MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **79** (2018) 717-742

ISSN 0340 - 6253

Maximum Balaban Index and Sum–Balaban Index of Tricyclic Graphs *

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(Received May 26, 2017)

Abstract

Balaban index and Sum-Balaban index were used in various quantitative structure-property relationship and quantitative structure activity relationship studies. In this paper, we characterize the graphs with the maximum Balaban index and maximum Sum-Balaban index of tricyclic graphs.

1 Introduction

Let G be a simple and connected graph with $|V(G)| = n$ and $|E(G)| = m$. If $m = n-1+c$, then G is called a c-cyclic graph. If $c = 0, 1, 2$ and 3, then G is a tree, unicyclic graph, bicyclic graph and tricyclic graph, respectively. Denote by $\mathcal{T}_n$ the set of all tricyclic graphs of order n.

*Research supported by Natural Science Foundation of Anhui Province (No.1508085MC55), Natural Science Foundation of Educational Government of Anhui Province (No.KJ2013A076) and key project of the Outstanding Young Talent Support Program of the University of Anhui Province (No.gxyqZD2016367)

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Let $N_G(u)$ be the neighbor vertex set of vertex u, then $d_G(u) = |N_G(u)|$ is called the degree of u, the distance between vertices u and v in G is denoted by $d_G(u,v)$, and $D_G(u) = \sum_{v \in V(G)} d_G(u,v)$ is the distance sum of vertex u in G. For a vertex $u \in V(G)$ by $G - u$ we denote the graph induced by $V(G) - \{u\}$.

The cyclomatic number μ of G is the minimum number of edges that must be removed from G in order to transform it to an acyclic graph. It is known that $\mu = |E(G)| - |V(G)| + 1 = m - n + 1$.

The Balaban index of a simple connected graph G is defined as

$$J(G) = \frac{m}{\mu + 1} \sum_{uv \in E(G)} \frac{1}{\sqrt{D_G(u) D_G(v)}}.$$

It was proposed by Balaban in [1, 2], which is also called the average distance-sum connectivity or J index. It appears to be a very useful molecular descriptor with attractive properties. In 2010, Balaban et al.[3] also proposed the study of the Sum-Balaban index $SJ(G)$ of a connected graph G, which is defined as

$$SJ(G) = \frac{m}{\mu + 1} \sum_{uv \in E(G)} \frac{1}{\sqrt{D_G(u) + D_G(v)}}.$$

Balaban index and Sum-Balaban index were used in various quantitative structure-property relationship (QSPR) and quantitative structure activity relationship (QSAR) studies. It has been shown that Balaban index has a strong correlation with the chemical properties of the chemical compound and other topological indices octanes. Mathematical propertices of Balaban index can be found in [4-9, 11-16]. Mathematical propertices of Sum-Balaban index can be found in [4, 9, 11, 13–15]. In this paper, we consider the Balaban index and Sum-Balaban index of tricyclic graphs in $\mathcal{T}_n$.

Let $\widehat{\mathcal{T}} = \{\widehat{\mathcal{T}}^i \mid 1 \le i \le 15\}$, where graphs $\widehat{\mathcal{T}}^i$ for $i = 1, 2, \ldots, 15$ are defined in Fig. 1. By [10], we known that for any $\mathcal{T} \in \mathcal{T}_n$, $\mathcal{T}$ can be obtained from an $\widehat{\mathcal{T}}^i$ $(1 \le i \le 15)$ by attaching trees to some of its vertices. We call $\widehat{\mathcal{T}}^i$ the base of $\mathcal{T}$.

We will establish the maximum Balaban index and maximum Sum-Balaban index among all tricyclic graphs.

2 Preliminaries

In this section, we will introduce some useful lemmas and graph transformations.

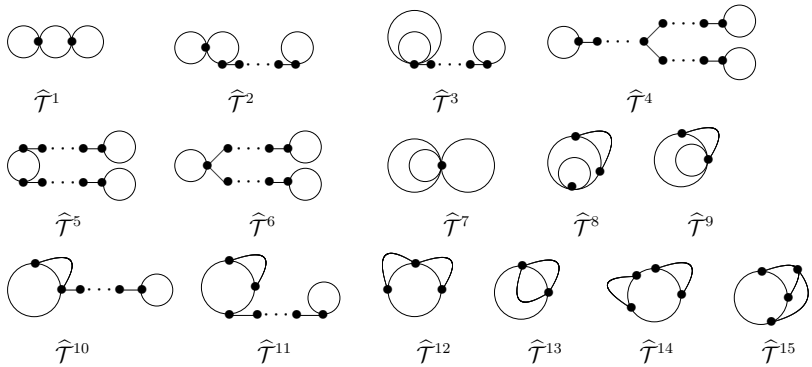


Fig. 1 The fifteen types of bases for tricyclic graphs

Lemma 2.1 ([8]). *Let $x, y, a \in R^+$ such that $x \geq y + a$. Then $\frac{1}{\sqrt{xy}} \geq \frac{1}{\sqrt{(x-a)(y+a)}}$, and the equality holds if and only if $x = y + a$.*

Lemma 2.2 ([14]). *Let $x_1, x_2, y_1, y_2 \in R^+$ such that $x_1 > y_1$ and $x_2 - x_1 = y_2 - y_1 > 0$. Then $\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{y_2}} < \frac{1}{\sqrt{x_2}} + \frac{1}{\sqrt{y_1}}$.*

2.1 Edge–lifting transformation

Let G_1 and G_2 be two graphs with $n_1 \geq 2$ and $n_2 \geq 2$ vertices, respectively. If G is the graph obtained from G_1 and G_2 by adding an edge between a vertex u_0 of G_1 and a vertex v_0 of G_2, and G' is the graph obtained by identifying u_0 of G_1 to v_0 of G_2 and adding a pendent edge to $u_0(v_0)$, then G' is called the edge-lifting transformation of G (see Fig. 2.1).

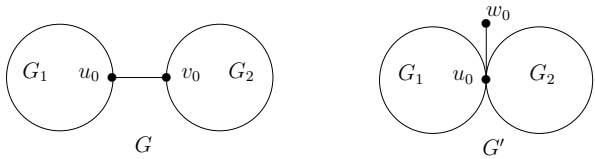


Fig. 2.1 The edge-lifting transformation

Lemma 2.3 ([5, 6]). *Let G' be the edge-lifting transformation of G. Then $J(G) < J(G')$ and $SJ(G) < SJ(G')$.*

Denote $\mathcal{T}_n^{(1)} = \{\mathcal{T}^1, \mathcal{T}^7, \mathcal{T}^8, \mathcal{T}^9, \mathcal{T}^{12}, \mathcal{T}^{13}, \mathcal{T}^{14}\}$ and $\mathcal{T}_n^{(2)} = \{\mathcal{T}^{15}\}$.

By Lemma 2.3, we can verify that if $\mathcal{T} \in \mathcal{T}_n$ attains the maximum Balaban index or maximum Sum-Balaban index of all graphs in $\mathcal{T}_n$, then the following two conditions hold.

(i) The base $\widehat{\mathcal{T}}$ of $\mathcal{T}$ is one of $\widehat{\mathcal{T}}_n^{(1)} \cup \widehat{\mathcal{T}}_n^{(2)}$.

(ii) The graph $\mathcal{T}$ is obtained from $\widehat{\mathcal{T}}$ by attaching some pendant edges.

Remark 2.4. *In order to determine the tricyclic graphs which attain the maximum Balaban index or maximum Sum-Balaban index of all graphs in $\mathcal{T}_n$, we just need to discuss the tricyclic graphs in $\mathcal{T}_n^{(1)} \cup \mathcal{T}_n^{(2)} = \{\mathcal{T}^1, \mathcal{T}^7, \mathcal{T}^8, \mathcal{T}^9, \mathcal{T}^{12}, \mathcal{T}^{13}, \mathcal{T}^{14}, \mathcal{T}^{15}\}$ (see Fig. 1).*

2.2 Cycle transformation([9])

Let $\hat{B}(p,q,t)$ be a bicyclic graph as shown in Fig.2.2, where $W_{v_i} = \{w \mid wv_i \in E(\hat{B}(p,q,t))$ and $d_{\hat{B}(p,q,t)}(w) = 1\}$ and $|W_{v_i}| = k_i$ for $1 \le i \le p$, and $W_{u_j} = \{w \mid wu_j \in E(\hat{B}(p,q,t))$ and $d_{\hat{B}(p,q,t)}(w) = 1\}$ and $|W_{u_j}| = l_j$ for $t+1 \le j \le q$.

If p is even and $p \ge 4$, then $\hat{B}'(p,q,t)$ is the graph obtained from $\hat{B}(p,q,t)$ by deleting the edge v_pv_{p-1} and all pendent vertices of v_p, meanwhile, adding the edge v_1v_{p-1} and k_p pendent edges to v_1.

If p is odd and $p \ge 5$, then $\hat{B}'(p,q,t)$ is the graph obtained from $\hat{B}(p,q,t)$ by deleting the edges $v_pv_{p-1}, v_{p-1}v_{p-2}$ and all pendent edges of v_p, v_{p-1}, meanwhile, adding the edges v_1v_{p-1}, v_1v_{p-2} and $k_p + k_{p-1}$ pendent edges to v_1.

We say that $\hat{B}'(p,q,t)$ is obtained from $\hat{B}(p,q,t)$ by the cycle transformation (see Fig. 2.2).

Lemma 2.5 ([9])**.** *Let $\hat{B} = \hat{B}(p,q,t) \in \hat{\mathcal{B}}_n$ with $p \ge q$ and $p \ge 4$, and $\hat{B}' = \hat{B}'(p,q,t)$ is obtained from $\hat{B}(p,q,t)$ by the cycle transformation (see Fig. 2.2). Then $J(\hat{B}) < J(\hat{B}')$, and $SJ(\hat{B}) < SJ(\hat{B}')$.*

Let $\mathcal{T} \in \mathcal{T}_n^{(1)}$, there exist a bicyclic subgraph of $\mathcal{T}$. We can obtained $\mathcal{G}^i (1 \le i \le 7)$ from $\mathcal{T}_n^{(1)}$ by repeating cycle transformation. Fig. 2.3 shows seven types of bases for $\widehat{\mathcal{G}}^i$, where $1 \le i \le 7$.

By Lemma 2.5, the following lemma is clear.

