H_{i0}ds\right), \\ w_{i2}(x) = (1 + c_{i0})\ w_{i1}(x) - c_{i0}\int\limits_0^1 K_i(x,s)s^{k_i}H_{i1}ds, \\ \vdots \\ w_{im}(x) = (1 + c_{i0})\ w_{i(m-1)}(x) - c_{i0}\int\limits_0^1 K_i(x,s)s^{k_i}H_{i(m-1)}ds, \quad m = 3, 4, \ldots. \end{cases} \tag{2.15}$$

The Mth-order approximate solutions of the problem (2.2) are given by

$$\phi_{iM}(x, c_{i0}) = \sum_{m=0}^{M} w_{im}(x, c_{i0}), \quad i = 1, 2. \tag{2.16}$$

To select the appropriate convergence control parameters c_{i0} has a big influence on the convergence region of series (2.16) and on the convergence rate as well [41, 44]. One of the methods for selecting the value of convergence control parameter is the so-called c_{i0}-curve and the horizontal line may be considered as the valid interval for c_{i0} [31, 45]. This method enables to determine the effective region of the convergence control parameter, however it does not give the possibility to determine the value ensuring the fastest convergence [41]. Another way to find the optimal value of the convergence-control parameters c_{i0} is obtained by minimizing the squared residual of governing equation

$$\Delta_{iM}(c_{i0}) = \int_0^1 [N_i(\phi_{iM}(x, c_{i0}))]^2 dx\ , \qquad i = 1, 2. \tag{2.17}$$

However, the exact squared residual error $\Delta_{iM}(c_{i0})$ is expensive to calculate when M is large. For speed up the calculations Liao [40, 41] suggested to replace the integral in formula (2.17) by its approximate value obtained by applying the quadrature rules. So, we approximate $\Delta_{iM}(c_{i0})$ by the so-called discrete averaged residual error defined by

$$\Delta_{iM}(c_{i0}) = \frac{1}{n}\sum_{j=1}^{n}[N_i(\phi_{iM}(x_j, c_{i0}))]^2, \quad i = 1, 2, \tag{2.18}$$

where $x_j = jh$, $h = x_j - x_{j-1}$. The optimal values c_{i0} will be obtained by solving

$$\frac{\partial \Delta_{iM}}{\partial c_{i0}} = 0, \qquad i = 1, 2 \tag{2.19}$$

and then those values will be substituted in (2.16) to get the optimal approximate solutions.

3 The HAM for BVP in catalytic diffusion reactions

Consider the particular case of the coupled Lane–Emden equations (1.1) with the quadratic and product nonlinearities as [25]:

$$\begin{cases} w_1''(x) + \dfrac{2}{x} w_1'(x) - k_{11} w_1^2(x) - k_{12} w_1(x) w_2(x) = 0, \\ w_2''(x) + \dfrac{2}{x} w_2'(x) - k_{21} w_1^2(x) - k_{22} w_1(x) w_2(x) = 0, \end{cases} \tag{3.1}$$

with boundary conditions

$$\begin{cases} w_1'(0) = 0, \quad w_1(1) = c_1, \\ w_2'(0) = 0, \quad w_2(1) = c_2, \end{cases} \tag{3.2}$$

The coupled Lane–Emden equations (3.1) and (3.2) occurs in catalytic diffusion reactions [25]. The parameters $c_1, c_2, k_{11}, k_{12}, k_{21}$ and k_{22} can be specified for the actual chemical reactions. In [25] authors studied the qualitative analysis for the solutions. In [26], Adomian decomposition method was applied to solve (3.1) and (3.2) by fixing parameters $c_1 = 1, c_2 = 2, k_{11} = k_{22} = 1, k_{12} = 2/5, k_{21} = 1/2$. All of the computations have been performed using the MATHEMATICA software.

