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Energy and Seidel Energy of Graphs

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Abstract

Let G be a simple graph of order n. Let $A(G)$ and $S(G)$ be the adjacency and Seidel matrix of G, respectively. Suppose that $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A(G)$ and $\theta_1, \ldots, \theta_n$ be the eigenvalues of $S(G)$. The Seidel energy and the energy of G is defined as $\mathcal{E}(S(G)) = |\theta_1| + \cdots + |\theta_n|$ and $\mathcal{E}(G) = |\lambda_1| + \cdots + |\lambda_n|$, respectively. Willem Haemers in [W. H. Haemers, Seidel switching and graph energy, *MATCH Commun. Math. Comput. Chem.* **68** (2012) 653–659] conjectured that $\mathcal{E}(S(G)) \geq \mathcal{E}(S(K_n)) = 2n - 2$, where K_n is the complete graph of order n. Let A and B be two matrices with real entries. In this paper we give a technique for comparing the sum of powers of the absolute eigenvalues of the matrices $A - B$ and $A + B$. As an application we prove that if G is a $p-$ regular graph with no eigenvalue in the interval $(-1, 0)$ and $p \neq \frac{n-1}{2}$, then for every $0 \leq \alpha \leq 2$, $|\theta_1|^\alpha + \cdots + |\theta_n|^\alpha \geq (n-1)^\alpha + (n-1)$ while for every $2 \leq \alpha \leq 4$, $|\theta_1|^\alpha + \cdots + |\theta_n|^\alpha \leq (n-1)^\alpha + (n-1)$. This implies that the Haemers conjecture is valid for every $p-$ regular graph G with no eigenvalue in the interval $(-1, 0)$ and $p \neq \frac{n-1}{2}$, where n is the order of G. As an another application we obtain similar inequalities related to $\mathcal{E}(S(G))$ and $\mathcal{E}(G)$. For every graph G, we conjectured that $\mathcal{E}(S(G)) \geq \mathcal{E}(G)$. We proved that this conjecture is valid if G is a $p-$ regular graph of order n, $p \neq \frac{n-1}{2}$ and G has no eigenvalue in the interval $(-1, 0)$. Moreover we show that Haemers conjecture implies our conjecture.

1 Introduction

Throughout this paper we will consider only simple graphs. Let $G = (V, E)$ be a simple graph. By $V(G)$ and $E(G)$ we denote the set of all vertices and edges of G, respectively.

Let $u, v \in V(G)$. By $e = uv$ we mean the edge of G with end points u and v. The *order* of G denotes the number of vertices of G. Let G and H be two graphs. The *Cartesian product* of G and H which is denoted by $G \times H$ is the graph with vertex set $\{(u,v) : u \in V(G)$ and v $\in$ V(H)$\}$ such that (u,v) and (u',v') are adjacent if and only if $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$. The *complete graph*, the *cycle*, and the *path* of order n, are denoted by K_n , C_n , and P_n, respectively. For every vertex $v \in V(G)$, the *degree* of v is the number of edges incident with v and is denoted by $deg_G(v)$. For simplicity we write $deg(v)$ instead of $deg_G(v)$. A $k-$regular graph is a graph such that its vertices have degree k. By $\Delta(G)$ we mean the maximum degree of vertices of G. For every graph G, by $\overline{G}$ we denote the *complement* graph of G.

Let $M_n(\mathbb{R})$ and $SM_n(\mathbb{R})$ be the set of all $n \times n$ matrices and symmetric matrices with real entries, respectively. It is well known that all eigenvalues of every symmetric matrix are real. For every matrix $C \in SM_n(\mathbb{R})$, by $Spec(C)$ we mean the multi-set of all eigenvalues of C. By $D = diag(d_1, \ldots, d_n)$ we mean the diagonal $n \times n$ matrix $D = [d_{ij}]$ such that $d_{ii} = d_i$, for every $1 \leq i \leq n$. We denote the $n \times n$ matrices $diag(1, \ldots, 1)$ and $[1]$ (that is all entries is 1) by I and J, respectively.