Lemma 2.6. *Let $\mathcal{T} \in \mathcal{G}^1 \in \mathcal{T}_n^{(1)}$, $\mathcal{T}'$ be obtained from $\mathcal{T}$ by the cycle transformation. Then $J(\mathcal{T}) < J(\mathcal{T}')$ and $SJ(\mathcal{T}) < SJ(\mathcal{T}')$.*

Remark 2.7. *In order to determine the tricyclic graphs which attain the maximum Balaban index or maximum Sum-Balaban index of all graphs in $\mathcal{T}_n$, we just need to discuss the tricyclic graphs in $\mathcal{T}_n^{(2)} \cup \mathcal{G}^i (1 \le i \le 7)$ (see Fig. 1 and Fig. 2.3).*

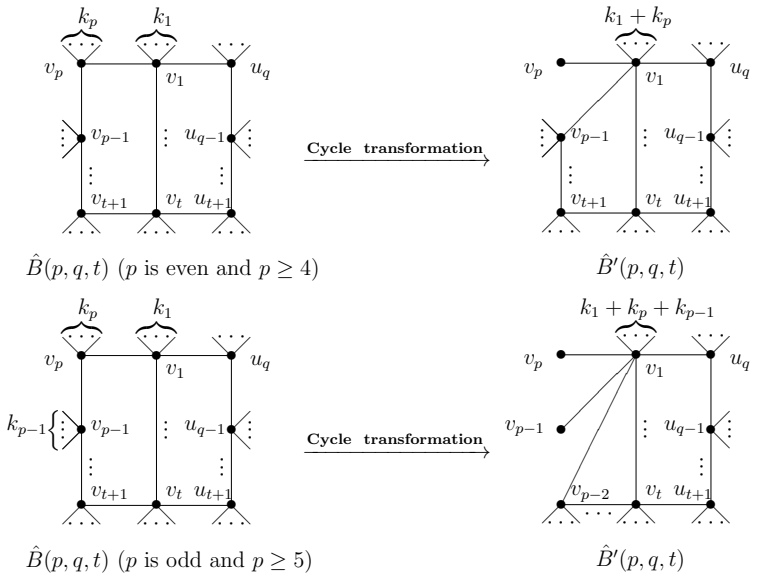


Fig. 2.2 The cycle transformation

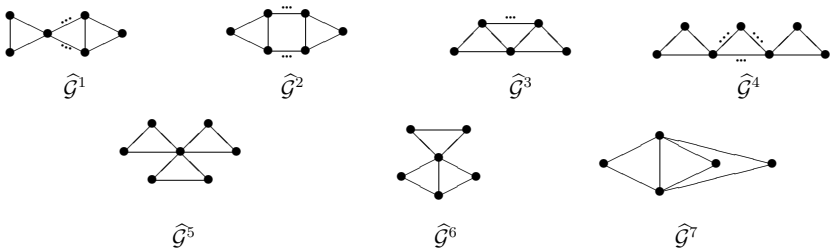


Fig. 2.3 The seven types of bases for $\mathcal{G}^i (1 \leq i \leq 7)$

2.3 Cycle transformation on a graph in $\mathcal{G}^i (1 \leq i \leq 7)$

Let $\mathcal{T} \in \mathcal{G}^1 \in \mathcal{T}_n^{(1)}$, $a + b \leq n - 6$. $V_1 = \{v_{1i}, v_{2j}\}$, where $1 \leq i \leq a$ and $1 \leq j \leq b$. $W_{v_x} = \{w \mid v_x \in V(\mathcal{T}), wv_x \in E(\mathcal{T}) \text{ and } d_{\mathcal{T}}(w) = 1\}$, $\mid W_{v_x} \mid = k_x$. $\mathcal{T}'$ is the graph obtained from $\mathcal{T}$ by deleting the edges v_2v_{1a}, v_3v_{2b}, $v_{1i}v_{1(i+1)}$ $(1 \leq i \leq a-1)$, $v_{2j}v_{2(j+1)}$ $(1 \leq j \leq b-1)$ and all pendent vertices of v_{1i} $(1 \leq i \leq a)$, v_{2j} $(1 \leq j \leq b)$, meanwhile, adding the edges v_1v_2, v_1v_3, $v_1v_{1i} (2 \leq i \leq a), v_1v_{2j} (2 \leq j \leq b)$ and $\sum_{i=1}^{a} k_{1i} + \sum_{j=1}^{b} k_{2j}$ pendent edges to v_1 (see Fig. 2.4). We say that $\mathcal{T}'$ is obtained from $\mathcal{T}$ by the cycle transformation.

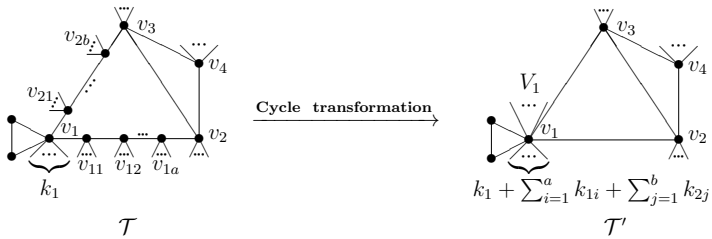


Fig. 2.4 The cycle transformation (when $a > 0, b > 0$ and $a + b \leq n - 6$)

Lemma 2.8. *Let $\mathcal{T} \in \mathcal{G}^1 \in \mathcal{T}_n^{(1)}$, $V_1 = \{v_{1i}, v_{2j}\}$, where $1 \leq i \leq a$ and $1 \leq j \leq b$, $a + b \leq n - 6$. $W_{v_x} = \{w \mid v_x \in V(\mathcal{T}), wv_x \in E(\mathcal{T})$ and $d_{\mathcal{T}}(w) = 1\}$, $\mid W_{v_x} \mid = k_x$. $\mathcal{T}'$ be obtained from $\mathcal{T}$ by the cycle transformation (see Fig. 2.4). Then $J(\mathcal{T}) < J(\mathcal{T}')$ and $SJ(\mathcal{T}) < SJ(\mathcal{T}')$.*

Proof. It can be check directly that

$$D_{\mathcal{T}}(v_x) \geq D_{\mathcal{T}'}(v_x), \text{ where } v_x \in V(\mathcal{T}) \setminus V_1;$$

$$D_{\mathcal{T}'}(v_{1i}) - D_{\mathcal{T}}(v_{1i}) \leq 2 + k_{1i} + k_2 + k_4, \text{ where } 1 \leq i \leq a;$$

$$D_{\mathcal{T}'}(v_{2j}) - D_{\mathcal{T}}(v_{2j}) \leq 2 + k_{2j} + k_3 + k_4, \text{ where } 1 \leq j \leq b;$$

$$D_{\mathcal{T}}(v_1) - D_{\mathcal{T}'}(v_1) \geq 3 + \sum_{i=1}^{a} k_{1i} + \sum_{j=1}^{b} k_{2j} + k_2 + k_3 + k_4;$$

$$D_{\mathcal{T}}(v_1) - D_{\mathcal{T}'}(v_1) > D_{\mathcal{T}'}(v_{1i}) - D_{\mathcal{T}}(v_{1i});$$

$$D_{\mathcal{T}}(v_1) - D_{\mathcal{T}'}(v_1) > D_{\mathcal{T}'}(v_{2j}) - D_{\mathcal{T}}(v_{2j});$$

$$D_{\mathcal{T}'}(v_i) - D_{\mathcal{T}'}(v_1) = n - 2, \text{ where } v_i \in V_1.$$

Then for the vertex $v_x, v_y \in V(\mathcal{T}) \setminus V_1$, we have

$$\frac{1}{\sqrt{D_{\mathcal{T}'}(v_x) D_{\mathcal{T}'}(v_y)}} \geq \frac{1}{\sqrt{D_{\mathcal{T}}(v_x) D_{\mathcal{T}}(v_y)}}, \text{ where } v_x, v_y \in V(\mathcal{T}) \setminus V_1. \tag{1}$$

$$\frac{1}{\sqrt{D_{\mathcal{T}'}(v_x) + D_{\mathcal{T}'}(v_y)}} \geq \frac{1}{\sqrt{D_{\mathcal{T}}(v_x) + D_{\mathcal{T}}(v_y)}}, \text{ where } v_x, v_y \in V(\mathcal{T}) \setminus V_1. \tag{2}$$

We following consider the edges $v_1v_{11}, v_1v_{21}, v_2v_{1a}, v_3v_{2b}, v_{1i}v_{1(i+1)}(1 \leq i \leq a-1)$, $v_{2j}v_{2(j+1)}(1 \leq j \leq b-1), v_{1i}v_x(1 \leq i \leq a), v_{2j}v_y(1 \leq j \leq b) \in E(\mathcal{T})$, where $v_x \in W_{v_{1i}}$, $v_y \in W_{v_{2j}}$.

Let $x = D_{\mathcal{T}'}(v_{11})$, $y = D_{\mathcal{T}'}(v_1)$, $a = 2 + k_{11} + k_2 + k_4 < n - 2$. Since $D_{\mathcal{T}'}(v_{11}) = D_{\mathcal{T}'}(v_1) + n - 2$, we have $x = y + n - 2 > y + a$. By Lemma 2.1, we have

$$\frac{1}{\sqrt{D_{\mathcal{T}'}(v_1)D_{\mathcal{T}'}(v_{11})}} > \frac{1}{\sqrt{D_{\mathcal{T}}(v_1)D_{\mathcal{T}}(v_{11})}}, \tag{3}$$

$$\frac{1}{\sqrt{D_{\mathcal{T}'}(v_1) + D_{\mathcal{T}'}(v_{11})}} > \frac{1}{\sqrt{D_{\mathcal{T}}(v_1) + D_{\mathcal{T}}(v_{11})}}. \tag{4}$$

Similarly, we have

$$\frac{1}{\sqrt{D_{\mathcal{T}'}(v_1)D_{\mathcal{T}'}(v_{21})}} > \frac{1}{\sqrt{D_{\mathcal{T}}(v_1)D_{\mathcal{T}}(v_{21})}}; \frac{1}{\sqrt{D_{\mathcal{T}'}(v_1)D_{\mathcal{T}'}(v_2)}} > \frac{1}{\sqrt{D_{\mathcal{T}}(v_{1a})D_{\mathcal{T}}(v_2)}};$$

$$\frac{1}{\sqrt{D_{\mathcal{T}'}(v_1)D_{\mathcal{T}'}(v_3)}} > \frac{1}{\sqrt{D_{\mathcal{T}}(v_{2b})D_{\mathcal{T}}(v_3)}};$$

$$\frac{1}{\sqrt{D_{\mathcal{T}'}(v_1)D_{\mathcal{T}'}(v_{1(i+1)})}} > \frac{1}{\sqrt{D_{\mathcal{T}}(v_{1i})D_{\mathcal{T}}(v_{1(i+1)})}}, \text{ where } 1 \le i \le a-1;$$

$$\frac{1}{\sqrt{D_{\mathcal{T}'}(v_1)D_{\mathcal{T}'}(v_{2(j+1)})}} > \frac{1}{\sqrt{D_{\mathcal{T}}(v_{2j})D_{\mathcal{T}}(v_{2(j+1)})}}, \text{ where } 1 \le j \le b-1;$$

$$\frac{1}{\sqrt{D_{\mathcal{T}'}(v_1)D_{\mathcal{T}'}(v_x)}} > \frac{1}{\sqrt{D_{\mathcal{T}}(v_{1i})D_{\mathcal{T}}(v_x)}}, \text{ where } 1 \le i \le a, v_x \in W_{v_{1i}};$$

$$\frac{1}{\sqrt{D_{\mathcal{T}'}(v_1)D_{\mathcal{T}'}(v_y)}} > \frac{1}{\sqrt{D_{\mathcal{T}}(v_{2j})D_{\mathcal{T}}(v_y)}}, \text{ where } 1 \le j \le b, v_y \in W_{v_{2j}}. \tag{5}$$

$$\frac{1}{\sqrt{D_{\mathcal{T}'}(v_1) + D_{\mathcal{T}'}(v_{21})}} > \frac{1}{\sqrt{D_{\mathcal{T}}(v_1) + D_{\mathcal{T}}(v_{21})}};$$

$$\frac{1}{\sqrt{D_{\mathcal{T}'}(v_1) + D_{\mathcal{T}'}(v_2)}} > \frac{1}{\sqrt{D_{\mathcal{T}}(v_{1a}) + D_{\mathcal{T}}(v_2)}};$$

$$\frac{1}{\sqrt{D_{\mathcal{T}'}(v_1) + D_{\mathcal{T}'}(v_3)}} > \frac{1}{\sqrt{D_{\mathcal{T}}(v_{2b}) + D_{\mathcal{T}}(v_3)}};$$

$$\frac{1}{\sqrt{D_{\mathcal{T}'}(v_1) + D_{\mathcal{T}'}(v_{1(i+1)})}} > \frac{1}{\sqrt{D_{\mathcal{T}}(v_{1i}) + D_{\mathcal{T}}(v_{1(i+1)})}}, \text{ where } 1 \le i \le a-1;$$

$$\frac{1}{\sqrt{D_{\mathcal{T}'}(v_1) + D_{\mathcal{T}'}(v_{2(j+1)})}} > \frac{1}{\sqrt{D_{\mathcal{T}}(v_{2j}) + D_{\mathcal{T}}(v_{2(j+1)})}}, \text{ where } 1 \le j \le b-1;$$

$$\frac{1}{\sqrt{D_{\mathcal{T}'}(v_1) + D_{\mathcal{T}'}(v_x)}} > \frac{1}{\sqrt{D_{\mathcal{T}}(v_{1i}) + D_{\mathcal{T}}(v_x)}}, \text{ where } 1 \le i \le a, v_x \in W_{v_{1i}};$$

$$\frac{1}{\sqrt{D_{\mathcal{T}'}(v_1) + D_{\mathcal{T}'}(v_y)}} > \frac{1}{\sqrt{D_{\mathcal{T}}(v_{2j}) + D_{\mathcal{T}}(v_y)}}, \text{ where } 1 \le j \le b, v_y \in W_{v_{2j}}. \tag{6}$$

By (1) (3) (5) and the definition of Balaban index, we have $J(\mathcal{T}') > J(\mathcal{T})$. By (2) (4) (6) and the definition of Sum-Balaban index, we have $SJ(\mathcal{T}') > SJ(\mathcal{T})$. ■

Fig. 2.5 shows cycle transformation on $\mathcal{T} \in \mathcal{G}^2$. Fig. 2.6 shows cycle transformation on $\mathcal{T} \in \mathcal{G}^3$. Fig. 2.7 shows cycle transformation on $\mathcal{T} \in \mathcal{G}^4$.