According to the HAM with Green's function (2.14), we have the following iteration formulation for (3.1) and (3.2) as

$$\begin{cases} w_{1m} - \chi_m w_{1(m-1)} = c_{10} \left[w_{1(m-1)} - (1-\chi_m) c_1 - \displaystyle\int_0^1 K_1(x,s)\, s^2\, H_{1(m-1)} ds \right], \\ w_{2m} - \chi_m w_{2(m-1)} = c_{20} \left[w_{2(m-1)} - (1-\chi_m) c_2 - \displaystyle\int_0^1 K_2(x,s)\, s^2\, H_{2(m-1)} ds \right], \end{cases} \tag{3.3}$$

where $K_i(x,s)$ are given below. For $k_i = 2,\ i = 1,2$

$$K_i(x,s) = \begin{cases} \dfrac{s^{1-k_i}-1}{1-k_i}, & x \le s, \\ \dfrac{x^{1-k_i}-1}{1-k_i}, & s \le x \end{cases} \tag{3.4}$$

3.1 For $c_1 = 1, c_2 = 2, k_{11} = k_{22} = 1, k_{12} = 2/5, k_{21} = 1/2$

Using (3.3) with $w_{10} = c_1, w_{20} = c_2$, and fixing the parameters $c_1 = 1, c_2 = 2, k_{11} = 1, k_{12} = 2/5, k_{21} = 1/2$ and $k_{22} = 1$, the 4th-order approximations are obtained as

$$\begin{aligned}\phi_{14} = 1 &+ \frac{9c_{10}}{10} + \frac{597c_{10}^2}{500} + \frac{55973c_{10}^3}{105000} + \frac{7c_{10}c_{20}}{120} + \frac{1613c_{10}^2c_{20}}{47250} + \frac{65c_{10}c_{20}^2}{3024} - \Big(\frac{9c_{10}}{10} \\ &+ \frac{33c_{10}^2}{25} + \frac{9611c_{10}^3}{15000} + \frac{c_{10}c_{20}}{12} + \frac{1409c_{10}^2c_{20}}{27000} + \frac{67c_{10}c_{20}^2}{2160}\Big)x^2 + \Big(\frac{63c_{10}^2}{500} + \frac{563c_{10}^3}{5000} \\ &+ \frac{c_{10}c_{20}}{40} + \frac{91c_{10}^2c_{20}}{4500} + \frac{7c_{10}c_{20}^2}{720}\Big)x^4 - \Big(\frac{173c_{10}^3}{35000} + \frac{137c_{10}^2c_{20}}{63000} + \frac{c_{10}c_{20}^2}{5040}\Big)x^6. \end{aligned} \tag{3.5}$$

$$\begin{aligned}\phi_{24} = 2 &+ \frac{5c_{20}}{4} + \frac{63c_{10}c_{20}}{200} + \frac{1961c_{10}^2c_{20}}{14000} + \frac{67c_{20}^2}{48} + \frac{673c_{10}c_{20}^2}{5040} + \frac{3139c_{20}^3}{6048} - \Big(\frac{5c_{20}}{4} \\ &+ \frac{9c_{10}c_{20}}{20} + \frac{413c_{10}^2c_{20}}{2000} + \frac{35c_{20}^2}{24} + \frac{713c_{10}c_{20}^2}{3600} + \frac{487c_{20}^3}{864}\Big)x^2 + \Big(\frac{27c_{10}c_{20}}{200} + \frac{141c_{10}^2c_{20}}{2000} \\ &+ \frac{c_{20}^2}{16} + \frac{83c_{10}c_{20}^2}{1200} + \frac{13c_{20}^3}{288}\Big)x^4 + \Big(-\frac{57c_{10}^2c_{20}}{14000} - \frac{13c_{10}c_{20}^2}{2800} - \frac{c_{20}^3}{2016}\Big)x^6. \end{aligned} \tag{3.6}$$

Applying (2.18) and (2.19), we obtain optimal values $c_{10} = -0.767463, c_{20} = -0.789762$ and hence the HAM approximations to the solutions are obtained as

$$\phi_{14}(x) = 0.780767 + 0.191485x^2 + 0.0244069x^4 + 0.00334088x^6. \tag{3.7}$$

$$\phi_{24}(x) = 1.68960 + 0.273372x^2 + 0.0326694x^4 + 0.00436072x^6. \tag{3.8}$$

and by setting $c_{10} = c_{20} = -1$, the ADM approximations to the solutions are obtained as

$$\psi_{14}(x) = 0.763625 + 0.220604x^2 + 0.00845556x^4 + 0.00731587x^6. \tag{3.9}$$

$$\psi_{24}(x) = 1.66822 + 0.30988x^2 + 0.0126944x^4 + 0.00921032x^6. \tag{3.10}$$