Suppose that $\{v_1, \ldots, v_n\}$ is the set of vertices of G. The *adjacency matrix* of G, $A(G) = [a_{ij}]$ is an $n \times n$ matrix, where $a_{ij} = 1$ if v_i and v_j are adjacent and $a_{ij} = 0$, otherwise. Thus $A(G)$ is a symmetric matrix with zeros on the diagonal and all the eigenvalues of A are real. By the eigenvalues of G we mean those of its adjacency matrix. Also by $Spec(G)$ we mean $Spec(A(G))$. We say that G is *integral* if every its eigenvalue is integral. The *energy* of a matrix $A \in M_n(\mathbb{C})$, where $M_n(\mathbb{C})$ is the set of all $n \times n$ matrices with complex entries, denoted by $\mathcal{E}(A)$, is defined as the sum of the absolute values of all eigenvalues of A. The *energy* of a graph G, denoted by $\mathcal{E}(G)$, is that of its adjacency matrix. In the other words, $\mathcal{E}(G) = \mathcal{E}(A(G))$. The energy of a graph was defined by Ivan Gutman in 1978. See [5–7] for more details about the energy of graphs.

Let G be a simple graph with vertex set $\{v_1, \ldots, v_n\}$. The Seidel matrix of G which is denoted by $S(G) = [s_{ij}]$ is a $n \times n$ matrix in which $s_{11} = \cdots = s_{nn} = 0$. Also for $i \neq j$, $s_{ij} = -1$ if v_i and v_j are adjacent, and $s_{ij} = 1$ otherwise. In the other words, $S(G) = A(\overline{G}) - A(G)$. In [8], Haemers defined the Seidel energy of G, $\mathcal{E}(S(G))$, as the sum of absolute value of the eigenvalues of $S(G)$. As an example the Seidel matrix of the complete graph K_n is $I - J$. Thus $Spec(S(K_n)) = \{1-n, \underbrace{1, \ldots, 1}_{n-1}\}$. Therefore

$\mathcal{E}(S(K_n)) = 2n-2$. In [8] it was conjectured that among all graphs of order n, the complete graph K_n has the minimum Seidel energy. Motivated by this conjecture we investigate the sum of powers of the eigenvalues of Seidel matrix of graphs. One of the result of this investigation implies the Haemers conjecture for some regular graphs.

The structure of this paper is the following. In the next section we investigate the parameter $\Lambda_\alpha(C)$ for some matrices C. In section 3 we investigate the Seidel energy and prove that Haemers conjecture is true for some regular graphs. In section 4 we obtain some relations about the Seidel energy and energy of graphs. In the last section we study graphs that have small eigenvalues.

2 The eigenvalues of the conjugate matrices

In this section we state a technique for comparing the sum of powers of eigenvalues of matrices. One can apply this method to obtain some inequities related to the energies of graphs such as energy, Seidel energy, Laplacian energy and signless Laplacian. For more details about Laplacian energy and signless Laplacian see [1, 2].

Let $C = [c_{ij}] \in M_n(\mathbb{R})$. We recall that $tr(C) = \sum_{i=1}^{n} c_{ii}$. We use trC^k instead of $tr(C^k)$. We note that $tr(C) = \sum_{i=1}^{n} \lambda_i$, where $\lambda_1, \ldots, \lambda_n$ are all eigenvalues of C. For every matrix $C \in SM_n(\mathbb{R})$, we define $\Lambda_\alpha(C)$ as $\Lambda_\alpha(C) = \sum_\lambda |\lambda|^\alpha$, where the summation taken over all non-zero eigenvalues of C. It is obvious that $\Lambda_\alpha(C) = \Lambda_{\frac{\alpha}{2}}(C^2)$ and $\frac{1}{t^\alpha}\Lambda_\alpha(C) = \Lambda_\alpha(\frac{1}{t}C)$, for every $t > 0$. Let A and B be two elements of $SM_n(\mathbb{R})$. Consider the matrices $A - B$ and $A + B$. We say that $A - B$ and $A + B$ are *conjugate with respect to the pair* (A, B). In this section we intend to compare $\Lambda_\alpha(A - B)$ and $\Lambda_\alpha(A + B)$ for some real number α (since $A - B$ and $A + B$ are symmetric, all their eigenvalues are real). We note that, for every real number $\alpha \geq 0$ and for every x, $|x| \leq 1$, the binomial series $\sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$ converges to $(1 + x)^\alpha$ (see [3, p. 419]). More precisely,

