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# Steiner Gutman Index

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**Abstract**

The concept of *Gutman index* $\mathrm{SGut}(G)$ of a connected graph $G$ was introduced in 1994. The Steiner distance in a graph, introduced by Chartrand et al. in 1989, is a natural generalization of the concept of classical graph distance. In this paper, we generalize the concept of Gutman index by Steiner distance. The *Steiner Gutman k-index* $\mathrm{SGut}_k(G)$ of $G$ is defined by $\mathrm{SGut}_k(G) = \sum_{\substack{S\subseteq V(G)\\ |S|=k}} \left[\prod_{v\in S} deg_G(v)\right] d_G(S)$, where $d_G(S)$ is the Steiner distance of $S$ and $deg_G(v)$ is the degree of $v$ in $G$. Expressions for $\mathrm{SGut}_k$ for some special graphs are obtained. We also give sharp upper and lower bounds of $\mathrm{SGut}_k$ of a connected graph, and get the expression of $\mathrm{SGut}_k(G)$ for $k = n, n-1$. Finally, we compare between $k$-center Steiner degree distance $SDD_k$ and $\mathrm{SGut}_k$ of graphs.

## 1 Introduction

In graph theory applied to chemical problems, a large number of molecular structure descriptors, so-called "topological indices", has been studied [27]. Many of these descriptors are defined in terms of vertex degrees; see [6, 16, 27]. Equally many of these descriptors are in terms of distance between vertices; see [27, 28]. There are also several degree-and-distance-based topological indices; see [11, 13, 14, 18].

Throughout this paper graph is connected. For a graph $G$, let $V(G)$, $E(G)$, and $m = |E(G)|$ denote the set of vertices, the set of edges, and the size of $G$, respectively.

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The minimum vertex degree is denoted by $\delta$ and the maximum by $\Delta$. Distance is one of the basic concepts of graph theory [4]. If $G$ is a connected graph and $u, v \in V(G)$, then the *distance* $d(u, v) = d_G(u, v)$ between $u$ and $v$ is the length of a shortest path connecting $u$ and $v$.

In [14], the *degree distance* of a graph $G$ is defined as

$$DD = DD(G) = \sum_{\{u,v\} \subseteq V(G)} [deg_G(u) + deg_G(v)] d_G(u, v),$$

where $deg_G(u)$ is the degree of the vertex $u \in V(G)$, and $d(u, v)$ is the distance between the vertices $u, v \in V(G)$. For more details on degree distance, we refer to [2, 3, 12, 25].

In [18], the *Gutman index* of a graph $G$ is defined as

$$\mathrm{SGut}(G) = \sum_{\{u,v\} \subseteq V(G)} [deg_G(u) deg_G(v)] d_G(u, v),$$

where $deg_G(u)$ is the degree of the vertex $u \in V(G)$, and $d(u, v)$ is the distance between the vertices $u, v \in V(G)$. For more details on Gutman index, we refer to [8, 12, 15, 27].

The Steiner distance of a graph, introduced by Chartrand et al. in 1989, is a natural and nice generalization of the concept of classical graph distance. For a graph $G(V, E)$ and a set $S \subseteq V(G)$ of at least two vertices, *an $S$-Steiner tree* or *a Steiner tree connecting $S$* (or simply, *an $S$-tree*) is a such subgraph $T(V', E')$ of $G$ that is a tree with $S \subseteq V'$. Let $G$ be a connected graph of order at least 2 and let $S$ be a nonempty set of vertices of $G$. Then the *Steiner distance* $d(S)$ among the vertices of $S$ (or simply the distance of $S$) is the minimum size of a connected subgraph whose vertex set contain $S$. Note that if $H$ is a connected subgraph of $G$ such that $S \subseteq V(H)$ and $|E(H)| = d_G(S)$, then $H$ is a tree. Clearly, $d_G(S) = \min\{|E(T)|\ ,\ S \subseteq V(T)\}$, where $T$ is subtree of $G$. Furthermore, if $S = \{u, v\}$, then $d_G(S) = d_G(u, v)$ is nothing new, but the classical distance between $u$ and $v$. Clearly, if $|S| = k$, then $d_G(S) \geq k - 1$.

If $v$ is a vertex of a connected graph $G$, then the *eccentricity* $\varepsilon(v)$ of $v$ is defined by $\varepsilon(v) = \max\{d(u, v) \mid u \in V(G)\}$. Furthermore, the *radius* $rad(G)$ and *diameter* $diam(G)$ of $G$ are defined by $rad(G) = \min\{\varepsilon(v) \mid v \in V(G)\}$ and $diam(G) = \max\{\varepsilon(v) \mid v \in V(G)\}$. Let $n$ and $k$ be integers such that $2 \leq k \leq n$. The *Steiner $k$-eccentricity* $\varepsilon_k(v)$ of a vertex $v$ of $G$ is defined by $\varepsilon_k(v) = \max\{d_G(S) \mid S \subseteq V(G),\ |S| = k, \text{and } v \in S\}$. The *Steiner $k$-radius* of $G$ is $srad_k(G) = \min\{\varepsilon_k(v) \mid v \in V(G)\}$, while the *Steiner $k$-diameter* of $G$ is $sdiam_k(G) = \max\{\varepsilon_k(v) \mid v \in V(G)\}$. Note that for every connected graph $G$,

$\varepsilon_2(v) = \varepsilon(v)$ for all vertices $v$ of $G$, $srad_2(G) = rad(G)$ and $sdiam_2(G) = diam(G)$. For more details on Steiner distance, we refer to [1, 5, 7, 9, 10, 26].