To examine the accuracy and applicability of the HAM, we define the residual and max-

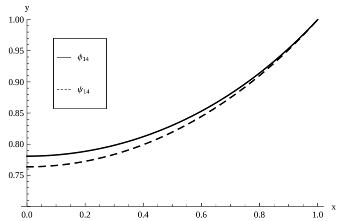

Figure 1 Plots of the HAM $\phi_{14}(x)$ and ADM $\psi_{14}(x)$ solutions

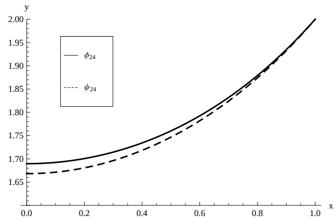

Figure 2 Plots of the HAM $\phi_{24}(x)$ and ADM $\psi_{24}(x)$ solutions

imum absolute residual errors as

$$Res_{iM}(x) = \left|\phi''_{iM} + \frac{2}{x}\phi'_{iM} - k_{11}\phi_{1M}^2 - k_{12}\phi_{1M}^2\phi_{2M}^2\right|, \qquad i = 1, 2 \tag{3.11}$$

$$R_{iM} = \max_{0\le x\le 1} Res_{iM}(x), \qquad i = 1, 2 \tag{3.12}$$

$$res_{iM}(x) = \left|\psi''_{iM} + \frac{2}{x}\psi'_{iM} - k_{11}\psi_{1M}^2 - k_{12}\psi_{1M}^2\psi_{2M}^2\right|, \qquad i = 1, 2 \tag{3.13}$$

$$r_{iM} = \max_{0\le x\le 1} res_{iM}(x), \quad i = 1, 2, \tag{3.14}$$

where ϕ_{iM} are ψ_{iM}, the HAM solutions and are the ADM solutions, respectively.

The numerical results of approximate solutions $(\phi_{i4}, \psi_{i4},\ i = 1, 2)$, the absolute residual errors $(Res_{i4}(x), res_{i4}(x),\ i = 1, 2)$, and the maximum absolute residual errors $(R_{i4}, r_{i4},\ i = 1, 2)$ obtained by the HAM and the ADM are given in Tables 1-3 for $c_1 = 1, c_2 = 2, k_{11} = k_{22} = 1, k_{12} = 2/5, k_{21} = 1/2$ and in Tables 4-6 for $c_1 = 1, c_2 = 2, k_{11} = k_{12} = k_{21} = k_{22} = 1$.

Table 1 The HAM and ADM approximations to solutions

x	ϕ_{14}	ψ_{14}	ϕ_{24}	ψ_{24}
0.0	0.780767047	0.763624868	1.689598095	1.668215608
0.1	0.782684342	0.765831758	1.692335084	1.671315683
0.2	0.788465717	0.772463013	1.700585517	1.680631694
0.3	0.798200840	0.783553024	1.714469358	1.696214314
0.4	0.812043169	0.799167888	1.734191780	1.718159052
0.5	0.830215964	0.819418576	1.760051017	1.746622830
0.6	0.853020705	0.844479370	1.792449348	1.781847192
0.7	0.880847918	0.874611567	1.831907230	1.824188148
0.8	0.914190405	0.910192446	1.879080563	1.874152645
0.9	0.953658877	0.951749513	1.934781102	1.932441674
1.0	1.000000000	1.000000000	2.000000000	2.000000000

Table 2 The absolute residual errors

x	$Res_{14}(x)$	$res_{14}(x)$	$Res_{24}(x)$	$res_{24}(x)$
0.0	0.011640586	0.230942914	0.016249657	0.766598470
0.1	0.011385744	0.226867128	0.015921395	0.763378418
0.2	0.010641294	0.214888858	0.014966847	0.754062831
0.3	0.009458900	0.195746198	0.013466018	0.739676243
0.4	0.007895555	0.170637696	0.011515827	0.721894603
0.5	0.005967766	0.141171286	0.009172894	0.702994203
0.6	0.003582957	0.109285297	0.006368090	0.685772693
0.7	0.000443210	0.077133580	0.002786970	0.673434203
0.8	0.004085554	0.046923639	0.002292183	0.669427479
0.9	0.011148438	0.020692579	0.010214085	0.677221825
1.0	0.022633272	8.41341E-17	0.023231283	0.700000000

