MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **80** (2018) 369-382

ISSN 0340 - 6253

Optimal Homotopy Analysis Method for Oxygen Diffusion in a Spherical Cell with Nonlinear Oxygen Uptake Kinetics

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(Received February 10, 2018)

Abstract

We investigate the diffusion of oxygen in a spherical cell including nonlinear uptake kinetics. We study the Lane-Emden boundary value problem, with Michaelis-Menten kinetics, to model the dimensionless oxygen concentration. Due to the presence of singularity, this problem pose difficulties in obtaining its solutions. To overcome the singular behavior at the spherical origin of the cell, Lane-Emden equation is transformed into an equivalent Fredholm integral equation. The optimal homotopy analysis method (OHAM) is applied to solve the Lane-Emden integral form for studying the concentration of oxygen within the spherical cell. To speed up the calculations, the discrete averaged residual error is used to obtain optimal value of the adjustable parameter c_0 to control the convergence of solution. Unlike, Adomian decomposition method (ADM) [1], the proposed method contains an adjustable parameter c_0 to control the convergence of series solutions.

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1 Introduction

In [2], Lin tried to predict the oxygen tension in a spherical cell by using an oxygen uptake kinetics of the Michaelis-Menten type. If V is the maximum reaction rate, P the oxygen tension and k_m the Michaelis-Menten constant, then unsteady state oxygen diffusion in a spherical cell can be expressed mathematically as

$$\frac{\partial P}{\partial t} = D\left(\frac{\partial^2 P}{\partial r^2} + \frac{2}{r}\frac{\partial P}{\partial r}\right) - \frac{VP}{P + k_m}, \tag{1.1}$$

and the initial and boundary conditions are given by

$$t = 0; \qquad P = 0, \tag{1.2}$$

$$r = 0; \qquad \frac{\partial P}{\partial r} = 0, \tag{1.3}$$

$$r = r_0; \qquad D\frac{\partial P}{\partial r} = h(P_0 - P), \tag{1.4}$$

in which D is the diffusion coefficient of oxygen in the protoplasm, r_0 the radius of cell, h the permeability of membrane, r the radial co-ordinate and t the time. For convenience of computational purpose, equation (1.1) and the initial and boundary conditions can be cast into dimensionless form by introducing the following dimensionless variables and parameters

$$C = \frac{P}{P_0}, \quad \tau = \frac{tD}{r_0^2}, \quad R = \frac{r}{r_0}, \quad \alpha = \frac{Vr_0^2}{P_0 D}, \quad K = \frac{k_m}{P_0}, \quad H = \frac{hr_0}{D}.$$

Equations (1.1)–(1.4) are then transformed into

$$\frac{\partial C}{\partial \tau} = \frac{\partial^2 C}{\partial R^2} + \frac{2}{R}\frac{\partial C}{\partial R} - \alpha\frac{C}{C + K_m}, \tag{1.5}$$

subject to the initial and boundary conditions

$$\tau = 0; \qquad C = 0, \tag{1.6}$$

$$R = 0; \qquad \frac{\partial C}{\partial R} = 0, \tag{1.7}$$

$$R = 1; \qquad \frac{\partial C}{\partial R} = H(1 - C). \tag{1.8}$$

In [3], McElwain considered the steady state and reduced (1.5)–(1.8) into nonlinear sin-

gular boundary value problems in the Lane-Emden form [1, 3–5] as

$$C''(R) + \frac{2}{R}C'(R) - \alpha\frac{C(R)}{C(R)+K} = 0, \quad 0 < R < 1, \quad \text{where } ' = \frac{d}{dR}, \tag{1.9}$$

subject to the boundary conditions

$$C'(0) = 0, \qquad C'(1) + HC(1) = H. \tag{1.10}$$

Here, the variables C and R represent the oxygen concentration and the radial distance, respectively. The parameters α, K and H are real constants that represent the maximum reaction rate, the Michaelis constant, which is the half-saturation concentration, and the permeability of the cell membrane, respectively [2, 3].

The boundary condition at $R = 0$ ensures that the oxygen distribution is symmetric at the center of the sphere whereas the boundary condition at $R = 1$ states that the flux of oxygen across the cell membrane, which is one less than the normalized concentration at the cell membrane [6, 7]. Since the metabolic reactions in a cell are catalyzed by enzymes, that is, it is plausible to express the oxygen uptake kinetics by the Michaelis-Menten equation: $\textit{oxygen uptake}=\alpha\frac{C(R)}{C(R)+K}$.