The following observation is easily seen.

**Observation 1.1** *Let $k$ be an integer such that $2 \leq k \leq n$. If $H$ is a spanning subgraph of $G$, then $sdiam_k(G) \leq sdiam_k(H)$.*

Li et al. [19] generalized the concept of Wiener index by Steiner distance. The *Steiner Wiener $k$-index* or *$k$-center Steiner Wiener index* $SW_k(G)$ of $G$ is defined by

$$SW_k(G) = \sum_{\substack{S \subseteq V(G) \\ |S|=k}} d_G(S)\,.$$

For $k = 2$, the above defined Steiner Wiener index coincides with the ordinary Wiener index. It is usual to consider $SW_k$ for $2 \leq k \leq n-1$, but the above definition implies $SW_1(G) = 0$ and $SW_n(G) = n-1$. For more details on the Steiner Wiener index, we refer to [19, 20, 24].

Recently, Gutman [17] generalized the concept of degree distance by Steiner degree distance. The *$k$-center Steiner degree distance* $SDD_k(G)$ of $G$ is defined by

$$SDD_k(G) = \sum_{\substack{S \subseteq V(G) \\ |S|=k}} \left( \sum_{v \in S} deg_G(v) \right) d_G(S)\,.$$

For more details on the Steiner degree distance, we refer to [18, 23].

We now generalize the concept of Gutman index by Steiner distance. The *Steiner Gutman $k$-index* $\mathrm{SGut}_k(G)$ of $G$ is defined by

$$\mathrm{SGut}_k(G) = \sum_{\substack{S \subseteq V(G) \\ |S|=k}} \left( \prod_{v \in S} deg_G(v) \right) d_G(S)\,.$$

The relations between the Steiner degree distance, Steiner Gutman index and Steiner Wiener index are shown in the following Table 1:

**Table 1.** Three Steiner distance parameters.

| Parameters | Definitions |
|---|---|
| Steiner Wiener $k$-index | $SW_k(G) = \sum_{\substack{S \subseteq V(G) \\ \lvert S\rvert=k}} d_G(S)$ |
| $k$-center Steiner Degree distance | $SDD_k(G) = \sum_{\substack{S \subseteq V(G) \\ \lvert S\rvert=k}} \left[\sum_{v \in S} deg_G(v)\right] d_G(S)$ |
| Steiner Gutman $k$-index | $\mathrm{SGut}_k(G) = \sum_{\substack{S \subseteq V(G) \\ \lvert S\rvert=k}} \left[\prod_{v \in S} deg_G(v)\right] d_G(S)$ |

In Section 2, we obtain the exact values of the Steiner Gutman $k$-index of the complete graph, complete bipartite graph, path and star. When $G$ is a connected graph or tree, we also get the expression of $\mathrm{SGut}_k(G)$ for $k = n, n-1$. In Section 3, we obtain sharp lower and upper bounds for $\mathrm{SGut}_k$ in terms of degree, or both order and size, or order. In Section 4, comparison between $SDD_k$ and $\mathrm{SGut}_k$ of graphs is given.

## 2 Results for some special graphs

Beginning this section, we note that the special case for $k = 2$ of all formulas derived here for the Steiner Gutman $k$-index, thus pertaining to the ordinary Gutman index, are well known and mentioned many times in the earlier literature. From the definition of Steiner Gutman $k$-index, one can easily obtain the following result:

**Proposition 2.1** *Let $K_n$ be the complete graph of order $n$, and let $k$ be an integer such that $2 \leq k \leq n$. Then*

$$\mathrm{SGut}_k(K_n) = \binom{n}{k}(n-1)^k(k-1).$$

By the similar method in [22], we can derive the following result for complete bipartite graphs.

**Theorem 2.1** *Let $K_{a,b}$ be the complete bipartite graph of order $a+b$ $(1 \leq a \leq b)$, and let $k$ be an integer such that $2 \leq k \leq a+b$. Then*

$$\mathrm{SGut}_k(K_{a,b}) = \begin{cases} ka^k\binom{b}{k} + kb^k\binom{a}{k} + (k-1)\sum_{x=1}^{k-1}\binom{a}{x}\binom{b}{k-x}b^x a^{k-x} & \text{if } 1 \leq k \leq a \\ ka^k\binom{b}{k} + (k-1)\sum_{x=1}^{a}\binom{a}{x}\binom{b}{k-x}b^x a^{k-x} & \text{if } a < k \leq b \\ (k-1)\sum_{x=1}^{a}\binom{a}{x}\binom{b}{k-x}b^x a^{k-x} & \text{if } b < k \leq a+b. \end{cases}$$

The following corollary is immediate from the above theorem.