Table 3 The maximum absolute residual errors

M	R_{1M}	r_{1M}	R_{2M}	r_{2M}
2	4.665E-01	8.666E-01	5.998E-01	1.56667
3	9.750E-02	4.428E-01	6.772E-02	7.100E-01
4	1.164E-02	2.309E-01	1.624E-02	7.665E-01
5	2.451E-03	1.267E-01	3.370E-02	7.100E-01
6	6.333E-04	7.155E-02	8.600E-04	7.100E-01
7	1.395E-04	4.160E-02	1.867E-04	7.100E-01
8	3.632E-05	2.467E-02	4.846E-05	7.100E-01
9	1.607E-05	1.490E-02	2.161E-05	7.100E-01

3.2 For $c_1 = 1, c_2 = 2, k_{11} = k_{12} = k_{21} = k_{22} = 1$

Taking $w_{10} = c_1 = 1$, and $w_{20} = c_2 = 2$, and fixing the parameters $k_{11} = k_{12} = k_{21} = k_{22} = 1$, we obtain

$$\phi_{14} = 1 + \frac{3c_{10}}{2} + \frac{11c_{10}^2}{5} + \frac{2741c_{10}^3}{2520} + \frac{7c_{10}c_{20}}{40} + \frac{41c_{10}^2c_{20}}{315} + \frac{65c_{10}c_{20}^2}{1008} - \left(\frac{3c_{10}}{2} + \frac{5c_{10}^2}{2} + \frac{491c_{10}^3}{360} + \frac{c_{10}c_{20}}{4} + \frac{73c_{10}^2c_{20}}{360} + \frac{67c_{10}c_{20}^2}{720}\right)x^2 + \left(\frac{3c_{10}^2}{10} + \frac{7c_{10}^3}{24} + \frac{3c_{10}c_{20}}{40} + \frac{c_{10}^2c_{20}}{12} + \frac{7c_{10}c_{20}^2}{240}\right)x^4 - \left(\frac{13c_{10}^3}{840} + \frac{3c_{10}^2c_{20}}{280} + \frac{c_{10}c_{20}^2}{1680}\right)x^6.$$

$$\phi_{24} = 2 + \frac{3c_{20}}{2} + \frac{7c_{10}c_{20}}{10} + \frac{893c_{10}^2c_{20}}{2520} + \frac{67c_{20}^2}{40} + \frac{769c_{10}c_{20}^2}{2520} + \frac{3139c_{20}^3}{5040} - \left(\frac{3c_{20}}{2} + c_{10}c_{20} + \frac{191c_{10}^2c_{20}}{360} + \frac{7c_{20}^2}{4} + \frac{163c_{10}c_{20}^2}{360} + \frac{487c_{20}^3}{720}\right)x^2 + \left(\frac{3c_{10}c_{20}}{10} + \frac{23c_{10}^2c_{20}}{120} + \frac{3c_{20}^2}{40} + \frac{19c_{10}c_{20}^2}{120} + \frac{13c_{20}^3}{240}\right)x^4 - \left(\frac{13c_{10}^2c_{20}}{840} + \frac{3c_{10}c_{20}^2}{280} + \frac{c_{20}^3}{1680}\right)x^6.$$

Applying (2.18) and (2.19), we obtain optimal values $c_{10} = -0.689796, c_{20} = -0.708697$ and hence the HAM approximations to the solutions are obtained as

$$\phi_{14}(x) = 0.674423 + 0.271204x^2 + 0.0454739x^4 + 0.00889876x^6,$$
$$\phi_{24}(x) = 1.67352 + 0.27178x^2 + 0.0455586x^4 + 0.00914259x^6,$$

and by setting $c_{10} = c_{20} = -1$, the ADM approximations to the solutions are obtained as

$$\psi_{14}(x) = 0.592659 + 0.409722x^2 - 0.0291667x^4 + 0.0267857x^6,$$
$$\psi_{24}(x) = 1.59266 + 0.409722x^2 - 0.0291667x^4 + 0.0267857x^6.$$

