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Steiner Degree Distance*

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Abstract

The degree distance $DD(G)$ of a connected graphs G was invented by Dobrynin and Kochetova in 1994. Recently, one of the present authors introduced the concept of k-center Steiner degree distance defined as

$$SDD_k(G) = \sum_{\substack{S\subseteq V(G)\\ |S|=k}} \left[\sum_{v\in S} deg_G(v)\right] d_G(S)$$

where $d_G(S)$ is the Steiner k-distance of S and $deg_G(v)$ is the degree of the vertex v in G. Expressions for SDD_k for some special graphs are obtained, as well as sharp upper and lower bounds of SDD_k of a connected graph. Some properties of SDD_k of trees are established.

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1 Introduction

All graphs in this paper are assumed to be undirected, finite, and simple. Let G be such a graph with vertex set $V(G)$ and edge set $E(G)$. Then the order and size of G are $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$. In other words, G has n vertices and m edges.

The degree $deg_G(u)$ of the vertex $u \in V(G)$ is number of first neighbors of this vertex.

Distance is one of the basic concepts of graph theory [7]. If G is connected and $u, v \in V(G)$, then the *distance* $d(u,v) = d_G(u,v)$ between u and v is the length of a shortest path connecting u and v. If v is a vertex of a connected graph G, then the *eccentricity* $\varepsilon(v)$ of v is defined as $\varepsilon(v) = \max\{d(u,v) \,|\, u \in V(G)\}$. Furthermore, the *radius* $rad(G)$ and *diameter* $diam(G)$ of G are $rad(G) = \min\{\varepsilon(v) \,|\, v \in V(G)\}$ and $diam(G) = \max\{\varepsilon(v) \,|\, v \in V(G)\}$. These latter two quantities are related by the inequalities $rad(G) \leq diam(G) \leq 2rad(G)$. More details on this subject can be found in [14].

We refer to [5] for graph theoretical notation and terminology not specified here.

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For a graph G with vertex set $V(G)$, the *degree distance* is defined as [13]

$$DD = DD(G) = \sum_{\{u,v\} \in V(G)} [deg_G(u) + deg_G(v)] d_G(u,v) \tag{1}$$

For more details on degree–and–distance–based graph invariant, we refer to [2–4, 6, 16, 17, 22, 24, 27].

The Wiener index $W(G)$ of the graph G is defined as

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u,v)\,. \tag{2}$$

Details on this oldest distance–based topological index can be found in numerous surveys, e.g., in [11, 12, 18, 19, 25, 26, 28].

The Steiner distance of a graph, introduced by Chartrand et al. in 1989 [9], is a natural and nice generalization of the concept of classical graph distance. For a subset S of the vertex set $V(G)$, consisting of at least two vertices, the *Steiner distance* $d(S)$ (or simply the distance of S) is the minimum size (number of edges) of a connected subgraphs whose vertex set contains S. This connected subgraph is necessarily a tree and is referred to as a *Steiner tree*.

Note that if $S = \{u, v\}$, then $d(S) = d(u, v)$ is nothing new, but the classical distance between u and v. Clearly, if $|S| = k$, then $d(S) \geq k - 1$.

Let n and k be integers such that $2 \leq k \leq n$. The *Steiner k-eccentricity* $\varepsilon_k(v)$ of a vertex v of G is $\varepsilon_k(v) = \max\{d(S) \,|\, S \subseteq V(G),\ |S| = k, \text{and } v \in S\}$. The *Steiner k-radius* of G is $srad_k(G) = \min\{\varepsilon_k(v) \,|\, v \in V(G)\}$, whereas the *Steiner k-diameter* of G is $sdiam_k(G) = \max\{\varepsilon_k(v) \,|\, v \in V(G)\}$. Note that for every connected graph G, $\varepsilon_2(v) = \varepsilon(v)$ for all vertices v of G, $srad_2(G) = rad(G)$ and $sdiam_2(G) = diam(G)$. For more details on Steiner distance, we refer to [1, 8–10, 14, 23].

The following result is immediate.

Observation 1. *If H is a connected spanning subgraph of G, then*

$$sdiam_k(G) \leq sdiam_k(H)$$

holds for all k, $2 \leq k \leq n$.

Li et al. [20] put forward a Steiner–distance–based generalization of the Wiener index concept. According to [20], the *k-center Steiner Wiener index* $SW_k(G)$ of the graph G is defined by

$$SW_k(G) = \sum_{\substack{S \subseteq V(G) \\ |S| = k}} d(S)\,. \tag{3}$$

For $k = 2$, the above defined Steiner Wiener index coincides with the ordinary Wiener index, Eq. (2). It is usual to consider SW_k for $2 \leq k \leq n - 1$, but the above definition would be applicable also in the cases $k = 1$ and $k = n$, implying $SW_1(G) = 0$ and $SW_n(G) = n - 1$.