**Corollary 2.1** *Let $S_n$ be the star of order $n$ $(n \geq 3)$, and let $k$ be an integer such that $2 \leq k \leq n$. Then*

$$\mathrm{SGut}_k(S_n) = (kn - 2k + 1)\binom{n-1}{k-1}.$$

For paths of order $n$, we have the following.

**Proposition 2.2** *Let $P_n$ be the path of order $n$, and let $k$ be an integer such that $2 \leq k \leq n-2$. Then*

$$\mathrm{SGut}_k(P_n) = 2^k(k-1)\binom{n}{k+1} + 2^{k-2}(n-1)\binom{n-2}{k-2}.$$

*Proof.* Let $P_n = v_1v_2\dots v_n$, and $G = P_n$. Note that $d_G(v_1) = d_G(v_n) = 1$ and $d_G(v_i) = 2$ for each $i$ $(2 \leq i \leq n-1)$. We first regard the degree of both $v_1$ and $v_n$ as 2. In this way, all the degree of $v_1, v_2, \dots, v_n$ are 2.

$$\sum_{\substack{S\subseteq V(G)\\|S|=k}} \left(\prod_{v\in S} deg_G(v)\right) d_G(S) = 2^k \sum_{\substack{S\subseteq V(G)\\|S|=k}} d_G(S) = 2^k SW_k(G)\,.$$

Next, let us compute how we add additional contributions for $2^k SW_k(G) - \mathrm{SGut}_k(G)$ by assuming $d_G(v_1) = d_G(v_n) = 2$. It is clear that

$$\begin{aligned} & 2^k\mathrm{SW}_k(G) - \mathrm{SGut}_k(G) \\ = \;& 2^{k-1} \sum_{\substack{S\subseteq V(G),|S|=k\\ v_1\in S,\ deg_G(v_n)=2}} d_G(S) + 2^{k-1} \sum_{\substack{S\subseteq V(G),|S|=k\\ v_n\in S,\ deg_G(v_1)=2}} d_G(S) - 2^{k-2} \sum_{\substack{S\subseteq V(G),|S|=k\\ v_1\in S,\ v_n\in S}} d_G(S), \end{aligned}$$

where $deg_G(v_1) = 2$ or $deg_G(v_n) = 2$ means that we regard the degree of $v_1$ or $v_n$ as 2.

For symmetry, we only need to compute $\sum_{\substack{S\subseteq V(G),|S|=k\\ v_1\in S,\ deg_G(v_1)=2}} d_G(S)$. Choose $S \subseteq V(G)$ and $|S| = k$. Without loss of generality, let $S = \{u_1, u_{i_2}, \dots, u_{i_k}\}$ where $1 < i_2 < \cdots < i_k \leq n$. Then $k \leq i_k \leq n$. Let $d_G(u_1, u_{i_k}) = j-1$. Since $d_G(S) = d_G(u_1, u_{i_k}) = j-1$, it follows that $k-1 \leq j-1 \leq n-1$, and hence $k \leq j \leq n$. Once the vertex $u_{i_k}$ is chosen, since $d_G(u_1, u_{i_k}) = j-1$, we have $\binom{j-2}{k-2}$ ways to choose $u_{i_2}, u_{i_3}, \dots, u_{i_k-1}$. Therefore, we have

$$\begin{aligned} & 2^{k-1} \sum_{\substack{S\subseteq V(G),|S|=k\\ v_1\in S,\ deg_G(v_n)=2}} d_G(S) + 2^{k-1} \sum_{\substack{S\subseteq V(G),|S|=k\\ v_n\in S,\ deg_G(v_1)=2}} d_G(S) \\ = \;& 2\cdot 2^{k-1} \sum_{\substack{S\subseteq V(G_1),|S|=k\\ v_1\in S}} d_G(S) = 2^k \sum_{k\leq j\leq n} (j-1)\binom{j-2}{k-2} \\ = \;& 2^k(k-1) \sum_{k\leq j\leq n} \binom{j-1}{k-1} = 2^k(k-1)\binom{n}{k}. \end{aligned}$$

We now turn our attention to compute $\sum_{\substack{S\subseteq V(G),|S|=k\\ v_1\in S,\ v_n\in S}} d_G(S)$. Without loss of generality, let $S = \{u_1, u_{i_2}, \dots, u_{i_{k-1}}, u_n\}$ where $1 < i_2 < \cdots < i_{k-1} < n$. Clearly, $d_G(S) = n-1$, and we have $\binom{n-2}{k-2}$ ways to choose $u_{i_2}, u_{i_3}, \ldots, u_{i_{k-1}}$. Therefore, we have

$$2^{k-2} \sum_{\substack{S\subseteq V(G),|S|=k \\ v_1\in S,\ v_n\in S}} d_G(S) \;=\; 2^{k-2}(n-1)\binom{n-2}{k-2}.$$

Therefore, we have

$$\begin{aligned}
\mathrm{SGut}_k(G) &= 2^k SW_k(G) - 2^k(k-1)\binom{n}{k} + 2^{k-2}(n-1)\binom{n-2}{k-2} \\
&= 2^k(k-1)\binom{n+1}{k+1} - 2^k(k-1)\binom{n}{k} + 2^{k-2}(n-1)\binom{n-2}{k-2} \\
&= 2^k(k-1)\binom{n}{k+1} + 2^{k-2}(n-1)\binom{n-2}{k-2}.
\end{aligned}$$

■

For $k = n$, the following result is immediate.