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \quad , \quad \text{for every } x, \quad |x| \leq 1 \ , \ \alpha \geq 0 \ . \tag{1}$$

As an example, let G be a simple graph of order n and size m. Let $A = A(G)$. Then $\Lambda_1(A) = \mathcal{E}(G)$ (the energy of G) and $\Lambda_2(A) = 2m$.

Lemma 1. *Let $D \in SM_n(\mathbb{R})$ and $Spec(D) = \{\mu_1, \ldots, \mu_n\}$. Let*

$$\ell \geq \max\left\{\frac{|\mu_1|}{\sqrt{2}}, \ldots, \frac{|\mu_n|}{\sqrt{2}}\right\}.$$

Then

$$\Lambda_\alpha(D) = \ell^\alpha \sum_{k=0}^{\infty} \binom{\alpha/2}{k} tr\left(\frac{1}{\ell^2}D^2 - I\right)^k . \tag{2}$$

Proof. First we prove the assertion for any matrix $C \in SM_n(\mathbb{R})$ such that all its eigenvalues lie in the interval $[-\sqrt{2}, \sqrt{2}]$ (in the other words $\ell = 1$). Let $C \in SM_n(\mathbb{R})$ and $Spec(C) = \{\lambda_1, \dots, \lambda_n\}$ such that $|\lambda_i^2 - 1| \leq 1$, for $1 \leq i \leq n$. We have

$$\Lambda_\alpha(C^2) = (\lambda_1^2)^\alpha + \cdots + (\lambda_n^2)^\alpha = (1 + \lambda_1^2 - 1)^\alpha + \cdots + (1 + \lambda_n^2 - 1)^\alpha .$$

Using Eq. (1) we can write

$$\Lambda_\alpha(C^2) = \sum_{k=0}^{\infty} \binom{\alpha}{k} (\lambda_1^2 - 1)^k + \cdots + \sum_{k=0}^{\infty} \binom{\alpha}{k} (\lambda_n^2 - 1)^k = \sum_{k=0}^{\infty} \binom{\alpha}{k} \sum_{i=1}^{n} (\lambda_i^2 - 1)^k .$$

Thus we obtain that

$$\Lambda_\alpha(C^2) = \sum_{k=0}^{\infty} \binom{\alpha}{k} tr(C^2 - I)^k \tag{3}$$

where I is the $n \times n$ diagonal matrix $diag(1, \dots, 1)$.

Now, we prove the Lemma for the matrix D. We note that for every $t > 0$, $\Lambda_\alpha(D) = t^\alpha \Lambda_{\alpha/2}\left(\frac{1}{t^2}D^2\right)$. Putting $C = \frac{1}{\ell}D$ in Eq. (3) completes the proof. □

Remark 2. *Let $D, E \in SM_n(\mathbb{R})$. Eq. (2) shows that for comparing $\Lambda_\alpha(D)$ and $\Lambda_\alpha(E)$ it is convenient to compare $tr(\frac{1}{\ell^2}D^2 - I)^k$ and $tr(\frac{1}{\ell^2}E^2 - I)^k$ (equivalently comparing $tr(D^2 - \ell^2 I)^k$ and $tr(E^2 - \ell^2 I)^k$).*

Now, we state the main result of this section.