The Lane–Emden equation has been used to model several phenomena in mathematical physics and astrophysics, chemical reaction systems, and biological population phenomena as described in [8–15] and some of the references cited therein. In [2], Lin presented a numerical approximation which was later re-examined in [3] by McElwain by using the fourth-order Runge–Kutta method and the bisection method to compute his approximations, which showed a substantially different result compared to that obtained in [2]. In [16], Hiltmann and Lory used the multiple shooting method to show that the problem has unique solution in the physically feasible range. In [7], Simpson and Ellery applied the Maclaurin series to derive their approximations and examined the convergence and limitations of those approximations. Moreover, in [1], the author used Volterra-integral form of the Lane-Emden equation and the Adomian decomposition method to solve the problem analytically. In [4], a nonlinear model representing oxygen diffusion accompanied by the Michaelis-Menten consumption kinetics inside a spherical cell was solved analytically by the differential transform method (DTM) and the modified Adomian decomposition method (MADM).

The purpose of this paper is to apply the optimal homotopy analysis method (OHAM) to find accurate approximate solutions of nonlinear Lane-Emden equation (1.1). To overcome the singular behavior at the spherical origin of the cell, the Lane-Emden equation is transformed into an equivalent Fredholm integral equation and then the OHAM is applied to get approximate solutions. The most significant feature of the OHAM is the optimal control of the convergence of solutions by a convergence-control parameter which ensures a very fast convergence. The OHAM has been proved to be reliable and efficient. In summary, the OHAM has the following advantages:

- Unlike ADM or MADM, the present approach does not require any additional computational work for unknown constants;
- Guarantee of convergence;
- Flexibility on choice of base function and initial guess of solution;
- Useful analytic tool to investigate highly nonlinear problems with multiple solutions, singularity and perturbed.

2 Description of the method

Our solution approach is very flexible and we will demonstrate this by studying a generalization of Eqs. (1.9)-(1.10) which we write as

$$C''(R) + \frac{2}{R}C'(R) - f(C) = 0, \quad 0 < R < 1 \tag{2.1}$$

subject to the boundary conditions

$$C'(0) = 0, \qquad C'(1) + HC(1) = H. \tag{2.2}$$

Comparing (1.9) and (2.1) we see that (2.1) is valid for any uptake model $f(C)$. By setting Michaelis-Menten nonlinearity

$$f(C) = \alpha \frac{C(R)}{C(R) + K},$$

we recover the original non dimensional model (1.9).

Our aim is now to solve the general problem (2.1). To do so, we first convert (2.1)-(2.2) into integral equation. Integrating (2.1) w.r.to R first from 0 to R and then from R to 1, then changing the order of integration, and applying the boundary conditions $C'(0) = 0$ and $C'(1) + HC(1) = H$, and with appropriate algebraic manipulations (for additional details [17]) we get the equivalent the Fredholm integral form as

$$C(R) = 1 + \int_0^1 G(R, s)s^2 f(C(s))ds, \tag{2.3}$$

$$\mathcal{N}[C(R)] = C(R) - 1 - \int_0^1 G(R, s)s^2 f(C(s))ds = 0 \tag{2.4}$$

where Green's function $G(R, s)$ is given by

$$G(R, s) = \begin{cases} \left(1 - \dfrac{1}{s}\right) - \dfrac{1}{H}, & 0 < R \leq s \\ \left(1 - \dfrac{1}{R}\right) - \dfrac{1}{H}, & 0 < s \leq R \end{cases} \tag{2.5}$$

According to homotopy analysis method or optimal homotopy asymptotic method [18–23], we use $q \in [0, 1]$ as an embedding parameter, the general zero-order deformation equation is constructed as

$$(1 - q)[\phi(R, q) - C_0(R)] = q\, c_0\, \mathcal{N}[\phi(R, q)]. \tag{2.6}$$

Here, $C_0(R)$ denotes an initial guess for the exact solution $C(R)$, $c_0 \neq 0$ is convergence-controller parameter, $\phi(R, q)$ is an unknown function and $\mathcal{N}[\phi(R, q)]$ is given by

$$\mathcal{N}[\phi(R, q)] = \phi(R, q) - 1 - \int_0^1 G(R, s)s^2 f(\phi(s, q))ds = 0. \tag{2.7}$$

The zero-order deformation (2.6) becomes $\phi(R, 0) = C_0(R)$ at $q = 0$, and it leads to $\mathcal{N}[\phi(R, 1)] = 0$, at $q = 1$, which is exactly the same as the original problem (2.3) provided that $\phi(R, 1) = C(R)$.