**Observation 2.1** *Let $G$ be the connected graph of order $n$. Then*

$$\mathrm{SGut}_n(G) = (n-1)\prod_{v\in V(G)} d_G(v).$$

In [21], Mao obtained the following result.

**Lemma 2.1** [21] *Let $G$ be a graph. Then $sdiam_{n-1}(G) = n-2$ if and only if $G$ is 2-connected.*

We now give the expression of $\mathrm{SGut}_{n-1}(G)$ for a graph $G$.

**Proposition 2.3** *Let $G$ be a connected graph of order $n$.*
(1) *If $G$ is 2-connected, then*

$$\mathrm{SGut}_{n-1}(G) = (n-2)\left(\prod_{v\in V(G)} d_G(v)\right)\left(\sum_{v\in V(G)} \frac{1}{d_G(v)}\right).$$

(2) *If $\kappa(G) = 1$, then*

$$\mathrm{SGut}_{n-1}(G) = \left(\prod_{v\in V(G)} d_G(v)\right)\left[(n-2)\left(\sum_{v\in V(G)} \frac{1}{deg_G(v)}\right) + \left(\sum_{i=1}^{p} \frac{1}{deg_G(v_i)}\right)\right]$$

*where $v_i$ $(1 \le i \le p)$ are all cut vertices of $G$.*

*Proof.* (1) Since $G$ is 2-connected, it follows from Lemma 2.1 that $d_G(S)=n-2$ for any $S\subseteq V(G)$ and $|S|=n-1$, and hence

$$\mathrm{SGut}_{n-1}(G)=(n-2)\sum_{\substack{S\subseteq V(G)\\|S|=n-1}}\left(\prod_{v\in S}deg_G(v)\right)=(n-2)\left(\prod_{v\in V(G)}d_G(v)\right)\left(\sum_{v\in V(G)}\frac{1}{d_G(v)}\right).$$

(2) Note that $v_1,v_2,\ldots,v_p$ are all the cut vertices in $G$. For any $S\subseteq V(G)$ and $|S|=n-1$, if $V(G)-S=\{v_i\}$ for some $i$ $(1\leq i\leq p)$, then it follows from Lemma 2.1 that $d_G(S)=n-1$; if $V(G)-S\neq\{v_i\}$ for each $i$ $(1\leq i\leq p)$, then it follows from Lemma 2.1 that $d_G(S)=n-2$. Let $U=\{v_{p+1},v_{p+2},\ldots,v_n\}$ be the set of non-cut vertices. Then

$$\begin{aligned}
&\mathrm{SGut}_{n-1}(G)\\
=\ &(n-2)\sum_{\substack{S\subseteq V(G),|S|=n-1,\\V(G)-S=\{v_i\},\ p+1\leq i\leq n}}\left(\prod_{v\in S}deg_G(v)\right)+(n-1)\sum_{\substack{S\subseteq V(G),|S|=n-1,\\V(G)-S=\{v_i\},\ 1\leq i\leq p}}\left(\prod_{v\in S}deg_G(v)\right)\\
=\ &(n-2)\left(\prod_{v\in V(G)}d_G(v)\right)\left(\sum_{i=p+1}^{n}\frac{1}{deg_G(v_i)}\right)+(n-1)\left(\prod_{v\in V(G)}d_G(v)\right)\left(\sum_{i=1}^{p}\frac{1}{deg_G(v_i)}\right)\\
=\ &(n-2)\left(\prod_{v\in V(G)}d_G(v)\right)\left(\sum_{v\in V(G)}\frac{1}{deg_G(v)}\right)+\left(\prod_{v\in V(G)}d_G(v)\right)\left(\sum_{i=1}^{p}\frac{1}{deg_G(v_i)}\right).
\end{aligned}$$

■

The following corollary is immediate.

**Corollary 2.2** *Let $T$ be a tree of order $n$. If $v_1,v_2,\ldots,v_p$ are all the pendent vertices in $T$, then*

$$\mathrm{SGut}_{n-1}(T)=(n-2)(n-p)\left(\prod_{i=1}^{p}\frac{1}{deg_T(v_i)}\right)+(n-1)\left(\prod_{i=1}^{p}\frac{1}{deg_T(v_i)}\right)\left(\sum_{i=1}^{p}\frac{1}{deg_T(v_i)}\right).$$

# 3 Lower and upper bounds for general graphs

In this section, we give upper and lower bounds of $\mathrm{SGut}_k(G)$.

## 3.1 Bounds in terms of degree

The following bounds are sharp for $\mathrm{SGut}_k(G)$.