In [20], Li et al. obtained the following results.

Lemma 2. [20] *Let S_n, P_n, K_n, and $K_{a,b}$ be the star, path, and complete graph of order n, and the complete bipartite graph of order $a + b$. Then for k being an integer such that $2 \leq k \leq n - 2$ or $2 \leq k \leq a + b - 2$,*

$$SW_k(S_n) = (n-1)\binom{n-1}{k-1} \tag{4}$$

$$SW_k(P_n) = (k-1)\binom{n+1}{k+1} \tag{5}$$

$$SW_k(K_n) = (k-1)\binom{n}{k} \tag{6}$$

$$SW_k(K_{a,b}) = \begin{cases} (k-1)\binom{a+b}{k} + \binom{a}{k} + \binom{b}{k} & \textit{if } \ 1 \leq k \leq a \\ (k-1)\binom{a+b}{k} + \binom{b}{k} & \textit{if } \ a < k \leq b \\ (k-1)\binom{a+b}{k} & \textit{if } \ b < k \leq a+b\,. \end{cases} \tag{7}$$

Recently, one of the present authors [15] offered an analogous generalization of the concept of degree distance, Eq. (1). Thus, the *k-center Steiner degree distance $SDD_k(G)$* of G is defined as

$$SDD_k(G) = \sum_{\substack{S\subseteq V(G)\\ |S|=k}} \left[\sum_{v\in S} deg_G(v)\right] d_G(S)\,. \tag{8}$$

In Section 2, we report expressions for the k-center Steiner degree distance of the star, path, as well as the complete and complete bipartite graphs. We also get general expressions for $SDD_k(G)$ for $k = n, n-1$. In Section 3, we obtain sharp lower and upper bounds for SDD_k.

2 Steiner degree distance of special graphs

Beginning this section, we note that in the special case $k = 2$, all formulas derived here for the k-center Steiner degree distance reduce to expressions for the ordinary degree distance.

Theorem 1. *Let S_n, P_n, K_n, and $K_{a,b}$ be the path, star, and complete graph of order n, and the complete bipartite graph of order $a+b$. Then for k being an integer such that $2 \leq k \leq n-2$ or $2 \leq k \leq a+b-2$,*

$$SDD_k(S_n) = (2kn-n-3k+2)\binom{n-1}{k-1} \tag{9}$$

$$SDD_k(P_n) = \frac{2(k-1)(kn-1)}{k+1}\binom{n}{k} \tag{10}$$

$$SDD_k(K_n) = n(n-1)^2\binom{n-2}{k-2} \tag{11}$$

$$SDD_k(K_{a,b}) = \begin{cases} 2ab(k-1)\binom{a+b-1}{k-1} + ka\binom{b}{k} + kb\binom{a}{k} & \textit{if } 1 \le k \le a \\[2ex] 2ab(k-1)\binom{a+b-1}{k-1} + ka\binom{b}{k} & \textit{if } a < k \le b \\[2ex] 2ab(k-1)\binom{a+b-1}{k-1} + k(k-1)a\binom{b}{k} & \textit{if } b < k \le a+b\,. \end{cases} \tag{12}$$

Proof. We first verify the expression for $SDD_k(S_n)$.

Let w be the center of S_n. For any $S \subseteq V(S_n)$ and $|S| = k$, if $w \in S$, then $d_{S_n}(S) = k-1$ and $\sum_{v\in S} deg_{S_n}(v) = (n-1)+(k-1) = n+k-2$. If $w \notin S$, then $d_{S_n}(S) = k$ and $\sum_{v\in S} deg_{S_n}(v) = k$. Therefore,

$$\begin{aligned} SDD_k(S_n) &= \sum_{\substack{S\subseteq V(S_n)\\ |S|=k\,,\,w\in S}} \left[\sum_{v\in S} deg_{S_n}(v)\right] d_{S_n}(S) + \sum_{\substack{S\subseteq V(S_n)\\ |S|=k\,,\,w\notin S}} \left[\sum_{v\in S} deg_{S_n}(v)\right] d_{S_n}(S) \\ &= \binom{n-1}{k-1}(n+k-2)(k-1) + \binom{n-1}{k}k^2 \\ &= \binom{n-1}{k-1}(n+k-2)(k-1) + \binom{n-1}{k-1}(n-k)k \end{aligned}$$

from which Eq. (9) directly follows.