Theorem 3. *Let $SM_n(\mathbb{R})$ be as mentioned above. Let $D, E \in SM_n(\mathbb{R})$. Suppose that $Spec(D) = \{\lambda_1, \dots, \lambda_n\}$ and $Spec(E) = \{\mu_1, \dots, \mu_n\}$. Let*

$$\ell_0 = \max\left\{\frac{|\lambda_1|}{\sqrt{2}}, \dots, \frac{|\lambda_n|}{\sqrt{2}}, \frac{|\mu_1|}{\sqrt{2}}, \dots, \frac{|\mu_n|}{\sqrt{2}}\right\}.$$

Suppose that for some $\ell \geq \ell_0$ we have:

i) If $k \geq 3$ is odd, then $tr(D^2 - \ell^2 I)^k \geq tr(E^2 - \ell^2 I)^k$.

ii) If $k \geq 0$ is even, then $tr(D^2 - \ell^2 I)^k \leq tr(E^2 - \ell^2 I)^k$.

Then the following hold:

1) If $0 \leq \alpha \leq 2$ and $tr(D^2 - \ell^2 I) \geq tr(E^2 - \ell^2 I)$, then $\Lambda_\alpha(D) \geq \Lambda_\alpha(E)$.

2) If $2 \le \alpha \le 4$ and $tr(E^2 - \ell^2 I) \ge tr(D^2 - \ell^2 I)$, then $\Lambda_\alpha(E) \ge \Lambda_\alpha(D)$.

Proof. It is easy to check that if $0 \le \beta \le 1$ and $k \ge 1$, then the sign of $\binom{\beta}{k}$ is $(-1)^{k+1}$. On the other hand if $1 \le \beta \le 2$ and $k \ge 2$, then the sign of $\binom{\beta}{k}$ is $(-1)^k$. Using Eq. (2) for the matrices D and E completes the proof. □

Remark 4. *Let $a_1, \dots, a_n$ be some real numbers. It is well known that*

$$\lim_{\alpha \to 0^+} \left(\frac{|a_1|^\alpha + \cdots + |a_n|^\alpha}{n} \right)^{1/\alpha} = |a_1 \cdots a_n|^{1/n} .$$

Let $X, Y \in SM_n(\mathbb{R})$. Suppose that $Spec(X) = \{x_1, \dots, x_n\}$ and $Spec(Y) = \{y_1, \dots, y_n\}$. Assume that $\Lambda_\alpha(X) \ge \Lambda_\alpha(Y)$ for every α, $\alpha_0 \ge \alpha > 0$. Using the above equality we obtain that $|x_1 \cdots x_n| \ge |y_1 \cdots y_n|$.

Theorem 3 shows that for comparing $\Lambda_\alpha(D)$ and $\Lambda_\alpha(E)$ it is sufficient to compare $tr(D^2 - \ell^2 I)^k$ and $tr(E^2 - \ell^2 I)^k$. In general it is difficult to indicate the sign of $tr(D^2 - \ell^2 I)^k - tr(E^2 - \ell^2 I)^k$ for every k. In sequel we intend to investigate the sign of $tr(D^2 - \ell^2 I)^k - tr(E^2 - \ell^2 I)^k$ for the matrices $A - B$ and $A + B$ (a conjugate pair with respect to (A, B)).

Let $A, B \in SM_n(\mathbb{R})$. Let ℓ be a positive large number (as mentioned in Theorem 3). We have

$$\begin{aligned} tr\big((A+B)^2 - \ell^2 I\big)^k - tr\big((A-B)^2 - \ell^2 I\big)^k &= tr(A^2 + B^2 + AB + BA - \ell^2 I)^k \\ &- tr(A^2 + B^2 - AB - BA - \ell^2 I)^k . \end{aligned}$$

Let $X = A^2 + B^2 - \ell^2 I$ and $Y = AB + BA$. Then

$$tr\big((A+B)^2 - \ell^2 I\big)^k - tr\big((A-B)^2 - \ell^2 I\big)^k = tr(X+Y)^k - tr(X-Y)^k .$$