**Theorem 3.1** *Let $G$ be a connected graph of order $n$. Then*

$$\delta(G)^k SW_k(G)\leq \mathrm{SGut}_k(G)\leq \Delta(G)^k SW_k(G),$$

*with equality if and only if $G$ is a regular graph.*

*Proof.* From the definition of Steiner degree distance, we have

$$\mathrm{SGut}_k(G) \;=\; \sum_{\substack{S\subseteq V(G)\\|S|=k}} \left(\prod_{v\in S} deg_G(v)\right) d_G(S) \le \sum_{\substack{S\subseteq V(G)\\|S|=k}} \Delta(G)^k d_G(S) = \Delta(G)^k SW_k(G)\,.$$

and

$$\mathrm{SGut}_k(G) \;=\; \sum_{\substack{S\subseteq V(G)\\|S|=k}} \left(\prod_{v\in S} deg_G(v)\right) d_G(S) \ge \sum_{\substack{S\subseteq V(G)\\|S|=k}} \delta(G)^k d_G(S) = \delta(G)^k SW_k(G)\,.$$

To show the sharpness of the upper and lower bounds, we consider a $r$-regular graph $G$. Then $\Delta(G) = \delta(G) = r$, and $\mathrm{SGut}_k(G) = r^k SW_k(G) = \Delta(G)^k SW_k(G) = \delta(G)^k SW_k(G)$. ■

Li et al. [19] obtained the following sharp bounds for $SW_k(G)$.

**Lemma 3.1** [19] *Let $G$ be a connected graph of order $n$, and let $k$ be an integer such that $2 \le k \le n$. Then*

$$\binom{n}{k}(k-1) \le SW_k(G) \le (k-1)\binom{n+1}{k+1}.$$

*Moreover, the lower bound is sharp.*

The following result can be easily seen.

**Proposition 3.1** *Let $G$ be a connected graph of order $n$, and let $k$ be an integer such that $2 \le k \le n$. Then*

$$\delta(G)^k\binom{n}{k}(k-1) \le \mathrm{SGut}_k(G) \le \Delta(G)^k(k-1)\binom{n+1}{k+1}.$$

*Moreover, the bounds are sharp.*

*Proof.* From Theorem 3.1 and Lemma 3.1, we have

$$\mathrm{SGut}_k(G) \ge \delta(G)^k SW_k(G) \ge \delta(G)^k\binom{n}{k}(k-1).$$

Similarly, from Theorem 3.1 and Lemma 3.1, we have

$$\mathrm{SGut}_k(G) \le \Delta(G)^k SW_k(G) \le \Delta(G)^k(k-1)\binom{n+1}{k+1}.$$

To show the sharpness of lower bound, we consider the complete graph $K_n$. Since $\delta(K_n) = n-1$, it follows from Proposition 2.1 that

$$\mathrm{SGut}_k(K_n) = \binom{n}{k}(n-1)^k(k-1) = \delta(K_n)^k\binom{n}{k}(k-1).$$

To show the sharpness of upper bound, we consider the path $P_2$. For $k = 2$, since $\Delta(P_2) = 1$, it follows from Proposition 2.2 that

$$\mathrm{SGut}_2(P_2) = 1 = \Delta(P_2)^k(k-1)\binom{n+1}{k+1}.$$

■

## 3.2 Bounds in terms of order and size

For graph $G$ having $n$ vertices and $m$ edges, we have the following upper and lower bounds of $\mathrm{SGut}_k(G)$.

**Theorem 3.2** *Let $G$ be a connected graph $n$ vertices and $m$ edges, and let $k$ be an integer with $2 \le k \le n$. Then*

$$(n-1)\left(\frac{2m}{k}\right)^k \binom{n-1}{k-1}^k \ge \mathrm{SGut}_k(G) \ge \begin{cases} 2m(k-1)\binom{n-1}{k-1} & \text{if } \delta(G) \ge 2 \\ (k-1)\binom{n}{k} & \text{if } \delta(G) = 1. \end{cases}$$

*Moreover, the upper and lower bounds are sharp.*

*Proof.* For any $S \subseteq V(G)$ and $|S| = k$, we have $k-1 \le d_G(S) \le n-1$, and hence

$$(k-1)\sum_{\substack{S\subseteq V(G)\\ |S|=k}}\left(\prod_{v\in S} deg_G(v)\right) \le \mathrm{SGut}_k(G) \le (n-1)\sum_{\substack{S\subseteq V(G)\\ |S|=k}}\left(\prod_{v\in S} deg_G(v)\right).$$

Let

$$M = \sum_{\substack{S\subseteq V(G)\\ |S|=k}}\left(\prod_{v\in S} deg_G(v)\right) = \sum_{\{v_1,v_2,\ldots,v_k\}\subseteq V(G)} deg_G(v_1)deg_G(v_2)\ldots deg_G(v_k).$$

and

$$N = \sum_{\{v_1,v_2,\ldots,v_k\}\subseteq V(G)} [deg_G(v_1) + deg_G(v_2) + \ldots + deg_G(v_k)].$$