Next we consider the Steiner degree distance of K_n.

Let $S \subseteq V(K_n)$ and $|S| = k$. Since K_n is a regular graph of degree $n-1$,

$$\sum_{v\in S} deg_{K_n}(v) = k(n-1)$$

implying

$$\begin{aligned} SDD_k(K_n) &= \sum_{\substack{S\subseteq V(K_n)\\ |S|=k}} \left[\sum_{v\in S} deg_{K_n}(v)\right] d_{K_n}(S) \\ &= k(n-1) \sum_{\substack{S\subseteq V(K_n)\\ |S|=k}} d_{K_n}(S) = k(n-1)\, SW_k(K_n)\,. \end{aligned}$$

Eq. (11) follows now from Lemma 2.

The proofs of Eqs. (10) and (12) are similar, yet somewhat more complicated, and are omitted. □

Observation 3. *Let G be the connected graph of order n and size m. Then directly from the definition of $SDD_n(G)$, Eq. (8), it follows*

$$SDD_n(G) = 2m(n-1).$$

In [21], Mao obtained the following result.

Lemma 4. [21] *Let G be a connected graph of order n. Then $sdiam_{n-1}(G) = n-2$ if and only if G is 2-connected.*

We now show how to compute $SDD_{n-1}(G)$. Denote the vertex–connectivity of the graph G by $\kappa(G)$.

Theorem 2. *Let G be a connected graph of order n and size m.*

(1) *If $\kappa(G) \geq 2$, then*

$$SDD_{n-1}(G) = 2m(n-2)(n-1)\,. \tag{13}$$

(2) *If $\kappa(G) = 1$, then*

$$SDD_{n-1}(G) = 2m(n^2 - 3n + 2 + p) - \sum_{i=1}^{p} deg_G(w_i) \tag{14}$$

where w_i $(1 \leq i \leq p)$ are the cut vertices of G.

Proof. (1) Since $\kappa(G) \geq 2$, all subgraphs of G with $n-1$ vertices have $\kappa \geq 1$, and therefore the Steiner distance of the respective $n-1$ vertices is $n-2$. Hence,

$$SDD_{n-1}(G) = (n-2) \sum_{\substack{S \subseteq V(G) \\ |S|=n-1}} \left[\sum_{v \in S} deg_G(v) \right]$$

For each $v \in V(G)$, there are $(n-1)$ $(n-1)$-element subsets of $V(G)$ such that each of these contains v. The contribution of vertex v to $SDD_{n-1}(G)$ is exactly $(n-1)deg_G(v)$. From the arbitrariness of v, we get Eq. (13).

(2) Note that $w_1, w_2, \ldots, w_p$ are the cut vertices of G. For any $S \subseteq V(G)$ and $|S| = n-1$, if $V(G) \setminus S = \{w_i\}$ for some i, then by Lemma 4, $d_G(S) = n-1$. If $V(G) \setminus S \neq \{w_i\}$ for each i $(1 \leq i \leq p)$, then Lemma 4 implies $d_G(S) = n-2$. Then $SDD_{n-1}(G)$ is equal to

$$(n-2) \sum_{\substack{S \subseteq V(G) \\ |S|=n-1 \\ V(G)\setminus S \neq \{w_i\}}} \left[\sum_{v \in S} deg_G(v) \right] + (n-1) \sum_{\substack{S \subseteq V(G) \\ |S|=n-1 \\ V(G)\setminus S = \{w_i\}}} \left[\sum_{v \in S} deg_G(v) \right]$$

$$= \quad (n-2)\left[2(n-p)m - \sum_{i=p+1}^{n} deg_G(w_i)\right] + (n-1)\left[2pm - \sum_{i=1}^{p} deg_G(w_i)\right]$$

which is equal to the right–hand side of Eq. (14). □

In [15], a result for SDD_3 of a tree T was obtained. Here we calculate $SW_{n-1}(T)$.

Theorem 3. *Let T be a tree of order n, possessing p pendent vertices. Then*

$$SDD_{n-1}(T) = 2(n-1)^3 - p(2n-3) \tag{15}$$

irrespective of any other structural detail of T.

Proof. Since $k = n-1$, the respective subsets S contain all except one vertices of T. If the vertex missing from S is pendent, then the vertices contained in S form a tree of order $n-1$. Therefore $d_T(S) = n-2$, and $\sum_{v \in S} deg_T(v) = 2m(T) - 1 = 2n-3$. There are p such subsets, contributing to SDD_{n-1} by $p \times (2n-4)(n-2) = 2p(n-2)^2$.