Using the same method as Lemma 2.8, the following lemma is clear.

Lemma 2.9. *Let $\mathcal{T} \in \mathcal{G}^i \in \mathcal{T}_n^{(1)}$ $(i = 2, 3, 4)$, $\mathcal{T}'$ be obtained from $\mathcal{T}$ by the cycle transformation (see Fig. 2.5, 2.6, 2.7). Then $J(\mathcal{T}) < J(\mathcal{T}')$ and $SJ(\mathcal{T}) < SJ(\mathcal{T}')$.*

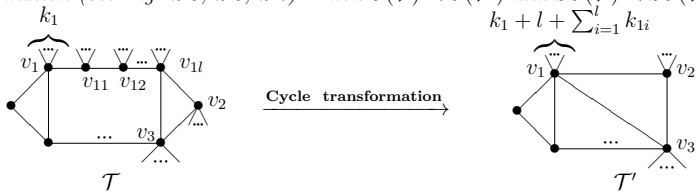


Fig. 2.5 The cycle transformation on $\mathcal{T} \in \mathcal{G}^2$

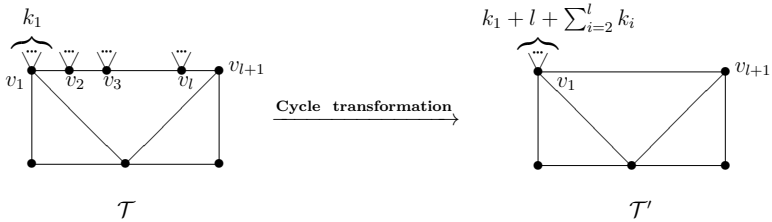


Fig. 2.6 The cycle transformation on $\mathcal{T} \in \mathcal{G}^3$

Fig. 2.7 The cycle transformation on $\mathcal{T} \in \mathcal{G}^4$

We can obtained $\mathcal{G}_2^i$ $(1 \le i \le 6)$ from $\mathcal{T}_n^{(1)}$ by repeating cycle transformation. Fig. 2.8 shows six types of bases for $\mathcal{G}_2^i$, where $1 \le i \le 6$.

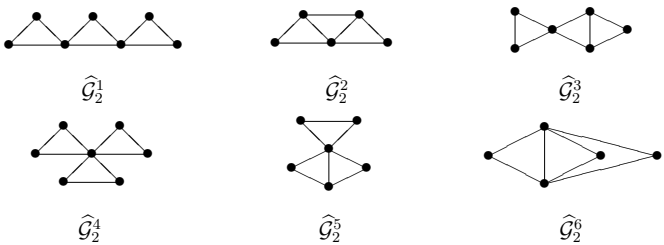


Fig. 2.8 The six types of bases for $\mathcal{G}_2^i$ $(1 \le i \le 6)$

2.4 Cycle transformation on a graph in $\mathcal{T}_n^{(2)}$

Let $\mathcal{T} \in \mathcal{T}_n^{(2)}$, $P_i \in \mathcal{T}$, where $1 \le i \le 6$. P_1 is the path from v_1 to v_2 and $\{v_3, v_4\} \notin V(P_1)$; P_2 is the path from v_1 to v_4 and $\{v_2, v_3\} \notin V(P_2)$; P_3 is the path from v_1 to v_3 and $\{v_2, v_4\} \notin V(P_3)$; P_4 is the path from v_2 to v_4 and $\{v_1, v_3\} \notin V(P_4)$; P_5 is the path from v_3 to v_4 and $\{v_1, v_2\} \notin V(P_5)$; P_6 is the path from v_2 to v_3 and $\{v_1, v_4\} \notin V(P_6)$ (see Fig. 2.9).

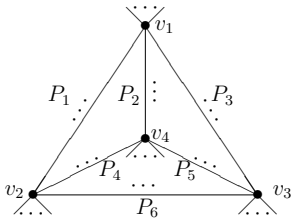


Fig. 2.9 Graph $\mathcal{T} \in \mathcal{T}_n^{(2)}$

Case 1. $|V(P_i)| \ge 3$, where $1 \le i \le 3$.

Let $W_{v_x} = \{w \mid v_x \in V(\mathcal{T}), wv_x \in E(\mathcal{T}) \text{ and } d_{\mathcal{T}}(w) = 1\}$, $\mid W_{v_x} \mid = k_x$. $\mathcal{T}'$ is the graph obtained from $\mathcal{T}$ by deleting the edges $v_{11}v_{12}, v_{21}v_{22}, v_{31}v_{32}$ (when $|V(P_1)| = 3$, $v_{12} = v_2$; when $|V(P_2)| = 3, v_{22} = v_4$; when $|V(P_3)| = 3, v_{32} = v_3$) and all pendent vertices of v_{11}, v_{21}, v_{31}, meanwhile, adding the edges $v_1v_{12}, v_1v_{22}, v_1v_{32}$ and $k_{11} + k_{21} + k_{31}$ pendent edges to v_1(see Fig. 2.10).

We say that $\mathcal{T}'$ is obtained from $\mathcal{T}$ by the cycle transformation.

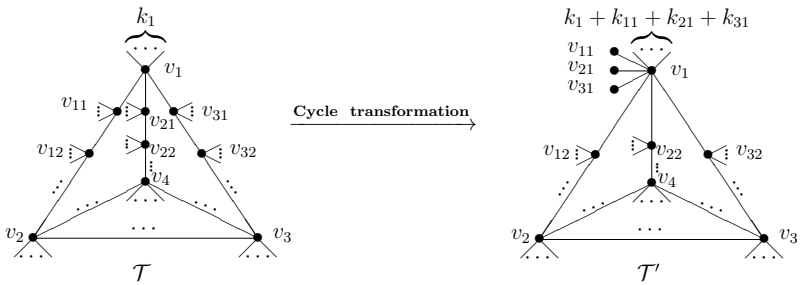


Fig. 2.10 The cycle transformation (when $|V(P_i)| \ge 3$, where $1 \le i \le 3$)

Lemma 2.10. *Let $\mathcal{T} \in \mathcal{T}_n^{(2)}$, $|V(P_i)| \ge 3$, where $1 \le i \le 3$. $W_{v_x} = \{w \mid v_x \in V(\mathcal{T}), wv_x \in E(\mathcal{T})$ and $d_{\mathcal{T}}(w) = 1\}$, $\mid W_{v_x} \mid = k_x$. $\mathcal{T}'$ is obtained from $\mathcal{T}$ by the cycle transformation (see Fig. 2.10). Then $J(\mathcal{T}) < J(\mathcal{T}')$ and $SJ(\mathcal{T}) < SJ(\mathcal{T}')$.*

Proof. It can be check directly that

$$D_{\mathcal{T}}(v_x) \geq D_{\mathcal{T}'}(v_x), \text{ where } v_x \in V(\mathcal{T}) \setminus \{v_{11}, v_{21}, v_{31}\},$$
$$D_{\mathcal{T}}(v_1) - D_{\mathcal{T}'}(v_1) > D_{\mathcal{T}'}(v_{11}) - D_{\mathcal{T}}(v_{11}),$$
$$D_{\mathcal{T}}(v_1) - D_{\mathcal{T}'}(v_1) > D_{\mathcal{T}'}(v_{21}) - D_{\mathcal{T}}(v_{21}),$$
$$D_{\mathcal{T}}(v_1) - D_{\mathcal{T}'}(v_1) > D_{\mathcal{T}'}(v_{31}) - D_{\mathcal{T}}(v_{31}),$$
$$D_{\mathcal{T}'}(v_i) - D_{\mathcal{T}'}(v_1) = n-2, \text{ where } v_i \in \{v_{11}, v_{21}, v_{31}\}.$$

Then for the vertex $v_x, v_y \in V(\mathcal{T}) \setminus \{v_{11}, v_{21}, v_{31}\}$, we have

$$\frac{1}{\sqrt{D_{\mathcal{T}'}(v_x)D_{\mathcal{T}'}(v_y)}} \geq \frac{1}{\sqrt{D_{\mathcal{T}}(v_x)D_{\mathcal{T}}(v_y)}}, \text{ where } v_x, v_y \in V(\mathcal{T}) \setminus \{v_{11}, v_{21}, v_{31}\}. \quad (7)$$

$$\frac{1}{\sqrt{D_{\mathcal{T}'}(v_x) + D_{\mathcal{T}'}(v_y)}} \geq \frac{1}{\sqrt{D_{\mathcal{T}}(v_x) + D_{\mathcal{T}}(v_y)}}, \text{ where } v_x, v_y \in V(\mathcal{T}) \setminus \{v_{11}, v_{21}, v_{31}\}. \quad (8)$$

For the edges $v_1v_{11}, v_{11}v_{12}, v_1v_{21}, v_{21}v_{22}, v_1v_{31}, v_{31}v_{32}, v_{11}v_x, v_{21}v_y, v_{31}v_z \in E(\mathcal{T})$, where $v_x \in W_{v_{11}}$, $v_y \in W_{v_{21}}$, $v_z \in W_{v_{31}}$. By Lemma 2.1, we have

$$\frac{1}{\sqrt{D_{\mathcal{T}'}(v_1)D_{\mathcal{T}'}(v_{i1})}} > \frac{1}{\sqrt{D_{\mathcal{T}}(v_1)D_{\mathcal{T}}(v_{i1})}}, \text{ where } i = 1,2,3;$$
$$\frac{1}{\sqrt{D_{\mathcal{T}'}(v_1)D_{\mathcal{T}'}(v_{i2})}} > \frac{1}{\sqrt{D_{\mathcal{T}}(v_{i1})D_{\mathcal{T}}(v_{i2})}}, \text{ where } i = 1,2,3;$$
$$\frac{1}{\sqrt{D_{\mathcal{T}'}(v_1)D_{\mathcal{T}'}(v_x)}} > \frac{1}{\sqrt{D_{\mathcal{T}}(v_{11})D_{\mathcal{T}}(v_x)}}, \text{ where } v_x \in W_{v_{11}};$$
$$\frac{1}{\sqrt{D_{\mathcal{T}'}(v_1)D_{\mathcal{T}'}(v_y)}} > \frac{1}{\sqrt{D_{\mathcal{T}}(v_{21})D_{\mathcal{T}}(v_y)}}, \text{ where } v_y \in W_{v_{21}};$$
$$\frac{1}{\sqrt{D_{\mathcal{T}'}(v_1)D_{\mathcal{T}'}(v_z)}} > \frac{1}{\sqrt{D_{\mathcal{T}}(v_{31})D_{\mathcal{T}}(v_z)}}, \text{ where } v_z \in W_{v_{31}}. \quad (9)$$
$$\frac{1}{\sqrt{D_{\mathcal{T}'}(v_1) + D_{\mathcal{T}'}(v_{i1})}} > \frac{1}{\sqrt{D_{\mathcal{T}}(v_1) + D_{\mathcal{T}}(v_{i1})}}, \text{ where } i = 1,2,3;$$
$$\frac{1}{\sqrt{D_{\mathcal{T}'}(v_1) + D_{\mathcal{T}'}(v_{i2})}} > \frac{1}{\sqrt{D_{\mathcal{T}}(v_{i1}) + D_{\mathcal{T}}(v_{i2})}}, \text{ where } i = 1,2,3;$$
$$\frac{1}{\sqrt{D_{\mathcal{T}'}(v_1) + D_{\mathcal{T}'}(v_x)}} > \frac{1}{\sqrt{D_{\mathcal{T}}(v_{11}) + D_{\mathcal{T}}(v_x)}}, \text{ where } v_x \in W_{v_{11}};$$
$$\frac{1}{\sqrt{D_{\mathcal{T}'}(v_1) + D_{\mathcal{T}'}(v_y)}} > \frac{1}{\sqrt{D_{\mathcal{T}}(v_{21}) + D_{\mathcal{T}}(v_y)}}, \text{ where } v_y \in W_{v_{21}};$$
$$\frac{1}{\sqrt{D_{\mathcal{T}'}(v_1) + D_{\mathcal{T}'}(v_z)}} > \frac{1}{\sqrt{D_{\mathcal{T}}(v_{31}) + D_{\mathcal{T}}(v_z)}}, \text{ where } v_z \in W_{v_{31}}. \quad (10)$$