Since

$$deg_G(v_1)deg_G(v_2)\ldots deg_G(v_k) \le \frac{1}{k^k}[deg_G(v_1) + deg_G(v_2) + \ldots + deg_G(v_k)]^k,$$

it follows that

$$\begin{aligned}
M &= \sum_{\{v_1,v_2,\ldots,v_k\}\subseteq V(G)} deg_G(v_1)deg_G(v_2)\ldots deg_G(v_k)\\
&\leq \frac{1}{k^k}\sum_{\{v_1,v_2,\ldots,v_k\}\subseteq V(G)} [deg_G(v_1)+deg_G(v_2)+\ldots+deg_G(v_k)]^k\\
&\leq \frac{1}{k^k}\left[\sum_{\{v_1,v_2,\ldots,v_k\}\subseteq V(G)} (deg_G(v_1)+deg_G(v_2)+\ldots+deg_G(v_k))\right]^k\\
&\leq \frac{1}{k^k}N^k .
\end{aligned}$$

For each $v \in V(G)$, there are $\binom{n-1}{k-1}$ $k$-subsets in $G$ such that each of them contains $v$. The contribution of vertex $v$ is exactly $\binom{n-1}{k-1}deg_G(v)$. From the arbitrariness of $v$, we have

$$N = \binom{n-1}{k-1}\sum_{v\in V(G)} deg_G(v) = 2m\binom{n-1}{k-1},$$

and hence

$$\mathrm{SGut}_k(G) \leq (n-1)M \leq (n-1)\frac{1}{k^k}N^k \leq \left(\frac{2m}{k}\right)^k\binom{n-1}{k-1}^k(n-1).$$

If $\delta(G)\geq 2$, then $deg_G(v_1)deg_G(v_2)\ldots deg_G(v_k) \geq deg_G(v_1)+deg_G(v_2)+\ldots+deg_G(v_k)$, and hence

$$\begin{aligned}
\mathrm{SGut}_k(G) &\geq (k-1)\sum_{\{v_1,v_2,\ldots,v_k\}\subseteq V(G)} deg_G(v_1)deg_G(v_2)\ldots deg_G(v_k)\\
&\geq (k-1)\sum_{\{v_1,v_2,\ldots,v_k\}\subseteq V(G)} [deg_G(v_1)+deg_G(v_2)+\ldots+deg_G(v_k)]\\
&= (k-1)N\\
&= 2m(k-1)\binom{n-1}{k-1}.
\end{aligned}$$

If $\delta(G)=1$, then

$$\mathrm{SGut}_k(G) \geq (k-1)\sum_{\{v_1,v_2,\ldots,v_k\}\subseteq V(G)} deg_G(v_1)deg_G(v_2)\ldots deg_G(v_k) \geq (k-1)\binom{n}{k}.$$

To show the sharpness of the lower bound for $\delta = 1$ and the upper bound, we let $G = P_2$. For $k = 2$, we have

$$(n-1)\left(\frac{2m}{k}\right)^k\binom{n-1}{k-1}^k = 1 = \mathrm{SGut}_k(P_2) = (k-1)\binom{n}{k}.$$

To show the sharpness of the lower bound for $\delta \geq 2$, we let $G = C_3$. For $k = 2$, we have

$$\mathrm{SGut}_k(C_3) = 12 = 2m(k-1)\binom{n-1}{k-1}.$$

■

## 3.3 Bounds in terms of order

In this subsection we prove that the complete graph $K_n$ gives the maximum Steiner Gutman $k$-index of graphs.

**Theorem 3.1** *Let $G$ be a connected graph of order $n$. If there is a positive integer $k > 0.618n$, then*

$$\mathrm{SGut}_k(G) \leq \binom{n}{k}(n-1)^k(k-1) \tag{1}$$

*with equality holding if and only if $G \cong K_n$.*

*Proof.* We have

$$k > 0.618(n-1) > \frac{(\sqrt{5}-1)(n-1)}{2} > \frac{-(n-1)+\sqrt{(n-1)^2+4(n^2-2n)}}{2},$$

that is,

$$(2k+n-1)^2 > (n-1)^2 + 4(n^2-2n),$$

that is,

$$\frac{n+1}{k+1} < 1 + \frac{k}{n-2} < \left(1+\frac{1}{n-2}\right)^k = \left(\frac{n-1}{n-2}\right)^k. \tag{2}$$

Again since $k > 0.618(n-1)$, then one can easily see that

$$k(k-1) \geq n-2, \quad \text{that is,} \quad \left(\frac{n-1}{n-2}\right)^k > 1 + \frac{k}{n-2} \geq \frac{k}{k-1}.$$

From the above, we get

$$(n-1)^k(k-1) > (n-2)^k k. \tag{3}$$

First we assume that $\Delta = n-1$. If $G \cong K_n$, then by Proposition 2.1, the equality holds in (1). Otherwise, $G \ncong K_n$. For any $S \subseteq V(G)$ and $|S| = k$, without loss of generality, we let $S = \{u_1, u_2, \ldots, u_k\}$. Since $\Delta = n-1$, we have two possibilities: (i) $deg_G(u_i) = n-1$, $u_i \in S$ and (ii) $deg_G(u_i) \leq n-2$, for any $u_i \in S$.