Expanding $\phi(R, q)$ in a Taylor series with respect to the parameter q, we obtain

$$\phi(R, q) = C_0(R) + \sum_{m=1}^{\infty} C_m(R)q^m, \tag{2.8}$$

where $C_m(R)$ is given by

$$C_m(R) = \frac{1}{m!}\frac{\partial^m}{\partial q^m}[\phi(R,q)]\bigg|_{q=0}. \tag{2.9}$$

The series (2.8) converges for $q = 1$ if $c_0 \neq 0$ is chosen properly and it becomes

$$\phi(R,1) \equiv C(R) = C_0(R) + \sum_{m=1}^{\infty} C_m(R), \tag{2.10}$$

which will be one of solutions of the problem (2.3).

Defining the vector $\overrightarrow{C}_m = \{C_0(R), C_1(R), \ldots, C_m(R)\}$ and differentiating Eq. (2.6), m times with respect to the parameter q, dividing it by $m!$, setting subsequently $q = 0$, the mth-order deformation equation is obtained

$$C_m(R) - \chi_m\, C_{m-1}(R) = c_0\; R_m(\overrightarrow{C}_{m-1}, R), \tag{2.11}$$

where χ_m is given by

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases} \tag{2.12}$$

and

$$\begin{aligned} R_m(\overrightarrow{C}_{m-1}, R) &= \frac{1}{(m-1)!}\left[\frac{\partial^{m-1}}{\partial q^{m-1}}\mathcal{N}[\phi(R,q)]\right]\bigg|_{q=0} \\ &= \frac{1}{(m-1)!}\left[\frac{\partial^{m-1}}{\partial q^{m-1}}\mathcal{N}\Big(\sum_{k=0}^{\infty} C_k q^k\Big)\right]\bigg|_{q=0} \\ &= C_{m-1}(R) - (1-\chi_m)\frac{c}{a} - \int_0^1 G(R,s)\; s^{\alpha}\; \mathcal{D}_{m-1}[f(\phi)]\; ds \end{aligned} \tag{2.13}$$

where $\mathcal{D}_{m-1)}[f(\phi)]$ is the $(m-1)$th-order homotopy-derivative operator [24] given by

$$\mathcal{D}_{m-1}[f(\phi)] = \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}} f\Big(\sum_{k=0}^{\infty} C_k q^k\Big)\bigg|_{q=0}. \tag{2.14}$$

Using (2.11) and (2.13), the mth-order deformation equation is simplified as

$$C_m(R) - \chi_m C_{m-1}(R) = c_0\left[C_{m-1}(R) - (1-\chi_m) - \int_0^1 G(R,s) s^2 \mathcal{D}_{m-1}[f(\phi)] ds\right]. \tag{2.15}$$

In this paper, we choose an initial guess $C_0(R) = 1$, the solution components $C_m,\ m = 1, 2, \ldots$ are successively obtained as

$$C_1(R) = c_0 \left[C_0(R) - 1 - \int_0^1 G(R,s)\ s^2\ \mathcal{D}_0[f(\phi)]ds \right] \tag{2.16}$$

$$C_m(R) = (1 + c_0)\ C_{m-1}(R) - c_0 \left[\int_0^1 G(R,s)\ s^2\ \mathcal{D}_{m-1}[f(\phi)]ds \right], \quad m = 2, 3, \ldots \tag{2.17}$$

The Mth-order approximate solution of the problem (2.3) is given by

$$\varphi_M(R, c_0) = C_0(R) + \sum_{m=1}^{M} C_m(R, c_0). \tag{2.18}$$

Appropriate selection of the convergence control parameter c_0 has a big influence on the convergence region of series (2.10) and on the convergence rate as well [24, 25]. One of the methods for selecting the value of convergence control parameter is the so-called c_0-curve and the horizontal line may be considered as the valid interval for c_0 [19, 26]. This method enables to determine the effective region of the convergence control parameter, however it does not give the possibility to determine the value ensuring the fastest convergence [24]. Another way to find the optimal value of the convergence-control parameter c_0 is obtained by minimizing the squared residual of governing equation