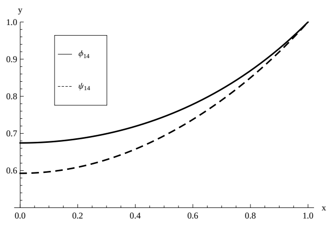

Figure 3 Plots of HAM $\phi_{14}(x)$ and ADM $\psi_{14}(x)$ solutions

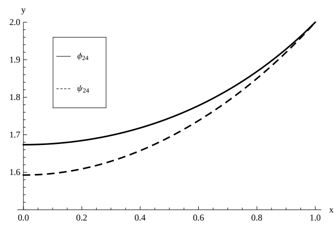

Figure 4 Plots of HAM $\phi_{24}(x)$ and ADM $\psi_{24}(x)$ solutions

Table 4 The HAM and ADM approximations to solutions

x	ϕ_{14}	ψ_{14}	ϕ_{24}	ψ_{24}
0.0	0.674423143	0.592658730	1.673518521	1.592658730
0.1	0.677139742	0.596753063	1.676240889	1.596753063
0.2	0.685344640	0.609002667	1.684463213	1.609002667
0.3	0.699206350	0.629317007	1.698354439	1.629317007
0.4	0.719016401	0.657577333	1.718207120	1.657577333
0.5	0.745205361	0.693684896	1.744453865	1.693684896
0.6	0.778365260	0.737628444	1.777690385	1.737628444
0.7	0.819278422	0.789571015	1.818705108	1.789571015
0.8	0.868952703	0.849956000	1.868515392	1.849956000
0.9	0.928663139	0.919632507	1.928410306	1.919632507
1.0	1.000000000	1.000000000	2.000000000	2.000000000

Table 5 The absolute residual errors

x	$Res_{14}(x)$	$res_{14}(x)$	$Res_{24}(x)$	$res_{24}(x)$
0.0	0.043719198	1.163185862	0.047175755	1.163185862
0.1	0.042789989	1.143631002	0.046264512	1.143631002
0.2	0.040067370	1.086028837	0.043608077	1.086028837
0.3	0.035715976	0.993549036	0.039407953	0.993549036
0.4	0.029906016	0.871406768	0.033895791	0.871406768
0.5	0.022646550	0.726730135	0.027166674	0.726730135
0.6	0.013530025	0.568313445	0.018923646	0.568313445
0.7	0.001363158	0.406196710	0.008108600	0.406196710
0.8	0.016352278	0.250993596	0.007616937	0.250993596
0.9	0.044140746	0.112865431	0.032593096	0.112865431
1.0	0.089549318	1.17961E-16	0.074158043	1.17961E-16

Table 6 The maximum absolute residual errors

M	R_{1M}	r_{1M}	R_{2M}	r_{2M}
2	1.0171100	2.00000	0.976732	2.000000
3	0.2902730	1.54514	0.142540	1.545140
4	4.371E-02	1.16319	4.717E-02	1.163190
5	2.824E-02	0.96358	1.272E-02	0.963584
6	4.308E-03	0.79717	4.546E-03	0.797179
7	1.188E-03	0.69863	1.263E-03	0.698631
8	1.288E-04	0.55557	1.367E-04	0.555577
9	1.721E-05	0.50333	1.795E-05	0.503336

Remark 3.1. One can note that in the Tables 6, we observe that the HAM (present method) gives stable and convergent solution.

4 Conclusion

We have examined a system of coupled Lane-Emden BVPs that models many physical and chemical phenomena such as catalytic diffusion reactions. We employed the HAM combined with the Green's function strategy [1]. Our approach enhances the computational efficiency while overcoming the difficulty of the singular behavior at the origin $x = 0$. The HAM was used systematically in a straightforward manner. The obtained results were supported by proper figures to show the power of the method and to show the enhancements over exiting techniques such as the ADM [26]

References

[1] R. Singh, Optimal homotopy analysis method for the non-isothermal reaction–diffusion model equations in a spherical catalyst, *J. Math. Chem.* **56** (2018) 2579–2590.

[2] O. W. Richardson, *The Emission of Electricity From Hot Bodies*, Longmans, Green and Company, London, 1921.

[3] A.M. Wazwaz, Solving the non-isothermal reaction–diffusion model equations in a spherical catalyst by the variational iteration method, *Chem. Phys. Lett.* **679** (2017) 132–136.