By (7) (9) and the definition of Balaban index, we have $J(\mathcal{T}') > J(\mathcal{T})$. By (8) (10) and the definition of Sum-Balaban index, we have $SJ(\mathcal{T}') > SJ(\mathcal{T})$. ■

By Lemma 2.10, we can obtained $\mathcal{F}^i$ $(1 \leq i \leq 6)$ from $\mathcal{T}_n^{(2)}$ by repeating cycle transformation. Fig. 2.11 shows six types of bases for $\mathcal{F}^i$, where $1 \leq i \leq 6$.

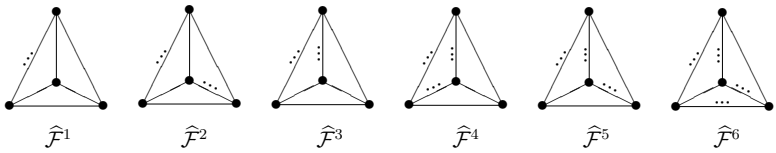


Fig. 2.11 The six types of bases for $\mathcal{F}^i (1 \leq i \leq 6)$

Case 2. $|V(P_1)| \geq 3$, $|V(P_i)| = 2$, where $2 \leq i \leq 6$ (see Fig. 2.11: $\widehat{\mathcal{F}}^1$).

Let $\mathcal{T} = \mathcal{F}^1$ and $5 \leq s \leq n$. $W_{v_x} = \{w \mid v_x \in V(\mathcal{T}), wv_x \in E(\mathcal{T})$ and $d_{\mathcal{T}}(w) = 1\}$, $\mid W_{v_x} \mid = k_x$. $\mathcal{T}'$ is the graph obtained from $\mathcal{T}$ by deleting the edges v_iv_{i+1} $(5 \leq i \leq s-1)$, v_2v_s and all pendent vertices of v_i $(5 \leq i \leq s)$, meanwhile, adding the edges v_1v_2, v_1v_i $(6 \leq i \leq s)$ and $\sum_{i=5}^{s} k_i$ pendent edges to v_1 (see Fig. 2.12).

We say that $\mathcal{T}'$ is obtained from $\mathcal{T}$ by the cycle transformation.

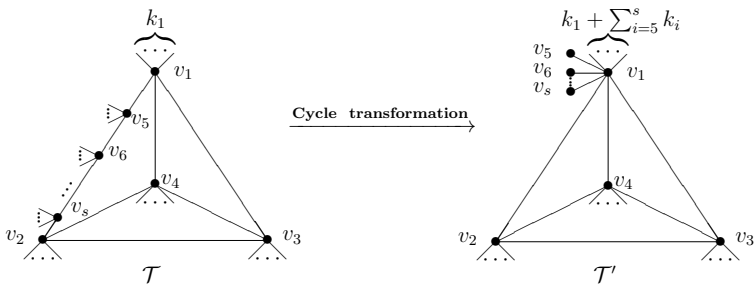


Fig. 2.12 The cycle transformation (when $|V(P_1)| \geq 3$, $|V(P_i)| = 2$, where $5 \leq s \leq n$, $2 \leq i \leq 6$)

Case 3. $|V(P_i)| \geq 3$, $|V(P_j)| = 2$, where $i = 1, 5$ and $j = 2, 3, 4, 6$, $a + b \leq n - 4$ (see Fig. 2.11: $\widehat{\mathcal{F}}^2$).

Let $\mathcal{T} = \mathcal{F}^2$, $W_{v_x} = \{w \mid v_x \in V(\mathcal{T}), wv_x \in E(\mathcal{T})$ and $d_{\mathcal{T}}(w) = 1\}$, $\mid W_{v_x} \mid = k_x$. $\mathcal{T}'$ is the graph obtained from $\mathcal{T}$ by deleting the edges v_2v_{1a}, v_4v_{2b}, $v_{1i}v_{1(i+1)}(1 \leq i \leq a-1), v_{2j}v_{2(j+1)}(1 \leq j \leq b-1)$ and all pendent vertices of $v_{1i}(1 \leq i \leq a), v_{2j}(1 \leq j \leq b)$,

meanwhile, adding the edges v_1v_2, v_3v_4, v_1v_{1i} $(2 \le i \le a), v_3v_{2j}$ $(2 \le j \le b)$ and $\sum_{i=1}^{a} k_{1i}$ pendent edges to v_1, $\sum_{j=1}^{b} k_{2j}$ pendent edges to v_3(see Fig. 2.13).

We say that $\mathcal{T}'$ is obtained from $\mathcal{T}$ by the cycle transformation.

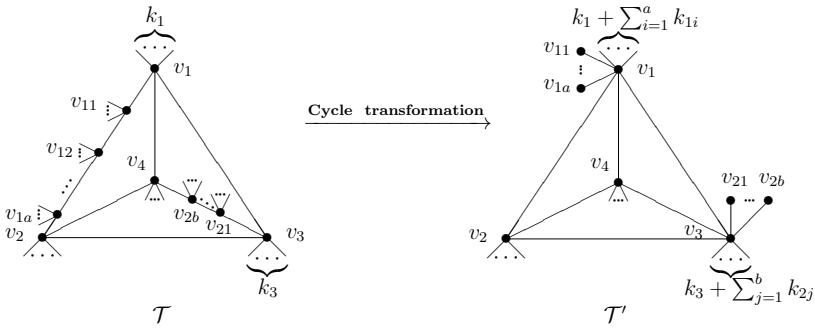


Fig. 2.13 The cycle transformation (when $|V(P_i)| \ge 3$, $|V(P_j)| = 2$, where $i = 1, 5$ and $j = 2, 3, 4, 6$, $a + b \le n - 4$)

Case 4. $|V(P_i)| \ge 3$, $|V(P_j)| = 2$, where $i = 1, 2$ and $3 \le j \le 6$, $a + b \le n - 4$ (see Fig. 2.11: $\widehat{\mathcal{F}}^3$).

Let $\mathcal{T} = \mathcal{F}^3$, $W_{v_x} = \{w \mid v_x \in V(\mathcal{T}), wv_x \in E(\mathcal{T})$ and $d_{\mathcal{T}}(w) = 1\}$, $\mid W_{v_x} \mid = k_x$. $\mathcal{T}'$ is the graph obtained from $\mathcal{T}$ by deleting the edges v_2v_{1a}, v_4v_{2b}, $v_{1i}v_{1(i+1)}$ $(1 \le i \le a-1)$, $v_{2j}v_{2(j+1)}$ $(1 \le j \le b-1)$ and all pendent vertices of v_{1i} $(1 \le i \le a)$, v_{2j} $(1 \le j \le b)$, meanwhile, adding the edges v_1v_2, v_1v_4, v_1v_{1i} $(2 \le i \le a)$, v_1v_{2j} $(2 \le j \le b)$ and $\sum_{i=1}^{a} k_{1i} + \sum_{j=1}^{b} k_{2j}$ pendent edges to v_1(see Fig. 2.14).

We say that $\mathcal{T}'$ is obtained from $\mathcal{T}$ by the cycle transformation.

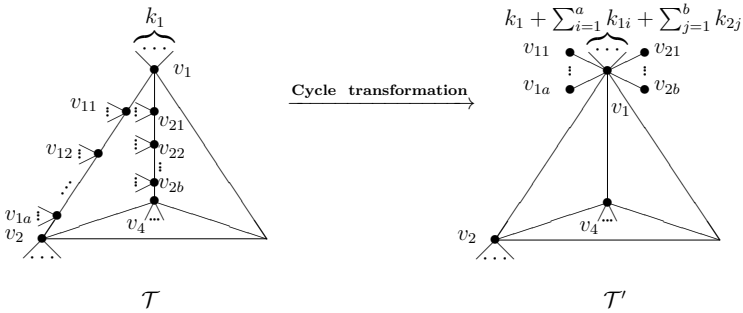


Fig. 2.14 The cycle transformation ($|V(P_i)| \ge 3$, $|V(P_j)| = 2$, where $a + b \le n - 4$, $i = 1, 2$ and $3 \le j \le 6$)

Case 5. If $|V(P_i)| \geq 3$, $|V(P_j)| = 2$, where $i = 1, 2, 4$ and $j = 3, 5, 6$, $a + b + c \leq n - 4$ (see Fig. 2.11: $\widehat{\mathcal{F}}^4$).

Let $\mathcal{T} = \mathcal{F}^4$, $W_{v_x} = \{w \mid v_x \in V(\mathcal{T}), wv_x \in E(\mathcal{T}) \text{ and } d_{\mathcal{T}}(w) = 1\}$, $\mid W_{v_x} \mid = k_x$. $\mathcal{T}'$ is the graph obtained from $\mathcal{T}$ by deleting the edges v_2v_{1a}, v_4v_{2b}, v_4v_{3c}, $v_{1i}v_{1(i+1)}$ $(1 \leq i \leq a-1), v_{2j}v_{2(j+1)}$ $(1 \leq j \leq b-1), v_{3l}v_{3(l+1)}$ $(1 \leq l \leq c-1)$ and all pendent vertices of v_{1i} $(1 \leq i \leq a), v_{2j}$ $(1 \leq j \leq b), v_{3l}$ $(1 \leq l \leq c)$, meanwhile, adding the edges v_1v_2, v_1v_4, v_2v_4, v_1v_{1i} $(2 \leq i \leq a), v_1v_{2j}$ $(2 \leq j \leq b), v_2v_{3l}$ $(2 \leq l \leq c)$ and $\sum_{i=1}^{a} k_{1i} + \sum_{j=1}^{b} k_{2j}$ pendent edges to v_1, $\sum_{l=1}^{c} k_{3l}$ pendent edges to v_2(see Fig. 2.15).

We say that $\mathcal{T}'$ is obtained from $\mathcal{T}$ by the cycle transformation.

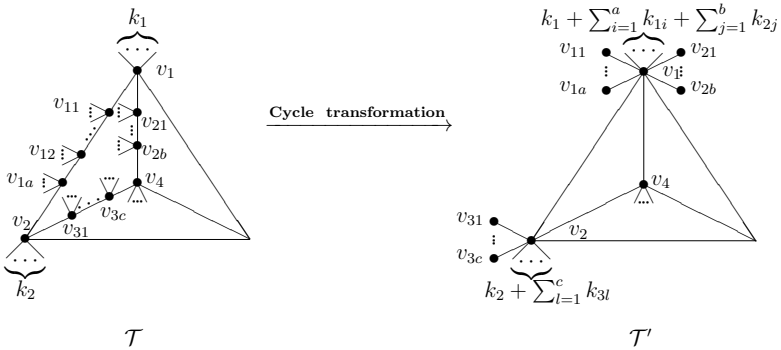


Fig. 2.15 The cycle transformation (when $|V(P_i)| \geq 3$, $|V(P_j)| = 2$, where $i = 1, 2, 4$ and $j = 3, 5, 6$, $a + b + c \leq n - 4$)

Case 6. $|V(P_i)| \geq 3$, $|V(P_j)| = 2$, where $i = 1, 2, 5$ and $j = 3, 4, 6$, $a + b + c \leq n - 4$ (see Fig. 2.11: $\widehat{\mathcal{F}}^5$).