*Case (i)* : $deg_G(u_i) = n-1$, $u_i \in S$. In this case the tree $T$ induced by the edges in $\{u_iu_1, u_iu_2, \ldots, u_iu_{i-1}, u_iu_{i+1}, \ldots, u_iu_k\}$ is a Steiner tree connecting $S$, which implies

$d(S) \leq k-1$. Since $|S| = k$, it follows that $d_G(S) \geq k-1$. Therefore, $d_G(S) = k-1$. For each $v \in V(G)$, we have $deg_G(v) \leq n-1$. Thus we have

$$\left(\prod_{v \in S} deg_G(v)\right) d_G(S) \leq (n-1)^k\,(k-1)$$

with equality holding if and only if $deg_G(v) = n-1$ for any $v \in S$.

*Case (ii)* : $deg_G(u_i) \leq n-2$, for any $u_i \in S$. Let $w$ be a vertex of degree $n-1$ as $\Delta = n-1$. In this case the tree $T$ induced by the edges in $\{wu_1, wu_2, \ldots, wu_k\}$ is a Steiner tree connecting $S$. One can easily see that $d(S) = k$. For each $v \in S$, we have $deg_G(v) \leq n-2$. Thus we have

$$\left(\prod_{v \in S} deg_G(v)\right) d_G(S) \leq (n-2)^k\,k$$

with equality holding if and only if $deg_G(v) = n-2$ for any $v \in S$.

Since $G \ncong K_n$, using (3) with the above results, we get

$$\mathrm{SGut}_k(G) = \sum_{\substack{S \subseteq V(G) \\ |S|=k}} \left(\prod_{v \in S} deg_G(v)\right) d_G(S) < \binom{n}{k}(n-1)^k(k-1)\,.$$

Next we assume that $\Delta \leq n-2$. By (2), we have

$$\frac{n+1}{k+1} < \left(\frac{n-1}{n-2}\right)^k \leq \left(\frac{n-1}{\Delta}\right)^k.$$

From Proposition 3.1 with the above result, we have

$$\mathrm{SGut}_k(G) \;<\; \frac{k+1}{n+1}\,(n-1)^k\,(k-1)\binom{n+1}{k+1} = \binom{n}{k}\,(k-1)\,(n-1)^k\,.$$

■

# 4 Comparison between $SDD_k$ and $\mathrm{SGut}_k$ of graphs

In this section we compare between $SDD_k$ and $\mathrm{SGut}_k$ of graphs.

**Example 1** *Let $S_n$ be the star of order $n$ $(n \geq 3)$, and let $k$ be an integer such that $2 \leq k \leq n$. Then*

$$\mathrm{SGut}_k(S_n) = (kn-2k+1)\binom{n-1}{k-1}.$$

**Example 2** [23] *Let $S_n$ be the star of order $n$. Then*

$$SDD_k(S_n) = (2kn - n - 3k + 2)\binom{n-1}{k-1}.$$

**Proposition 4.1** *Let $G$ be a connected graph of order $n$ with minimum degree $\delta$, and let $k$ be an integer with $3 \leq k \leq n$. Let $r$ be the number of the pendant vertices in $G$.*

(1) *If $\delta(G) \geq 2$, then*

$$\mathrm{SDD}_k(G) \leq \mathrm{SGut}_k(G).$$

(2) *If $\delta(G) = 1$, $r + 3 \leq k$ and $2^{k-r} \geq 2(k-r) + r$, then*

$$\mathrm{SDD}_k(G) \leq \mathrm{SGut}_k(G).$$

(3) *If $\delta(G) = 1$, and $r = n - 1$, then*

$$\mathrm{SDD}_k(G) > \mathrm{SGut}_k(G).$$

*Proof.* (1) For any $S \subseteq V(G)$ and $|S| = k$, we let $S = \{v_1, v_2, \ldots, v_k\}$. Let $deg_G(v_i) = x_i$ for $i$ $(1 \leq i \leq k)$. To show $\mathrm{SDD}_k(G) \leq \mathrm{SGut}_k(G)$, from the arbitrariness of $S$ and the definitions, it suffices to show that $\prod_{i=1}^{k} x_i \geq \sum_{i=1}^{k} x_i$. Let $f(x_1, x_2, \ldots, x_k) = x_1 x_2 \ldots x_k - (x_1 + x_2 + \ldots + x_k)$. Since $\delta(G) \geq 2$, it follows that $x_i \geq 2$ for each $i$ $(1 \leq i \leq k)$, and hence

$$\frac{\partial f}{\partial x_i} = x_1 x_2 \ldots x_{i-1} x_{i+1} \ldots x_k - 1 > 0$$

Therefore, $f(x_1, x_2, \ldots x_k)$ is a monotone increasing function. Clearly, $f(x_1, x_2, \ldots, x_k) \geq f(2, 2, \ldots, 2) = 2^k - 2k \geq 0$. So, we have $\mathrm{SDD}_k(G) \leq \mathrm{SGut}_k(G)$.