If the vertex of T, not present in S, is non-pendent, then the vertices contained in S cannot form a tree, and the respective Steiner tree must contain all the n vertices of T. Therefore, $d_T(S) = n-1$, and $\sum_{v \in S} deg_T(v) = 2m(T) - d_T(w)$, where $w \in V(G) \setminus S$. There are $n-p$ such subsets, contributing to SDD_{n-1} by

$$\begin{aligned}
& 2(n-1)(n-p)m(T) - (n-1) \sum_{d_T(w) \geq 2} d_T(w) \\
= \quad & 2(n-1)(n-p)m(T) - 2(n-1)[2m(T) - 2p] \\
= \quad & 2(n-1)^2\,(n-p) - (n-1)(2n-2-p) \\
= \quad & 2(n-1)^3 - p(n-1)(2n-3)
\end{aligned}$$

which straightforwardly leads to Eq. (15) . □

3 Lower and upper bounds for general graphs

Denote by $\delta(G)$ and $\Delta(G)$ the smallest and greatest vertex degree of the graph G. The bounds stated as Proposition 5, follow immediately from the the definitions of the Steiner Wiener index, Eq. (3), and Steiner degree distance, Eq. (8).

Proposition 5. *Let G be a connected graph of order n. Then*

$$k\delta(G)SW_k(G) \le SDD_k(G) \le k\Delta(G)SW_k(G)$$

holds for all k, $2 \le k \le n$, with equality if and only if G is a regular graph.

Theorem 4. *Let G be a connected graph of order n and size m. Then*

$$2m\binom{n-1}{k-1}(k-1) \le SDD_k(G) \le 2m\binom{n-1}{k-1}(n-1) \tag{16}$$

holds for all k, $2 \le k \le n$.

Proof. For any $S \subseteq V(G)$ and $|S| = k$, we have $k-1 \le d_G(S) \le n-1$, and hence

$$(k-1) \sum_{\substack{S\subseteq V(G)\\ |S|=k}} \left[\sum_{v\in S} deg_G(v)\right] \le SDD_k(G) \le (n-1) \sum_{\substack{S\subseteq V(G)\\ |S|=k}} \left[\sum_{v\in S} deg_G(v)\right]. \tag{17}$$

Let

$$M = \sum_{\substack{S\subseteq V(G)\\ |S|=k}} \left[\sum_{v\in S} deg_G(v)\right].$$

For each $v \in V(G)$, there are $\binom{n-1}{k-1}$ k-subsets in G containing the vertex v. The contribution of v to M is thus $\binom{n-1}{k-1} deg_G(v)$. From the arbitrariness of v, we have

$$M = \binom{n-1}{k-1} \sum_{v\in V(G)} deg_G(v) = 2m\binom{n-1}{k-1}$$

which substituted back into (17) yields (16). □

Li et al. [20] obtained the following sharp bounds for the Steiner Wiener index.

Lemma 6. [20] *Let G be a connected graph of order n. Then*

$$\binom{n}{k}(k-1) \le SW_k(G) \le (k-1)\binom{n+1}{k+1}$$

holds for all k, $2 \le k \le n$. Moreover, the lower bound is sharp.

Proposition 7. *Let G be a connected graph of order n. Then*

$$k\delta(G)\binom{n}{k}(k-1) \le SDD_k(G) \le k\Delta(G)(k-1)\binom{n+1}{k+1}$$

holds for all k, $2 \le k \le n$. Moreover, the lower bound is sharp.

Proof. From Proposition 5 and Lemma 6, we have

$$SDD_k(G) \geq k\delta(G)\, SW_k(G) \geq k\delta(G)\binom{n}{k}(k-1)$$

and

$$SDD_k(G) \leq k\Delta(G)\, SW_k(G) \leq k\Delta(G)(k-1)\binom{n+1}{k+1}.$$

In order to show the sharpness of the lower bound, consider the complete graph K_n. Since $\Delta(K_n) = n-1$, it follows from Eqs. (6) and (11) that

$$SDD_k(K_n) = n(n-1)^2\binom{n-2}{k-2} = \binom{n}{k}k(n-1)(k-1) = k\delta(K_n)\, SW_k(K_n).$$

To show the sharpness of the upper bound, we consider the path P_2. Since $\delta(P_2) = 1$, it follows from Eqs. (5) and (10) that

$$SDD_2(P_n) = DD(P_n) = 2 = 2\Delta(P_2)\, SW_2(P_2).$$

□

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