[4] K. Reger, R. Van Gorder, Lane–Emden equations of second kind modelling thermal explosion in infinite cylinder and sphere, *Appl. Math. Mech.* **34** (2013) 1439–1452.

[5] R. A. Van Gorder, Exact first integrals for a Lane–Emden equation of the second kind modeling a thermal explosion in a rectangular slab, *New Astronomy* **16** (2011) 492–497.

[6] A. M. Wazwaz, A new algorithm for solving differential equations of Lane–Emden type, *Appl. Math. Comput.* **118** (2001) 287–310.

[7] M. Inc, D. Evans, The decomposition method for solving of a class of singular two–point boundary value problems, *Int. J. Comput. Math.* **80** (2003) 869–882.

[8] S. Khuri, A. Sayfy, A novel approach for the solution of a class of singular boundary value problems arising in physiology, *Math. Comput. Modell.* **52** (2010) 626–636.

[9] A. Ravi Kanth, K. Aruna, He's variational iteration method for treating nonlinear singular boundary value problems, *Comput. Math. Appl.* **60** (2010) 821–829.

[10] A. Ebaid, A new analytical and numerical treatment for singular two–point boundary value problems via the Adomian decomposition method, *J. Comput. Appl. Math.* **235** (2011) 1914–1924.

[11] A. M. Wazwaz, R. Rach, Comparison of the Adomian decomposition method and the variational iteration method for solving the Lane–Emden equations of the first and second kinds, *Kybernetes* **40** (2011) 1305–1318.

[12] A. M. Wazwaz, The variational iteration method for solving systems of equations of Emden–Fowler type, *Int. J. Comp. Math.* **88** (2011) 3406–3415.

[13] M. Danish, S. Kumar, S. Kumar, A note on the solution of singular boundary value problems arising in engineering and applied sciences: Use of OHAM, *Comput. Chem. Eng.* **36** (2012) 57–67.

[14] A. M. Wazwaz, R. Rach, J. S. Duan, Adomian decomposition method for solving the volterra integral form of the Lane–Emden equations with initial values and boundary conditions, *Appl. Math. Comput.* **219** (2013) 5004–5019.

[15] R. Singh, J. Kumar, G. Nelakanti, Numerical solution of singular boundary value problems using Green's function and improved decomposition method, *J. Appl. Math. Comp.* **43** (2013) 409–425.

[16] R. Singh, N. Das, J. Kumar, The optimal modified variational iteration method for the Lane–Emden equations with Neumann and Robin boundary conditions, *Eur. Phys. J. Plus* **132** (2017) #251.

[17] R. Singh, A. M. Wazwaz, J. Kumar, An efficient semi–numerical technique for solving nonlinear singular boundary value problems arising in various physical models, *Int. J. Comput. Math.* **93** (2016) 1330–1346.

[18] R. Singh, J. Kumar, An efficient numerical technique for the solution of nonlinear singular boundary value problems, *Comput. Phys. Commun.* **185** (2014) 1282–1289.

[19] R. Singh, J. Kumar, The Adomian decomposition method with Green's function for solving nonlinear singular boundary value problems, *J. Appl. Math. Comput.* **44** (2014) 397–416.

[20] R. Singh, A. M. Wazwaz, Numerical solution of the time dependent Emden–Fowler equations with boundary conditions using modified decomposition method, *Appl. Math. Inf. Sci.* **10** (2016) 403–408.

[21] N. Das, R. Singh, A. M. Wazwaz, J. Kumar, An algorithm based on the variational iteration technique for the Bratu–type and the Lane–Emden problems, *J. Math. Chem.* **54** (2016) 527–551.

[22] R. Singh, S. Singh, A. M. Wazwaz, A modified homotopy perturbation method for singular time dependent Emden–Fowler equations with boundary conditions, *J. Math. Chem.* **54** (2016) 918–931.

[23] A. M. Wazwaz, R. Rach, J. S. Duan, Variational iteration method for solving oxygen and carbon substrate concentrations in microbial floc particles, *MATCH Commun. Math. Comp. Chem.* **76** (2016) 511–523.