Case 6.1. $|V(P_1)| \geq 4$ or $|V(P_5)| \geq 4$.

Let $\mathcal{T} = \mathcal{F}^5$, $V_1 = \{v_{2j}, v_{3l}\}$, where $1 \leq j \leq b, 1 \leq l \leq c$; $W_{v_x} = \{w \mid v_x \in V(\mathcal{T}), wv_x \in E(\mathcal{T}) \text{ and } d_{\mathcal{T}}(w) = 1\}$, $\mid W_{v_x} \mid = k_x$; $M_1 = \sum_{j=1}^{b} k_{2j} + \sum_{l=1}^{c} k_{3l}$. $\mathcal{T}'$ is the graph obtained from $\mathcal{T}$ by deleting the edges v_2v_{1a}, v_1v_{2b}, v_3v_{3c}, $v_{1i}v_{1(i+1)}$ $(1 \leq i \leq a-1), v_{2j}v_{2(j+1)}$ $(1 \leq j \leq b-1), v_{3l}v_{3(l+1)}$ $(1 \leq l \leq c-1)$ and all pendent vertices of v_{1i} $(1 \leq i \leq a), v_{2j}$ $(1 \leq j \leq b), v_{3l}$ $(1 \leq l \leq c)$, meanwhile, adding the edges v_1v_2, v_1v_4, v_3v_4, v_1v_{1i} $(2 \leq i \leq a), v_4v_{2j}$ $(2 \leq j \leq b), v_4v_{3l}$ $(2 \leq l \leq c)$ and $\sum_{i=1}^{a} k_{1i}$ pendent edges to v_1, $\sum_{j=1}^{b} k_{2j} + \sum_{l=1}^{c} k_{3l}$ pendent edges to v_4(see Fig. 2.16).

We say that $\mathcal{T}'$ is obtained from $\mathcal{T}$ by the cycle transformation.

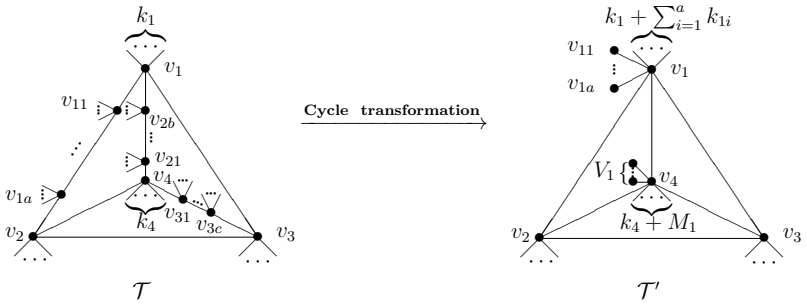


Fig. 2.16 The cycle transformation ($|V(P_1)| \geq 4$, $|V(P_i)| \geq 3$, $|V(P_j)| = 2$, where $i = 2, 5$ and $j = 3, 4, 6$, $a + b + c \leq n - 4$)

Case 6.2. $|V(P_2)| \geq 4$.

Let $\mathcal{T} = \mathcal{F}^5$, $W_{v_x} = \{w \mid v_x \in V(\mathcal{T}), wv_x \in E(\mathcal{T}) \text{ and } d_{\mathcal{T}}(w) = 1\}$, $\mid W_{v_x} \mid = k_x$. $\mathcal{T}'$ is the graph obtained from $\mathcal{T}$ by deleting the edges $v_2v_{1a}, v_4v_{2b}, v_4v_{3c}, v_{1i}v_{1(i+1)}(1 \leq i \leq a-1)$, $v_{2j}v_{2(j+1)}$ $(1 \leq j \leq b-1), v_{3l}v_{3(l+1)}$ $(1 \leq l \leq c-1)$ and all pendent vertices of v_{1i} $(1 \leq i \leq a), v_{2j}$ $(1 \leq j \leq b), v_{3l}$ $(1 \leq l \leq c)$, meanwhile, adding the edges v_1v_2, v_1v_4, v_3v_4, $v_1v_{1i}(2 \leq i \leq a), v_1v_{2j}$ $(2 \leq j \leq b), v_3v_{3l}$ $(2 \leq l \leq c)$ and $\sum_{i=1}^{a} k_{1i} + \sum_{j=1}^{b} k_{2j}$ pendent edges to v_1, $\sum_{l=1}^{c} k_{3l}$ pendent edges to v_3(see Fig. 2.17).

We say that $\mathcal{T}'$ is obtained from $\mathcal{T}$ by the cycle transformation.

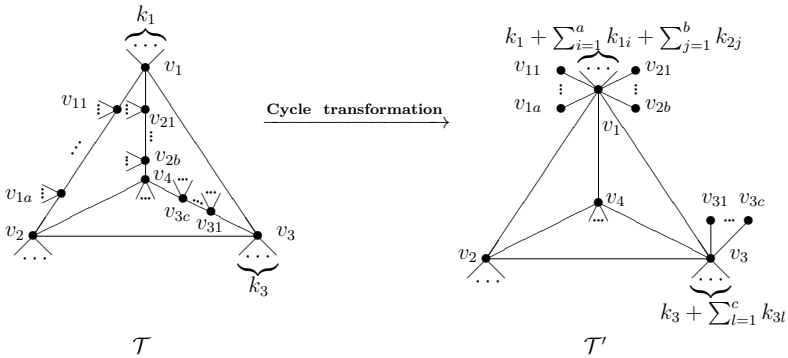


Fig. 2.17 The cycle transformation ($|V(P_2)| \geq 4$, $|V(P_i)| \geq 3$, $|V(P_j)| = 2$, where $i = 1, 5$ and $j = 3, 4, 6$, $a + b + c \leq n - 4$)

Case 6.3. $|V(P_i)| = 3$, $|V(P_j)| = 2$, where $i = 1, 2, 4$ and $j = 3, 5, 6$ (see Fig. 2.11: $\widehat{\mathcal{F}}^5$).

Case 6.3.1. $k_1 > 0$ or $k_5 > 0$.

Let $\mathcal{T} = \mathcal{F}^5$, $W_{v_x} = \{w \mid v_x \in V(\mathcal{T}), wv_x \in E(\mathcal{T}) \text{ and } d_{\mathcal{T}}(w) = 1\}$, $\mid W_{v_x} \mid = k_x$. $\mathcal{T}'$ is the graph obtained from $\mathcal{T}$ by deleting the edges v_2v_5, v_3v_7, v_4v_6, and all pendent vertices of v_5, v_6, v_7, meanwhile, adding the edges v_1v_2, v_1v_4, v_3v_4 and $k_5 + k_6$ pendent edges to v_1, k_7 pendent edges to v_4(see Fig. 2.18).

We say that $\mathcal{T}'$ is obtained from $\mathcal{T}$ by the cycle transformation.

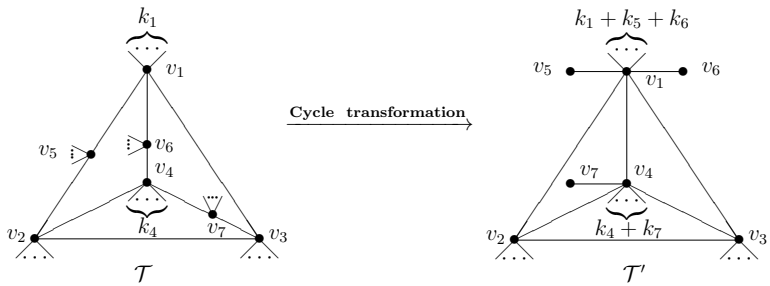


Fig. 2.18 The cycle transformation (when $|V(P_i)| = 3$, $|V(P_j)| = 2$, $k_1 > 0$ or $k_5 > 0$, where $i = 1, 2, 5$ and $j = 3, 4, 6$)

Case 6.3.2. $k_4 > 0$ or $k_7 > 0$.

Let $\mathcal{T} = \mathcal{F}^5$, $\mathcal{T}'$ is the graph obtained from $\mathcal{T}$ by deleting the edges v_1v_6, v_2v_5, v_3v_7, and all pendent vertices of v_5, v_6, v_7, meanwhile, adding the edges v_1v_2, v_1v_4, v_3v_4 and k_5 pendent edges to v_1 and $k_6 + k_7$ pendent edges to v_4 (see Fig. 2.19).

We say that $\mathcal{T}'$ is obtained from $\mathcal{T}$ by the cycle transformation.

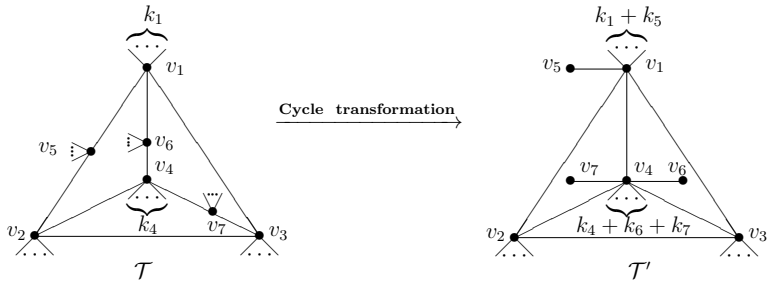


Fig. 2.19 The cycle transformation (when $|V(P_i)| = 3$, $|V(P_j)| = 2$, $k_4 > 0$ or $k_7 > 0$, where $i = 1, 2, 5$ and $j = 3, 4, 6$)

Case 6.3.3. $k_6 > 0$.

Let $\mathcal{T} = \mathcal{F}^5$, $\mathcal{T}'$ is the graph obtained from $\mathcal{T}$ by deleting the edges v_1v_5, v_1v_6, v_3v_7

and all pendent vertices of v_5, v_6, v_7, meanwhile, adding the edges v_1v_2, v_1v_4, v_3v_4 and k_5 pendent edges to v_2 and $k_6 + k_7$ pendent edges to v_4(see Fig. 2.20).

We say that $\mathcal{T}'$ is obtained from $\mathcal{T}$ by the cycle transformation.

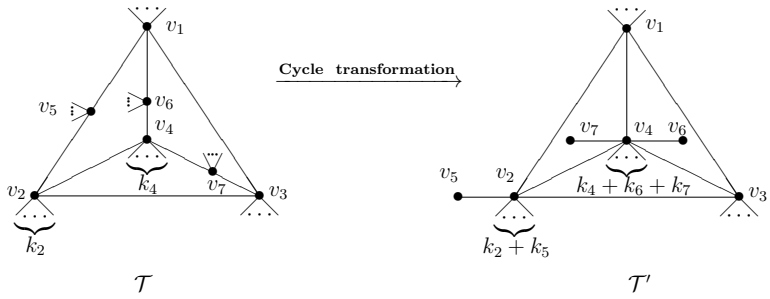


Fig. 2.20 The cycle transformation (when $|V(P_i)| = 3$, $|V(P_j)| = 2$, $k_6 > 0$, where $i = 1, 2, 5$ and $j = 3, 4, 6$)

Case 6.3.4. $k_i = 0$, $k_2 \geq k_3 \geq 0$, where $i = 1, 4, 5, 6, 7$.

Let $\mathcal{T} = \mathcal{F}^5$, $\mathcal{T}'$ is the graph obtained from $\mathcal{T}$ by deleting the edges $v_1v_5, v_1v_6, v_4v_6, v_3v_7$ and v_4v_7, meanwhile, adding the edges $v_1v_2, v_1v_4, v_2v_6, v_2v_7$ and v_3v_4(see Fig. 2.21).