(2) For any $S \subseteq V(G)$ and $|S| = k$, we let $S = \{v_1, v_2, \ldots, v_k\}$. Let $deg_G(v_i) = x_i$ for $i$ $(1 \leq i \leq k)$. To show $\mathrm{SDD}_k(G) \leq \mathrm{SGut}_k(G)$, from the arbitrariness of $S$ and the definitions, it suffices to show that $\prod_{i=1}^{k} x_i \geq \sum_{i=1}^{k} x_i$. Since there are $r \geq 1$ pendant vertices in $G$, we can assume that $x_1 = x_2 = \ldots = x_r = 1$. Then $x_i \geq 2$ for each $i$ $(r+1 \leq i \leq k)$. Let $f(x_{r+1}, x_{r+2}, \ldots, x_k) = x_{r+1} x_{r+2} \ldots x_k - (x_{r+1} + x_{r+2} + \ldots + x_k) - r$. Then

$$\frac{\partial f}{\partial x_i} = x_{r+1} x_{r+2} \ldots x_{i-1} x_{i+1} \ldots x_k - 1 > 0.$$

Therefore, $f(x_{r+1}, x_{r+2}, \ldots, x_k)$ is monotone increasing. Clearly, $f(x_{r+1}, x_{r+2}, \ldots, x_k) \geq f(2, 2, \ldots, 2) = 2^{k-r} - 2(k-r) - r \geq 0$ as $2^{k-r} \geq 2(k-r) + r$. Therefore we have $\mathrm{SDD}_k(G) \leq \mathrm{SGut}_k(G)$.

(3) If $\delta(G) = 1$ and $r = n - 1$, then $G = K_{1,n-1}$ is a star of order $n$. From Examples 1 and 2, we have $\mathrm{SDD}_k(G) > \mathrm{SGut}_k(G)$. ■

**Proposition 4.2** *Let $G$ be a connected graph of order $n$ with maximum degree $\Delta$, and let $k$ be an integer with $2 \leq k \leq n$.*
(1) *If $\Delta(G) \leq k-1$, then*
$$\mathrm{SDD}_k(G) \leq \mathrm{SGut}_k(K_n).$$
(2) *If $\Delta(G) = n-1$, then*
$$\mathrm{SDD}_k(G) \leq \mathrm{SGut}_k(K_n).$$

*Proof.* (1) For any $S \subseteq V(G)$ and $|S| = k$, since $\Delta(G) \leq k-1$, we have
$$\left[\sum_{v\in S} deg_G(v)\right] d_G(S) \leq (k-1)^k d_G(S) \leq (k-1)^k(n-1).$$
For this $S$ in complete graph $K_n$,
$$\left[\prod_{v\in S} deg_{K_n}(v)\right] d_{K_n}(S) = (n-1)^k(k-1).$$
Since $k \leq n$, it follows that $(k-1)^k(n-1) \leq (n-1)^k(k-1)$. From the arbitrariness of $S$, we have $\mathrm{SDD}_k(G) \leq \mathrm{SGut}_k(K_n)$.

(2) If $\Delta(G) = n-1$, then there exists a vertex $v$ such that $deg_G(v) = n-1$. For any $S \subseteq V(G)$ and $|S| = k$, we have $v \in S$ or $v \notin S$. If $v \in S$, then $d_G(S) = k-1$, and hence
$$\left[\sum_{v\in S} deg_G(v)\right] d_G(S) \leq (n-1)^k(k-1).$$
From the arbitrariness of $S$, we have $\mathrm{SDD}_k(G) \leq \mathrm{SGut}_k(K_n)$.

If $v \notin S$, then $d_G(S) = k-1$ or $d_G(S) = k$. If $d_G(S) = k-1$, then it is true as the case $v \in S$. If $d_G(S) = k$, then $G[S]$ is not connected, and there is at most one vertex of degree $k-2$ in $G[S]$. Therefore, there are at most two vertices in $S$ of degree less than or equal to $n-2$ and all the other vertices in $S$ are of degree at most $n-3$. Then
$$\left[\sum_{v\in S} deg_G(v)\right] d_G(S) \leq [(n-3)^{k-2} + 2(n-2)]k.$$
It suffices to show that $[(n-3)^{k-2} + 2(n-2)]k \leq (n-1)^k(k-1)$, that is,
$$\frac{k}{n-1}\left[\frac{(n-3)^{k-2}}{(n-1)^{k-1}} + \frac{2(n-2)}{(n-1)^{k-1}}\right] \leq \frac{(n-3)^{k-2}}{(n-1)^{k-1}} + \frac{2(n-2)}{(n-1)^{k-1}} \leq k-1. \tag{4}$$
One can easily see that (4) is true for $3 \leq k \leq n-1$. For $k = n$, one can easily prove that the result holds as $d_G(S) = n-1$. For $k = 2$, we have
$$\left[\sum_{v\in S} deg_G(v)\right] d_G(S) \leq 4(n-2) \leq (n-1)^2$$

as $n \geq 3$. This completes the proof of the result. ■

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