We have

$$tr(X+Y)^k - tr(X-Y)^k = 2 \sum_{X_i \in \{X,Y\}} tr(X_1 \cdots X_k) \tag{4}$$

where in the summation the number of Y is odd. Now, suppose that $AB = BA$. It is well known that every symmetric matrix is upper triangulable. Since $A, B \in SM_n(\mathbb{R})$ and $AB = BA$, then A and B are simultaneously upper triangulable. In the other worlds, there exists an invertible matrix $P \in M_n(\mathbb{R})$ such that $P^{-1}AP$ and $P^{-1}BP$ are upper triangular. Moreover we can say that there exists an invertible matrix $P \in M_n(\mathbb{R})$ such that $P^{-1}AP, P^{-1}BP, P^{-1}XP$ and $P^{-1}YP$ are upper triangular. Note that if $C = [c_{ij}]$ and $D = [d_{ij}]$ are two upper triangular $n \times n$ matrices, then $tr(CD) = c_{11}d_{11} + \cdots + c_{nn}d_{nn}$. Since $tr(X+Y)^k - tr(X-Y)^k = tr P^{-1}(X+Y)^k P - P^{-1} tr(X-Y)^k P$, $tr(X+Y)^k - tr(X-Y)^k = tr(P^{-1}XP + P^{-1}YP)^k - tr(P^{-1}XP - P^{-1}YP)^k$. Therefore if $P^{-1}XP = [x_{ij}]$ and $P^{-1}YP = [y_{ij}]$, then by Eq. (4) one obtains that

$$tr(X+Y)^k - tr(X-Y)^k = 2 \sum_{s \text{ is odd}} \binom{k}{s} \sum_{i=1}^{n} x_{ii}^{k-s}\, y_{ii}^{s} \,. \tag{5}$$

Suppose that $P^{-1}AP = [a_{ij}]$ and $P^{-1}BP = [b_{ij}]$. Thus $Spec(A) = \{a_{11}, \ldots, a_{nn}\}$ and $Spec(B) = \{b_{11}, \ldots, b_{nn}\}$. Therefore $x_{ii} = a_{ii}^2 + b_{ii}^2 - \ell^2$ and $y_{ii} = 2a_{ii}b_{ii}$ for $1 \le i \le n$. Using Eq. (5) we obtain

$$\begin{aligned} & tr\big((A+B)^2 - \ell^2 I\big)^k - tr\big((A-B)^2 - \ell^2 I\big)^k \\ = \; & 2 \sum_{s \text{ is odd}} \binom{k}{s} \sum_{i=1}^{n} (a_{ii}^2 + b_{ii}^2 - \ell^2)^{k-s}\, (2a_{ii}\, b_{ii})^s \,. \end{aligned} \tag{6}$$

3 The Seidel energy of graphs and Haemers conjecture

Let G be a simple graph with vertex set $\{v_1, \ldots, v_n\}$. As we mentioned in the introduction, the Seidel matrix of G, $S(G) = [s_{ij}]$, is a $n \times n$ matrix in which $s_{11} = \cdots = s_{nn} = 0$. Also for $i \neq j$, $s_{ij} = -1$ if v_i and v_j are adjacent, and $s_{ij} = 1$ otherwise. Let $Spec(S(G)) = \{\theta_1, \ldots, \theta_n\}$. In [8] Haemers defined the Seidel energy of G as $\mathcal{E}(S(G)) = \sum_{i=1}^{n} |\theta_i|$. Since $S(K_n) = I - J$, $Spec(S(K_n)) = \{1-n, \underbrace{1, \ldots, 1}_{n-1}\}$. Therefore $\mathcal{E}(S(K_n)) = 2n - 2$. Haemers conjectured that among all simple graphs of order n, the complete graph K_n has the minimum Siedel energy.

Conjecture 5. [8] *For every graph G of order n, $\mathcal{E}(S(G)) \geq \mathcal{E}(S(K_n)) = 2n - 2$.*

In this section not only we prove that for some families of regular graph the conjecture is true but also we obtain a stronger result. Let G be a graph. We can write $S(G) = A(\overline{G}) - A(G)$. On the other hand $-S(K_n) = A(K_n) = A(\overline{G}) + A(G)$. In the other words

$S(G)$ and $S(K_n)$ are conjugate with respect to $(A(\overline{G}), A(G))$. This motivated us to apply the mentioned method in section 2 and investigate $\Lambda_\alpha(S(G))$ and $\Lambda_\alpha(S(K_n))$. We note that $\lambda \in Spec(S(G))$ if and only if $-\lambda \in Spec(S(\overline{G}))$. Thus $\Lambda_\alpha(S(G)) = \Lambda_\alpha(S(\overline{G}))$.