We say that $\mathcal{T}'$ is obtained from $\mathcal{T}$ by the cycle transformation.

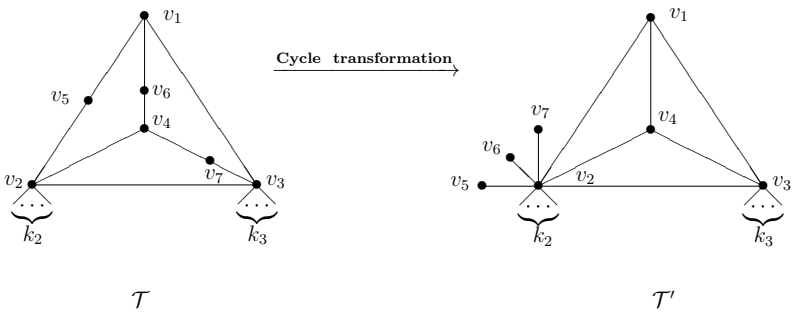


Fig. 2.21 The cycle transformation (when $|V(P_i)| = 3$, $|V(P_j)| = 2$, $k_l = 0$, $k_2 \geq k_3 \geq 0$, where $i = 1, 2, 5$, $j = 3, 4, 6$, $l = 1, 4, 5, 6, 7$)

Case 6.3.5. $k_i = 0$, $k_3 > k_2 \geq 0$, where $i = 1, 4, 5, 6, 7$.

Let $\mathcal{T} = \mathcal{F}^5$, $\mathcal{T}'$ is the graph obtained from $\mathcal{T}$ by deleting the edges $v_1v_5, v_2v_5, v_1v_6, v_4v_6$ and v_4v_7, meanwhile, adding the edges $v_1v_2, v_1v_4, v_3v_4, v_3v_5$ and v_3v_6(see Fig. 2.22).

We say that $\mathcal{T}'$ is obtained from $\mathcal{T}$ by the cycle transformation.

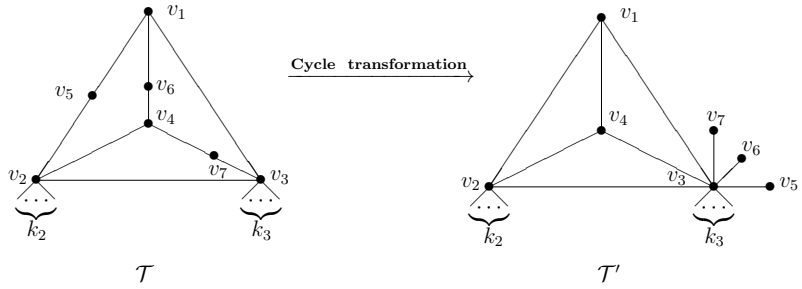


Fig. 2.22 The cycle transformation (when $|V(P_i)| = 3$, $|V(P_j)| = 2$, $k_l = 0$, $k_3 > k_2 \geq 0$, where $i = 1, 2, 5$, $j = 3, 4, 6$, $l = 1, 4, 5, 6, 7$)

Case 7. $|V(P_i)| \geq 3$ and $|V(P_j)| = 2$, where $i = 1, 2, 5, 6$ and $j = 3, 4$ (see Fig. 2.11: $\widehat{\mathcal{F}}^6$).

Let $\mathcal{T} = \mathcal{F}^6$, $\mathcal{T}'$ is the graph obtained from $\mathcal{T}$ by deleting the edges $v_{11}v_{12}, v_{21}v_{22}, v_{31}v_{32}, v_{41}v_{42}$ (might $v_{12} = v_2, v_{22} = v_4, v_{32} = v_4, v_{42} = v_2$) and all pendent vertices of $v_{11}, v_{21}, v_{31}, v_{41}$, meanwhile, adding the edges $v_1v_{12}, v_1v_{22}, v_3v_{32}, v_3v_{42}$ and $k_{11} + k_{21}$ pendent edges to v_1 and $k_{31} + k_{41}$ pendent edges to v_3(see Fig. 2.23).

We say that $\mathcal{T}'$ is obtained from $\mathcal{T}$ by the cycle transformation.

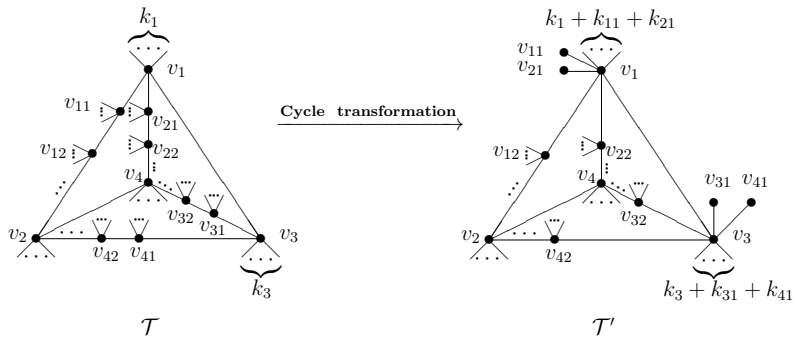


Fig. 2.23 The cycle transformation (when $|V(P_i)| \geq 3$, $|V(P_j)| = 2$, where $i = 1, 2, 5, 6$ and $j = 3, 4$)

Using the same method as Lemma 2.10, the following lemma is clear.

Lemma 2.11. *Let $\mathcal{T} = \mathcal{F}^i \in \mathcal{T}_n^{(2)}$, $1 \leq i \leq 6$. $\mathcal{T}'$ is obtained from $\mathcal{T}$ by the cycle transformation (see Fig. 2.12-2.23). Then $J(\mathcal{T}) < J(\mathcal{T}')$ and $SJ(\mathcal{T}) < SJ(\mathcal{T}')$.*

By Lemmas 2.10 and 2.11, we can obtained $\mathcal{G}_2^7$ from $\mathcal{T}_n^{(2)}$ by repeating cycle transformation. Fig. 2.24 shows the base for $\mathcal{G}_2^7$.

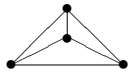


Fig. 2.24 The graph $\widehat{\mathcal{G}}_2^7$

Remark 2.12. *By Remark 2.11 and Lemmas 2.10, 2.11, we now only need to consider the Balaban indices and Sum-Balaban indices of graphs $\mathcal{G}_2^i$, where $1 \leq i \leq 7$ (see Fig. 2.8 and Fig. 2.24).*

2.5 Cycle–lifting transformation

Let G_1,G_2 and G_3 be three graphs with $n_1 \geq 2$, $n_2 \geq 2$ and $n_3 \geq 1$ vertices, respectively. If G is the graph obtained from G_1, G_2 and G_3 by adding an edge between v_1 and v_2, v_1 and v_3, v_2 and v_3, G' is the graph obtained by deleting the edges $v_2w \in G_2$ and adding the edges v_1w(see Fig. 2.25).

We say that G' is obtained from G by the cycle-lifting transformation.

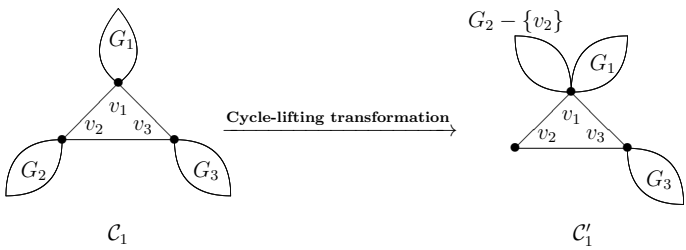


Fig. 2.25 The cycle-lifting transformation

Lemma 2.13. *Let $\mathcal{C}_1'$ be the cycle-lifting transformation of $\mathcal{C}_1$ (see Fig.2.25). Then $J(\mathcal{C}_1) < J(\mathcal{C}_1')$ and $SJ(\mathcal{C}_1) < SJ(\mathcal{C}_1')$.*

Proof. Let $V(\mathcal{C}_1) = \{v_1, v_2, v_3, \cdots, v_n\}$. It can be check directly that

$$D_{\mathcal{C}_1}(v_x) \geq D_{\mathcal{C}_1'}(v_x) \text{ for } v_x \in V(\mathcal{C}_1) \setminus \{v_2\},$$

$$D_{\mathcal{C}_1'}(v_2) - D_{\mathcal{C}_1}(v_2) = D_{\mathcal{C}_1}(v_1) - D_{\mathcal{C}_1'}(v_1) > 0,$$

$$D_{\mathcal{C}_1'}(v_2) > \max\{D_{\mathcal{C}_1}(v_1), D_{\mathcal{C}_1}(v_2)\} > D_{\mathcal{C}_1'}(v_1).$$

For the vertices $v_x, v_y \in V(\mathcal{C}_1) \setminus \{v_2\}$, it is easy to see that

$$\frac{1}{\sqrt{D_{\mathcal{C}_1'}(v_x)D_{\mathcal{C}_1'}(v_y)}} \geq \frac{1}{\sqrt{D_{\mathcal{C}_1}(v_x)D_{\mathcal{C}_1}(v_y)}}, \tag{11}$$

$$\frac{1}{\sqrt{D_{\mathcal{C}_1'}(v_x)+D_{\mathcal{C}_1'}(v_y)}} \geq \frac{1}{\sqrt{D_{\mathcal{C}_1}(v_x)+D_{\mathcal{C}_1}(v_y)}}. \tag{12}$$

For $v_1v_2 \in E(\mathcal{C}_1)$, letting $x = D_{\mathcal{C}_1'}(v_2)$, $y = D_{\mathcal{C}_1'}(v_1)$, $a = D_{\mathcal{C}_1'}(v_2) - D_{\mathcal{C}_1}(v_2) = D_{\mathcal{C}_1}(v_1) - D_{\mathcal{C}_1'}(v_1) > 0$, then $x > y + a$. By Lemma 2.1, we have

$$\frac{1}{\sqrt{D_{\mathcal{C}_1'}(v_1)D_{\mathcal{C}_1'}(v_2)}} > \frac{1}{\sqrt{D_{\mathcal{C}_1}(v_1)D_{\mathcal{C}_1}(v_2)}}, \tag{13}$$

$$\frac{1}{\sqrt{D_{\mathcal{C}_1'}(v_1)+D_{\mathcal{C}_1'}(v_2)}} = \frac{1}{\sqrt{D_{\mathcal{C}_1}(v_1)+D_{\mathcal{C}_1}(v_2)}}. \tag{14}$$

For $v_2v_3, v_1v_3 \in E(\mathcal{C}_1)$, letting $x_2 = D_{\mathcal{C}_1'}(v_2)$, $x_1 = D_{\mathcal{C}_1}(v_2)$, $y_2 = D_{\mathcal{C}_1}(v_1)$, $y_1 = D_{\mathcal{C}_1'}(v_1)$, then $x_1 > y_1$ and $x_2 - x_1 = y_2 - y_1 > 0$. By Lemma 2.2, we have

$$\frac{1}{\sqrt{D_{\mathcal{C}_1'}(v_2)}} + \frac{1}{\sqrt{D_{\mathcal{C}_1'}(v_1)}} > \frac{1}{\sqrt{D_{\mathcal{C}_1}(v_2)}} + \frac{1}{\sqrt{D_{\mathcal{C}_1}(v_1)}}.$$

Meanwhile, $D_{\mathcal{C}_1}(v_3) = D_{\mathcal{C}_1'}(v_3)$, then

$$\frac{1}{\sqrt{D_{\mathcal{C}_1'}(v_2)D_{\mathcal{C}_1'}(v_3)}} + \frac{1}{\sqrt{D_{\mathcal{C}_1'}(v_1)D_{\mathcal{C}_1'}(v_3)}} > \frac{1}{\sqrt{D_{\mathcal{C}_1}(v_2)D_{\mathcal{C}_1}(v_3)}} + \frac{1}{\sqrt{D_{\mathcal{C}_1}(v_1)D_{\mathcal{C}_1}(v_3)}}. \tag{15}$$