$$E_M(c_0) = \int_0^1 (\mathcal{N}[\varphi_M(R, c_0)])^2 dR. \tag{2.19}$$

The squared residual error defined by (2.19) is a kind of measurement of the accuracy of the Mth-order approximation. However, the exact squared residual error is expensive to calculate when M is large. For speed up the calculations Liao [24, 27] suggested to replace the integral in formula (2.19) by its approximate value obtained by applying the quadrature rules. So, we approximate E_M by the discrete averaged residual error defined by

$$E_M(c_0) \approx \frac{1}{n} \sum_{j=1}^{n} (\mathcal{N}[\varphi_M(jh, c_0)])^2, \tag{2.20}$$

where $0 = R_1 < R_2 < \ldots R_{j-1} < R_j < \ldots < R_n = 1$ with $R_j = jh$, $h = R_j - R_{j-1}$. The optimal value c_0 is obtained by solving $\frac{dE_M}{dc_0} = 0$, and optimal value will satisfy

$E_M(\hat{c}_0) < E_M(c_0)$. Having computed the optimal value $\hat{c}_0$ and substituting in (2.18), the approximate solution will be obtained.

3 Numerical Results

In this paper, we consider the effects of the biologically relevant parameters on the dimensionless oxygen concentration $C(R, \alpha, K, H)$ by using OHAM and ADMGF approximations to perform parametric simulations. To examine the accuracy and applicability of the OHAM, we define the residual and maximum absolute residual errors as

$$E^M_{res}(R) = \left|\varphi''_M + \frac{2}{R}\varphi'_M - \alpha\frac{\varphi_M}{\varphi_M + K}\right| \quad \text{and} \quad E^M = \max_{0\le R\le 1} |E^M_{res}(R)| \tag{3.1}$$

$$e^M_{res}(R) = \left|\psi''_M + \frac{2}{x}\psi'_M - \alpha\frac{\psi_M}{\psi_M + K}\right| \quad \text{and} \quad e^M = \max_{0\le R\le 1} |e^M_{res}(R)|, \tag{3.2}$$

where φ_M is the Mth-order OHAM solution and $\psi_M = \varphi_M(R, -1)$ is the Adomian decomposition method with Green's function (ADMGF) solution. Note that the optimal homotopy analysis method (2.15) reduce to ADMGF when parameter $c_0 = -1$ (for details [17]).

3.1 For $H = 5$

Applying (2.15) with an initial guess $C_0(R) = 1$, we approximate solution $\varphi_M(R)$. Using (2.19), we obtain optimal values $[\hat{c}_0 = -1.045949, -1.010201]$ for $(\alpha = 0.76129, K = 0.03119)$ and $[\hat{c}_0 = -0.920111, -0.909001]$ for $(\alpha = 4, K = 3)$ with $M = 5$ and $M = 10$, respectively. In Tables 1 and 2 we give the comparison of the absolute residual errors and the approximate solutions obtained by OHAM and ADMGF, respectively. Tables show a rapid rate of convergence for low-stage approximations by both the methods and the residual error approaches to zero.

Table 1 Results of residual error and solution $\alpha = 0.76129$, $K = 0.03119$, $H = 5$ in [17, 28]

R	e^5_{res}	E^5_{res}	e^{10}_{res}	E^{10}_{res}	ψ_{10}(ADMGF)	φ_{10}(OHAM)
0.0	2.91E-06	8.32E-07	2.13E-10	1.14E-10	0.82848329	0.82848329
0.1	2.80E-06	7.95E-07	1.95E-10	1.04E-10	0.829706092	0.829706092
0.2	2.49E-06	6.94E-07	1.50E-10	7.76E-11	0.833374734	0.833374734
0.3	2.03E-06	5.54E-07	9.42E-11	4.61E-11	0.839489914	0.839489914
0.4	1.50E-06	4.07E-07	4.56E-11	2.00E-11	0.848052785	0.848052785
0.5	9.93E-07	2.83E-07	1.45E-11	4.59E-12	0.859064927	0.859064927
0.6	5.66E-07	2.01E-07	6.60E-13	1.23E-12	0.872528320	0.87252832
0.7	2.63E-07	1.59E-07	2.52E-12	1.82E-12	0.888445306	0.888445306
0.8	8.70E-08	1.49E-07	1.77E-12	9.98E-13	0.906818548	0.906818548
0.9	1.13E-08	1.52E-07	7.45E-13	4.01E-13	0.927650988	0.927650988
1.0	5.97E-09	1.54E-07	2.54E-13	1.69E-13	0.950945798	0.950945798