[24] B. Muatjetjeja, C. M. Khalique, Noether, partial noether operators and first integrals for the coupled Lane–Emden system, *Math. Comput. Appl.* **15** (2010) 325–333.

[25] D. Flockerzi, K. Sundmacher, On coupled Lane–Emden equations arising in dusty fluid models, *J. Phys.* **268** (2011) #012006.

[26] R. Rach, J. S. Duan, A. M. Wazwaz, Solving coupled Lane–Emden boundary value problems in catalytic diffusion reactions by the Adomian decomposition method, *J. Math. Chem.* **52** (2014) 255–267.

[27] A. M. Wazwaz, R. Rach, J. S. Duan, A study on the systems of the Volterra integral forms of the Lane–Emden equations by the Adomian decomposition method, *Math. Meth. Appl. Sci.* **37** (2014) 10–19.

[28] T. C. Hao, F. Z. Cong, Y. F. Shang, An efficient method for solving coupled Lane–Emden boundary value problems in catalytic diffusion reactions and error estimate, *J. Math. Chem.* **56** (2018) 269–2706

[29] S. J. Liao, *The proposed homotopy analysis technique for the solution of nonlinear problems*, Ph. D. Thesis, Shanghai Jiao Tong Univ., 1992.

[30] S. J. Liao, An approximate solution technique not depending on small parameters: a special example, *Int. J. Non–Line. Mech.* **30** (1995) 371–380.

[31] S. J. Liao, *Beyond Perturbation: Introduction to the Homotopy Analysis Method*, CRC Press, Boca Raton, 2003.

[32] S. J. Liao, On the homotopy analysis method for nonlinear problems, *Appl. Math. Comput.* **147** (2004) 499–513.

[33] S. J Liao, Y. Tan, A general approach to obtain series solutions of nonlinear differential equations, *Stud. Appl. Math.* **119** (2007) 297–354.

[34] S. J Liao, Series solution of nonlinear eigenvalue problems by means of the homotopy analysis method, *Nonlin. Anal. Real. World Appl.* **10** (2009) 2455–2470.

[35] V. Marinca, N. Herişanu, Application of optimal homotopy asymptotic method for solving nonlinear equations arising in heat transfer, *Int. Commun. Heat. Mass Trans.* **35** (2008) 710–715.

[36] V. Marinca, N. Herişanu, I. Nemeş, Optimal homotopy asymptotic method with application to thin film flow, *Central Eur. J. Phys.* **6** (2008) #648.

[37] V. Marinca, N. Herişanu, C. Bota, B. Marinca, An optimal homotopy asymptotic method applied to the steady flow of a fourth–grade fluid past a porous plate, *Appl. Math. Lett.* **22** (2009) 245–251.

[38] N. Herişanu, V. Marinca, Accurate analytical solutions to oscillators with discontinuities and fractional–power restoring force by means of the optimal homotopy asymptotic method, *Comput. Math. Appl.* **60** (2010) 1607–1615.

[39] Z. Niu, C. Wang, A one–step optimal homotopy analysis method for nonlinear differential equations, *Commun. Nonlin. Sci. Num. Simul.* **15** (2010) 2026–2036.

[40] S. J. Liao, An optimal homotopy–analysis approach for strongly nonlinear differential equations, *Commun. Nonlin. Sci. Num. Simul.* **15** (2010) 2003–2016.

[41] S. J. Liao, *Homotopy Analysis Method in Nonlinear Differential Equations*, Springer, Heidelberg, 2012.

[42] H. S. Nik, S. Effati, S. S. Motsa, M. Shirazian, Spectral homotopy analysis method and its convergence for solving a class of nonlinear optimal control problems, *Num. Alg.* **65** (2014) 171–194.

[43] R. Singh, A. M. Wazwaz, Optimal homotopy analysis method for oxygen diffusion in a spherical cell with nonlinear oxygen uptake kinetics, *MATCH Commun. Math. Comp. Chem.* **80** (2018) 369–382.

[44] Z. M. Odibat, A study on the convergence of homotopy analysis method, *Appl. Math. Comput.* **217** (2010) 782–789.

[45] E. Hetmaniok, D. Słota, T. Trawiński, R. Wituła, Usage of the homotopy analysis method for solving the nonlinear and linear integral equations of the second kind, *Num. Alg.* **67** (2014) 163–185.