Let $x_2 = D_{\mathcal{C}_1'}(v_2) + D_{\mathcal{C}_1'}(v_3)$, $x_1 = D_{\mathcal{C}_1}(v_2) + D_{\mathcal{C}_1}(v_3)$, $y_2 = D_{\mathcal{C}_1}(v_1) + D_{\mathcal{C}_1}(v_3)$, $y_1 = D_{\mathcal{C}_1'}(v_1) + D_{\mathcal{C}_1'}(v_3)$. Then $x_1 > y_1$ and $x_2 - x_1 = y_2 - y_1 > 0$. By Lemma 2.2, we have

$$\frac{1}{\sqrt{D_{\mathcal{C}_1'}(v_2)+D_{\mathcal{C}_1'}(v_3)}} + \frac{1}{\sqrt{D_{\mathcal{C}_1'}(v_1)+D_{\mathcal{C}_1'}(v_3)}} > \frac{1}{\sqrt{D_{\mathcal{C}_1}(v_2)+D_{\mathcal{C}_1}(v_3)}} + \frac{1}{\sqrt{D_{\mathcal{C}_1}(v_1)+D_{\mathcal{C}_1}(v_3)}}. \tag{16}$$

For each edge $v_2v_x \in E(G_2)$, we have $D_{\mathcal{C}_1}(v_2) > D_{\mathcal{C}_1'}(v_1)$, and $D_{\mathcal{C}_1}(v_x) \geq D_{\mathcal{C}_1'}(v_x)$, where $v_x \in V(G) \setminus \{v_2\}$, then

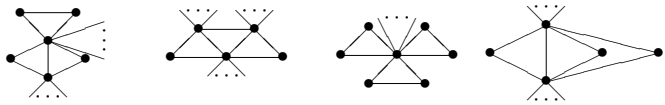
$$\frac{1}{\sqrt{D_{\mathcal{C}_1'}(v_1)D_{\mathcal{C}_1'}(v_x)}} > \frac{1}{\sqrt{D_{\mathcal{C}_1}(v_2)D_{\mathcal{C}_1}(v_x)}}, \tag{17}$$

$$\frac{1}{\sqrt{D_{\mathcal{C}_1'}(v_1)+D_{\mathcal{C}_1'}(v_x)}} > \frac{1}{\sqrt{D_{\mathcal{C}_1}(v_2)+D_{\mathcal{C}_1}(v_x)}}. \tag{18}$$

By (11),(13),(15),(17) and the definition of Balaban index, we have $J(\mathcal{C}_1') > J(\mathcal{C}_1)$.

By (12),(14),(16),(18) and the definition of Sum-Balaban index, we have $SJ(\mathcal{C}_1') > SJ(\mathcal{C}_1)$. ■

We can obtained $\mathcal{G}_3^i$ $(1 \leq i \leq 4)$ from $\mathcal{G}_2^i$ $(1 \leq i \leq 6)$ by repeating cycle-lifting transformation (see Fig. 2.26).



$\mathcal{G}_3^1$ $\mathcal{G}_3^2$ $\mathcal{G}_3^3$ $\mathcal{G}_3^4$

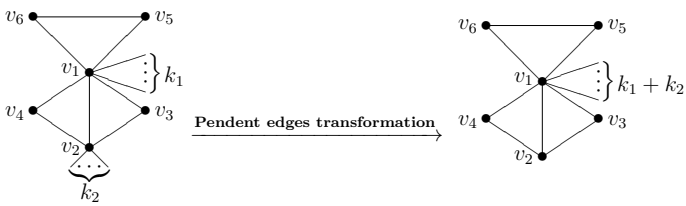
Fig. 2.26 Graphs $\mathcal{G}_3^i$ $(1 \leq i \leq 4)$

Remark 2.14. *By Lemma 2.13, we now only need to consider the Balaban indices and Sum-Balaban indices of graphs $\mathcal{G}_3^i$ and $\mathcal{G}_2^7$, where $1 \leq i \leq 4$ (see Fig. 2.24 and Fig. 2.26).*

2.6 Pendent edges transformation on $\mathcal{G}_3^i$ and $\mathcal{G}_2^7$ $(i = 1, 2, 4)$

2.6.1 Pendent edges transformation on $\mathcal{G}_3^1$

Let $C_1 = v_1v_2v_3$, $C_2 = v_1v_2v_4$, $C_3 = v_1v_5v_6$, $W_{v_i} = \{w \mid wv_i \in E(\mathcal{G}_3^1) \text{ and } d_{\mathcal{G}_3^1}(w) = 1\}$, and $|W_{v_i}| = k_i$ for $1 \leq i \leq 2$. The graph $\mathcal{G}_3^{1'}$ is obtained from $\mathcal{G}_3^1$ by deleting the pendent edges of v_2, and adding k_2 pendent edges to v_1. We say that $\mathcal{G}_3^{1'}$ is obtained from $\mathcal{G}_3^1$ by pendent edges transformation (see Fig. 2.27).



$\mathcal{G}_3^1$ $\mathcal{G}_3^{1'}$

Fig. 2.27 The pendent edges transformation on $\mathcal{G}_3^1$

Lemma 2.15. *Let $G' = \mathcal{G}_3^{1'}$ be the pendent edges transformation of $G = \mathcal{G}_3^1$ and $k_2 > 0$ (see Fig. 2.27). Then $J(G) < J(G')$ and $SJ(G) < SJ(G')$.*

Proof. It can be check directly that

$$D_G(v_x) \geq D_{G'}(v_x), \text{ where } v_x \in V(G) \setminus \{v_2\},$$

$$D_{G'}(v_2) - D_G(v_2) = D_G(v_1) - D_{G'}(v_1) = k_2 > 0,$$

$$D_G(v_2) > D_{G'}(v_1).$$

Case 1. $v_x, v_y \in V(G) \setminus \{v_2\}$.

For the vertex $v_x, v_y \in V(G) \setminus \{v_2\}$, we have

$$\frac{1}{\sqrt{D_{G'}(v_x)D_{G'}(v_y)}} \geq \frac{1}{\sqrt{D_G(v_x)D_G(v_y)}}, \text{ where } v_x, v_y \in V(G) \setminus \{v_2\}. \tag{19}$$

$$\frac{1}{\sqrt{D_{G'}(v_x) + D_{G'}(v_y)}} \geq \frac{1}{\sqrt{D_G(v_x) + D_G(v_y)}}, \text{ where } v_x, v_y \in V(G) \setminus \{v_2\}. \tag{20}$$

Case 2. $v_1v_2 \in E(G)$.

Let $x = D_{G'}(v_2)$, $y = D_{G'}(v_1)$, $a = D_{G'}(v_2) - D_G(v_2) = k_2 > 0$. Then $x > y + a$. By Lemma 2.1, we have

$$\frac{1}{\sqrt{D_{G'}(v_1)D_{G'}(v_2)}} > \frac{1}{\sqrt{D_G(v_1)D_G(v_2)}}, \tag{21}$$

$$\frac{1}{\sqrt{D_{G'}(v_1) + D_{G'}(v_2)}} = \frac{1}{\sqrt{D_G(v_1) + D_G(v_2)}}. \tag{22}$$

Case 3. $v_1v_3, v_2v_3 \in E(G)$.

Let $x_2 = D_{G'}(v_2)$, $x_1 = D_G(v_2)$, $y_2 = D_G(v_1)$, $y_1 = D_{G'}(v_1)$. Then $x_1 > y_1$ and $x_2 - x_1 = y_2 - y_1 > 0$. By Lemma 2.2, we have

$$\frac{1}{\sqrt{D_{G'}(v_2)}} + \frac{1}{\sqrt{D_{G'}(v_1)}} > \frac{1}{\sqrt{D_G(v_2)}} + \frac{1}{\sqrt{D_G(v_1)}}.$$

Meanwhile, $D_G(v_3) = D_{G'}(v_3)$, then

$$\frac{1}{\sqrt{D_{G'}(v_2)D_{G'}(v_3)}} + \frac{1}{\sqrt{D_{G'}(v_1)D_{G'}(v_3)}} > \frac{1}{\sqrt{D_G(v_2)D_G(v_3)}} + \frac{1}{\sqrt{D_G(v_1)D_G(v_3)}}. \tag{23}$$

Let $x_2 = D_{G'}(v_2) + D_{G'}(v_3)$, $x_1 = D_G(v_2) + D_G(v_3)$, $y_2 = D_G(v_1) + D_G(v_3)$, $y_1 = D_{G'}(v_1) + D_{G'}(v_3)$. Then $x_1 > y_1$ and $x_2 - x_1 = y_2 - y_1 > 0$. By Lemma 2.2, we have

$$\frac{1}{\sqrt{D_{G'}(v_2) + D_{G'}(v_3)}} + \frac{1}{\sqrt{D_{G'}(v_1) + D_{G'}(v_3)}} > \frac{1}{\sqrt{D_G(v_2) + D_G(v_3)}} + \frac{1}{\sqrt{D_G(v_1) + D_G(v_3)}}. \tag{24}$$

Case 4. $v_1v_4, v_2v_4 \in E(G)$. Since $D_G(v_4) = D_{G'}(v_4) = D_G(v_3) = D_{G'}(v_3)$, we have

$$\frac{1}{\sqrt{D_{G'}(v_2)D_{G'}(v_4)}} + \frac{1}{\sqrt{D_{G'}(v_1)D_{G'}(v_4)}} > \frac{1}{\sqrt{D_G(v_2)D_G(v_4)}} + \frac{1}{\sqrt{D_G(v_1)D_G(v_4)}}, \tag{25}$$

$$\frac{1}{\sqrt{D_{G'}(v_2) + D_{G'}(v_4)}} + \frac{1}{\sqrt{D_{G'}(v_1) + D_{G'}(v_4)}} > \frac{1}{\sqrt{D_G(v_2) + D_G(v_4)}} + \frac{1}{\sqrt{D_G(v_1) + D_G(v_4)}}. \tag{26}$$

Case 5. $v_2w \in E(G)$, where $w \in W_{v_2}$.

Since $D_G(v_2) > D_{G'}(v_1)$, $D_G(w) > D_{G'}(w)$, where $v_x \in W_{v_2}$, we have

$$\frac{1}{\sqrt{D_{G'}(v_1)D_{G'}(w)}} > \frac{1}{\sqrt{D_G(v_2)D_G(w)}}, \text{ where } w \in W_{v_2}, \tag{27}$$

$$\frac{1}{\sqrt{D_{G'}(v_1) + D_{G'}(v_i)}} > \frac{1}{\sqrt{D_G(v_2) + D_G(w)}}, \text{ where } w \in W_{v_2}. \tag{28}$$

By (19) (21) (23) (25) (27) and the definition of Balaban index, we have $J(G') > J(G)$. By (20) (22) (24) (26) (28) and the definition of Sum-Balaban index, we have $SJ(G') > SJ(G)$. ■

2.6.2 Pendent edges transformation on $\mathcal{G}_3^2$

Let $C_1 = v_1v_2v_3$, $C_2 = v_1v_2v_4$, $C_3 = v_1v_3v_5$, $W_{v_i} = \{w \mid wv_i \in E(\mathcal{G}_3^2) \text{ and } d_{\mathcal{G}_3^2}(w) = 1\}$ and $|W_{v_i}| = k_i$ for $1 \le i \le 3$. Choose any $i \in \{2, 3\}$. The graph $\mathcal{G}_3^{2'}$ is obtained from $\mathcal{G}_3^2$ by deleting the pendent edges of v_i, and adding k_i pendent edges to v_1. We say that $\mathcal{G}_3^{2'}$ is obtained from $\mathcal{G}_3^2$ by pendent edges transformation (see Fig. 2.28).