Table 2 Results of residual error and solution when $\alpha = 4$, $K = 3$, $H = 5$

R	e^5_{res}	E^5_{res}	e^{10}_{res}	E^{10}_{res}	ψ_{10}(ADMGF)	φ_{10}(OHAM)
0.0	0	0	0	0	0.794152518	0.794152518
0.1	1.20E-06	1.52E-07	5.99E-11	5.72E-12	0.795548497	0.795548497
0.2	2.41E-06	2.26E-07	1.17E-10	9.14E-12	0.799743414	0.799743414
0.3	3.62E-06	1.62E-07	1.68E-10	8.92E-12	0.806758213	0.806758213
0.4	4.75E-06	6.44E-08	2.08E-10	5.13E-12	0.816627817	0.816627817
0.5	5.71E-06	4.32E-07	2.29E-10	1.09E-12	0.829401148	0.829401148
0.6	6.35E-06	8.58E-07	2.26E-10	8.16E-12	0.845141155	0.845141155
0.7	6.52E-06	1.20E-06	1.98E-10	1.44E-11	0.863924839	0.863924839
0.8	6.12E-06	1.26E-06	1.52E-10	1.78E-11	0.885843276	0.885843276
0.9	5.12E-06	8.16E-07	9.91E-11	1.48E-11	0.911001636	0.911001636
1.0	3.64E-06	3.71E-07	5.16E-11	4.40E-13	0.939519180	0.939519180

3.2 For $K = 2, H = 4$

We fix the Michaelis constant $K = 2$ and permeability of the cell membrane parameter $H = 4$, we consider the influence of the maximum reaction rate for different values of α on the dimensionless oxygen concentration $C(R, \alpha, K = 2, H = 4)$ which is approximated by OHAM and ADMGF methods explained in section 2. The numerical results of the maximum residual errors and approximate solutions are shown in Tables 3 and 4.

Table 3 Maximum absolute residual error e^M and E^M when $K = 2, H = 4$

α	e^5	E^5	e^{10}	E^{10}
0.5	8.21E-11	1.51E-12	1.21E-14	2.40E-16
1.5	4.90E-08	7.91E-09	1.10E-13	6.50E-15
2.5	1.39E-06	3.08E-08	2.44E-11	5.13E-13
3.5	1.08E-05	2.45E-07	9.08E-10	7.63E-12
4.5	3.81E-02	8.77E-03	1.38E-08	7.75E-10

Table 4 numerical results of approximate solutions when $K = 2$, $H = 4$

R	$\varphi_5, (\alpha = 0.1)$	$\varphi_5, (\alpha = 0.5)$	$\varphi_5, (\alpha = 1.5)$	$\varphi_5, (\alpha = 2.5)$	$\varphi_5, (\alpha = 3.5)$
0.0	0.959175169	0.882446456	0.811973631	0.747370694	0.688240945
0.1	0.959445297	0.883211957	0.813177164	0.748958305	0.690162386
0.2	0.960255869	0.885510117	0.816792326	0.753729956	0.695941026
0.3	0.961607438	0.889345907	0.822832797	0.761712100	0.705619806
0.4	0.963500929	0.894727606	0.831321355	0.772948772	0.719270237
0.5	0.965937637	0.901666790	0.842289836	0.787501525	0.736992313
0.6	0.968919225	0.910178318	0.855779082	0.805449312	0.758914353
0.7	0.972447723	0.920280312	0.871838869	0.826888339	0.785192780
0.8	0.976525529	0.931994135	0.890527815	0.851931857	0.816011781
0.9	0.981155406	0.945344363	0.911913264	0.880709902	0.851582850
1.0	0.986340481	0.960358747	0.936071155	0.913368955	0.892144159

3.3 For $\alpha = 2, H = 4$

Now by fixing the maximum reaction rate $\alpha = 2$ and permeability of the cell membrane parameter $H = 4$, we consider the influence of the Michaelis constant for different values of K on the dimensionless oxygen concentration $C(R, \alpha = 2, K, H = 4)$ which is approximated by OHAM and ADMGF methods. The numerical results of the maximum residual errors and approximations of solutions are shown in Table 5 and 6.