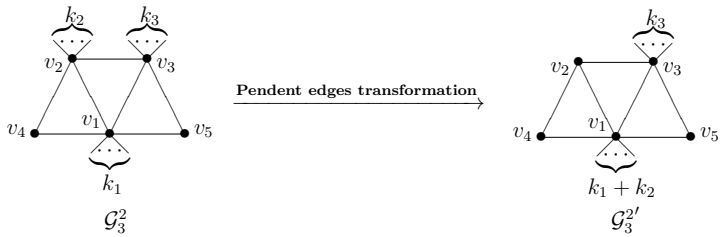


Fig. 2.28 The pendent edges transformation on $\mathcal{G}_3^2$(choose i=2)

2.6.3 Pendent edges transformation on $\mathcal{G}_3^4$.

Let $C_1 = v_1v_2v_3$, $C_2 = v_1v_2v_4$, $C_3 = v_1v_2v_5$, $W_{v_i} = \{w \mid wv_i \in E(\mathcal{G}_3^4) \text{ and } d_{\mathcal{G}_3^4}(w) = 1\}$, $|W_{v_i}| = k_i$ for $1 \le i \le 2$. The graph $\mathcal{G}_3^{4'}$ is obtained from $\mathcal{G}_3^4$ by deleting the pendent edges of v_2, and adding k_2 pendent edges to v_1(see Fig. 2.29).

We say that $\mathcal{G}_3^{4'}$ is obtained from $\mathcal{G}_3^4$ by pendent edges transformation.

2.6.4 Pendent edges transformation on $\mathcal{G}_2^7$

Let $C_1 = v_1v_2v_3$, $C_2 = v_1v_3v_4$, $C_3 = v_2v_3v_4$, $W_{v_i} = \{w \mid wv_i \in E(\mathcal{G}_2^7) \text{ and } d_{\mathcal{G}_2^7}(w) = 1\}$, $|W_{v_i}| = k_i$ for $1 = 1, 2$. The graph $\mathcal{G}_2^{7'}$ is obtained from $\mathcal{G}_2^7$ by deleting the pendent edges of v_2, and adding k_2 pendent edges to v_1. We say that $\mathcal{G}_2^{7'}$ is obtained from $\mathcal{G}_2^7$ by pendent edges transformation (see Fig. 2.30).

Using the same method as Lemma 2.15, the following lemma is clear.

Lemma 2.16. *Let $\mathcal{G}_3^{i\prime}$ be the pendent edges transformation of $\mathcal{G}_3^i$ and $i \in \{2, 4\}$ (see Fig. 2.28, 2.29). Then $J(\mathcal{G}_3^i) < J(\mathcal{G}_3^{i\prime})$ and $SJ(\mathcal{G}_3^i) < SJ(\mathcal{G}_3^{i\prime})$.*

Lemma 2.17. *Let $\mathcal{G}_2^{7\prime}$ be the pendent edges transformation of $\mathcal{G}_2^7$ (see Fig. 2.30). Then $J(\mathcal{G}_2^7) < J(\mathcal{G}_2^{7\prime})$ and $SJ(\mathcal{G}_2^7) < SJ(\mathcal{G}_2^{7\prime})$.*

Fig. 2.29 The pendent edges transformation on $\mathcal{G}_3^4$

Fig. 2.30 The pendent edges transformation on $\mathcal{G}_2^7$ (choose i=2)

We can obtained G_1, G_2, G_3, G_4, G_5 from $\mathcal{G}_3^i (1 \leq i \leq 4)$ and $\mathcal{G}_2^7$ by repeating cycle-lifting transformation and pendent edges transformation (see Fig. 2.31).

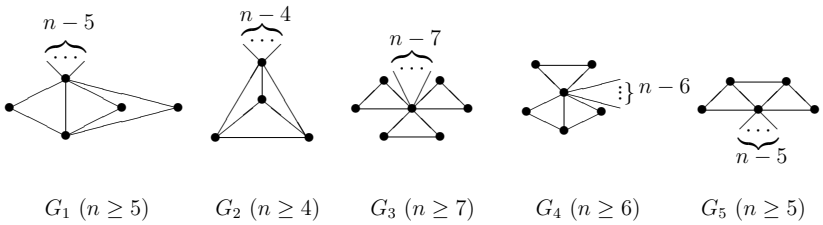


Fig. 2.31 Graphs $G_i (1 \leq i \leq 5)$

3 Maximum Balaban index and sum–Balaban index of tricyclic graphs

Remark 3.1. *From the discussions of Section 2, for any tricyclic graph $\mathcal{T} \in \mathcal{T}_n$, we finally get the graph G_i $(i = 1, 2, 3, 4, 5)$ from $\mathcal{T}$ by edge-lifting transformation, cycle transformation, cycle-lifting transformation, pendent edges transformation, or any combination of these, where graphs G_i $(i = 1, 2, 3, 4, 5)$ are defined in Fig. 2.31.*

From the discussions of Section 2, we have

$$J(\mathcal{T}) \leq \max\{J(G_i)\} \text{ and } SJ(\mathcal{T}) \leq \max\{SJ(G_i)\}, \text{ where } 1 \leq i \leq 5.$$

We will prove

$$J(\mathcal{T}) \leq \max\{J(G_i)\} = \begin{cases} J(G_2), & \text{if } n = 4; \\ J(G_1), & \text{if } n \geq 5. \end{cases}$$

and

$$SJ(\mathcal{T}) \leq \max\{SJ(G_i)\} = \begin{cases} SJ(G_2), & \text{if } n = 4; \\ SJ(G_1), & \text{if } n \geq 5. \end{cases}$$

That is to say, G_1 and G_2 attain the maximum Balaban index and Sum-Balaban index of all graphs in $\mathcal{T}_n$.

Theorem 3.2. *Let G_i $(1 \leq i \leq 5)$ be defined in Fig. 2.31, $n \geq 4$. Then*

(i) $\max\{J(G_i)\}$

$$= \begin{cases} J(G_2) = \frac{3n+6}{4\sqrt{2n^2-7n+5}} + \frac{3n+6}{8n-20} + \frac{n^2-2n-8}{4\sqrt{2n^2-5n+3}}, & \text{if } n = 4; \\ J(G_1) = \frac{n+2}{4\sqrt{2n^2-8n+6}} + \frac{3n+6}{4\sqrt{2n^2-6n+4}} + \frac{3n+6}{8\sqrt{n^2-5n+6}} + \frac{n^2-3n-10}{4\sqrt{2n^2-5n+3}}, & \text{if } n \geq 5. \end{cases}$$

(ii)

$$\max\{SJ(G_i)\} = \begin{cases} J(G_2) = \frac{3n+6}{4\sqrt{3n-6}} + \frac{3n+6}{4\sqrt{4n-10}} + \frac{n^2-2n-8}{4\sqrt{3n-4}}, & \text{if } n = 4; \\ J(G_1) = \frac{n+2}{4\sqrt{3n-7}} + \frac{3n+6}{4\sqrt{3n-5}} + \frac{3n+6}{4\sqrt{4n-10}} + \frac{n^2-3n-10}{4\sqrt{3n-4}}. & \text{if } n \geq 5. \end{cases}$$

Proof. Obviously, when $n = 4$,

$$\max\{J(G_i)\} = \max\{J(G_2)\}, \max\{SJ(G_i)\} = \max\{SJ(G_2)\}.$$

We following consider $n \geq 4$.

(i) It can be check directly that

$$\begin{aligned}
J(G_1) &= \tfrac{n+2}{4}\Big[\tfrac{1}{\sqrt{(n-1)(2n-6)}} + \tfrac{3}{\sqrt{(n-1)(2n-4)}} + \tfrac{3}{\sqrt{(2n-6)(2n-4)}} + \tfrac{n-5}{\sqrt{(n-1)(2n-3)}}\Big]; \\
J(G_2) &= \tfrac{n+2}{4}\Big[\tfrac{3}{\sqrt{(n-1)(2n-5)}} + \tfrac{3}{2n-5} + \tfrac{n-4}{\sqrt{(n-1)(2n-3)}}\Big]; \\
J(G_3) &= \tfrac{n+2}{4}\Big[\tfrac{6}{\sqrt{(n-1)(2n-4)}} + \tfrac{3}{2n-4} + \tfrac{n-7}{\sqrt{(n-1)(2n-3)}}\Big]; \\
J(G_4) &= \tfrac{n+2}{4}\Big[\tfrac{1}{\sqrt{(n-1)(2n-5)}} + \tfrac{4}{\sqrt{(n-1)(2n-4)}} + \tfrac{1}{2n-4} + \tfrac{2}{\sqrt{(2n-4)(2n-5)}} + \tfrac{n-6}{\sqrt{(n-1)(2n-3)}}\Big]; \\
J(G_5) &= \tfrac{n+2}{4}\Big[\tfrac{2}{\sqrt{(n-1)(2n-5)}} + \tfrac{2}{\sqrt{(n-1)(2n-4)}} + \tfrac{1}{2n-5} + \tfrac{2}{\sqrt{(2n-5)(2n-4)}} + \tfrac{n-5}{\sqrt{(n-1)(2n-3)}}\Big].
\end{aligned}$$

Then $\max\{J(G_i)\}$

$$= \begin{cases} J(G_2) = \frac{3n+6}{4\sqrt{2n^2-7n+5}} + \frac{3n+6}{8n-20} + \frac{n^2-2n-8}{4\sqrt{2n^2-5n+3}}, & \text{if } n = 4; \\ J(G_1) = \frac{n+2}{4\sqrt{2n^2-8n+6}} + \frac{3n+6}{4\sqrt{2n^2-6n+4}} + \frac{3n+6}{8\sqrt{n^2-5n+6}} + \frac{n^2-3n-10}{4\sqrt{2n^2-5n+3}}, & \text{if } n \geq 5. \end{cases}$$

(ii) It can be check directly that

$$\begin{aligned} SJ(G_1) &= \tfrac{n+2}{4}\left(\tfrac{1}{\sqrt{3n-7}} + \tfrac{3}{\sqrt{3n-5}} + \tfrac{3}{\sqrt{4n-10}} + \tfrac{n-5}{\sqrt{3n-4}}\right); \\ SJ(G_2) &= \tfrac{n+2}{4}\left(\tfrac{3}{\sqrt{3n-6}} + \tfrac{3}{\sqrt{4n-10}} + \tfrac{n-4}{\sqrt{3n-4}}\right); \\ SJ(G_3) &= \tfrac{n+2}{4}\left(\tfrac{6}{\sqrt{3n-5}} + \tfrac{3}{\sqrt{4n-8}} + \tfrac{n-7}{\sqrt{3n-4}}\right); \\ SJ(G_4) &= \tfrac{n+2}{4}\left(\tfrac{1}{\sqrt{3n-6}} + \tfrac{4}{\sqrt{3n-5}} + \tfrac{1}{\sqrt{4n-8}} + \tfrac{2}{\sqrt{4n-9}} + \tfrac{n-6}{\sqrt{3n-4}}\right); \\ SJ(G_5) &= \tfrac{n+2}{4}\left(\tfrac{2}{\sqrt{3n-6}} + \tfrac{2}{\sqrt{3n-5}} + \tfrac{1}{\sqrt{4n-10}} + \tfrac{2}{\sqrt{4n-9}} + \tfrac{n-5}{\sqrt{3n-4}}\right). \end{aligned}$$

Then

$$\max\{SJ(G_i)\} = \begin{cases} J(G_2) = \frac{3n+6}{4\sqrt{3n-6}} + \frac{3n+6}{4\sqrt{4n-10}} + \frac{n^2-2n-8}{4\sqrt{3n-4}}, & \text{if } n = 4; \\ J(G_1) = \frac{n+2}{4\sqrt{3n-7}} + \frac{3n+6}{4\sqrt{3n-5}} + \frac{3n+6}{4\sqrt{4n-10}} + \frac{n^2-3n-10}{4\sqrt{3n-4}}. & \text{if } n \geq 5. \end{cases}$$

The theorem holds. ■

Acknowledgment: The authors are very grateful to the anonymous referees for many valuable suggestions, which greatly improved the quality of this paper.

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