Table 5 Maximum absolute residual error when $\alpha = 2, H = 4$

K	e^5	E^5	e^{10}	E^{10}
0.1	2.83E-04	2.16E-06	7.91E-07	1.47E-09
0.5	2.85E-05	3.24E-07	1.14E-08	3.72E-10
1.5	1.43E-06	5.52E-07	3.67E-11	7.31E-13
2.5	2.47E-07	3.68E-09	9.28E-13	8.15E-14
3.5	9.13E-08	6.40E-09	5.88E-14	7.82E-15

Table 6 Results of approximate solution when $\alpha = 2$, $H = 4$

R	$\varphi_5, (K = 0.1)$	$\varphi_5, (K = 0.5)$	$\varphi_5, (K = 1.5)$	$\varphi_5, (K = 2.5)$	$\varphi_5, (K = 3.5)$
0.0	0.56480555	0.696364355	0.817189609	0.867425946	0.895656092
0.1	0.567638453	0.698305227	0.818365494	0.868284779	0.896335413
0.2	0.576144680	0.704135966	0.821897088	0.870863551	0.898374852
0.3	0.590346397	0.713880791	0.827796192	0.875169074	0.901778840
0.4	0.610279291	0.727579575	0.836082433	0.881212700	0.906554761
0.5	0.635990799	0.745287118	0.846783197	0.889010311	0.912712958
0.6	0.667537855	0.767072173	0.859933541	0.898582316	0.920266736
0.7	0.704984339	0.793016253	0.875576070	0.909953634	0.929232365
0.8	0.748398470	0.823212259	0.893760798	0.923153679	0.939629085
0.9	0.797850337	0.857762978	0.914544965	0.938216347	0.951479115
1.0	0.853409775	0.896779499	0.937992841	0.955179981	0.964807654

3.4 For $\alpha = 2, K = 4$

Finally, we now fix the parameters $\alpha = 2$ and $K = 4$, we consider the influence of permeability of the cell membrane parameter for the different values of H. The numerical results of the maximum residual errors and approximations of solutions are shown in Tables 7 and 8.

Table 7 Maximum absolute residual error when $\alpha = 2, K = 4$

H	e^5	E^5	e^{10}	E^{10}
0.2	8.11E-03	1.08E-04	2.68E-04	8.04E-07
0.5	1.05E-04	1.63E-06	6.29E-08	2.52E-09
1.0	4.64E-06	9.97E-07	1.24E-10	1.04E-13
3.0	1.02E-07	2.99E-09	4.33E-14	5.34E-15
10	1.23E-08	3.18E-10	1.41E-14	7.96E-16

Table 8 Results of approximate solution when $\alpha = 2$, $K = 4$

R	$\varphi_5, (H = 0.2)$	$\varphi_5, (H = 0.5)$	$\varphi_5, (H = 1.0)$	$\varphi_5, (H = 3.0)$	$\varphi_5, (H = 10)$
0.0	0.564805550	0.696364355	0.817189609	0.867425946	0.895656092
0.1	0.567638453	0.698305227	0.818365494	0.868284779	0.896335413
0.2	0.576144680	0.704135966	0.821897088	0.870863551	0.898374852
0.3	0.590346397	0.713880791	0.827796192	0.875169074	0.901778840
0.4	0.610279291	0.727579575	0.836082433	0.881212700	0.906554761
0.5	0.635990799	0.745287118	0.846783197	0.889010311	0.912712958
0.6	0.667537855	0.767072173	0.859933541	0.898582316	0.920266736
0.7	0.704984339	0.793016253	0.875576070	0.909953634	0.929232365
0.8	0.748398470	0.823212259	0.893760798	0.923153679	0.939629085
0.9	0.797850337	0.857762978	0.914544965	0.938216347	0.951479115
1.0	0.853409775	0.896779499	0.937992841	0.955179981	0.964807654

4 Conclusions

In this paper, we have examined the Lane-Emden equation that describes the diffusion of oxygen in an idealized spherical cell including the effects of nonlinear oxygen uptake as modeled by Michaelis-Menten kinetics [2]. To overcome the singular behavior at the spherical origin of the cell, Lane-Emden equation is transformed into an equivalent Fredholm integral equation. The numerical results obtained by present method are better than the results obtained by other methods such as the ADMGF [17], as shown in Tables 1-8. Unlike ADM, the proposed OHAM does not require the computation of unknown constant and provides direct scheme for obtaining the approximation of the solution. Unlike ADMGF and MADM [1, 17], the OHAM always gives better convergent series solution. To speed up the calculations, the discrete averaged residual error has been used to obtain optimal value of the adjustable parameter c_0 to control the convergence of solution. The evaluated approximations show enhancements over existing techniques where the minimal size of the obtained errors emphasize these improvements.

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