Remark 6. *It is well known that if λ is an eigenvalue of a graph G, then $|\lambda| \leq \Delta(G)$, where $\Delta(G)$ is the maximum degree of vertices of G.*

Now, we state the main theorem of this section.

Theorem 7. *Let G be a $p-$ regular graph of order n with no eigenvalue in the interval $(-1, 0)$. Let $p \neq \frac{n-1}{2}$. Then the following hold:*

i) If $0 \leq \alpha \leq 2$, then $\Lambda_\alpha(S(G)) \geq \Lambda_\alpha(S(K_n)) = (n-1)^\alpha + (n-1)$.

ii) If $2 \leq \alpha \leq 4$, then $\Lambda_\alpha(S(G)) \leq \Lambda_\alpha(S(K_n)) = (n-1)^\alpha + (n-1)$.

Proof. Let A and $\overline{A}$ be the adjacency matrix of G and $\overline{G}$, respectively. Since G is regular, $A\overline{A} = \overline{A}A$. Thus there exists an invertible matrix $P \in M_n(\mathbb{R})$ such that $P^{-1}AP$ and $P^{-1}\overline{A}P$ are upper triangular. Let $P^{-1}AP = [\lambda_{ij}]$ and $P^{-1}\overline{A}P = [\mu_{ij}]$. Thus $Spec(A) = \{\lambda_{11}, \ldots, \lambda_{nn}\}$ and $Spec(\overline{A}) = \{\mu_{11}, \ldots, \mu_{nn}\}$. Since $P^{-1}AP + P^{-1}\overline{A}P = P^{-1}(A+\overline{A})P = P^{-1}(J-I)P$, thus $Spec(P^{-1}(J-I)P) = \{\lambda_{11}+\mu_{11}, \ldots, \lambda_{nn}+\mu_{nn}\}$. On the other hand $Spec(P^{-1}(J-I)P) = Spec(J-I) = \{n-1, \underbrace{-1, \ldots, -1}_{n-1}\}$. Therefore we may assume that $\lambda_{11} + \mu_{11} = n-1$ and $\lambda_{ii} + \mu_{ii} = -1$, for $i = 2, \ldots, n$. By Remark 6, $\lambda_{11} \leq p$ and $\mu_{11} \leq n-p-1$. Thus the equality $\lambda_{11} + \mu_{11} = n-1$ implies that $\lambda_{11} = p$ and $\mu_{11} = n-p-1$. Also we obtain $\mu_{ii} = -1 - \lambda_{ii}$, for $i = 2, \ldots, n$. In Theorem 3, let $D = A - \overline{A}$, $E = A + \overline{A}$ and $\ell = \sqrt{p^2 + (n-p-1)^2}$. Since $P^{-1}DP = P^{-1}AP - P^{-1}\overline{A}P$, $Spec(D) = \{\lambda_{11} - \mu_{11}, \ldots, \lambda_{nn} - \mu_{nn}\}$. Similarly, $Spec(E) = \{\lambda_{11} + \mu_{11}, \ldots, \lambda_{nn} + \mu_{nn}\}$. For every $\lambda \in Spec(D) \cup Spec(E)$, $|\lambda| \leq p + (n-1-p) = n-1$. It is easy to see that $\ell \geq \frac{n-1}{\sqrt{2}} \geq \frac{|\lambda|}{\sqrt{2}}$. Note that $\lambda_{ii}^2 + \mu_{ii}^2 \leq p^2 + (n-1-p)^2 = \ell^2$. Since $tr(D^2 - \ell^2 I)^k - tr(E^2 - \ell^2 I)^k = tr((A-\overline{A})^2 - \ell^2 I)^k - tr((A+\overline{A})^2 - \ell^2 I)^k$, by Eq. (6) we obtain that

$$tr(D^2 - \ell^2 I)^k - tr(E^2 - \ell^2 I)^k = 2 \sum_{s \text{ is odd}} \binom{k}{s} \sum_{i=1}^{n} (\lambda_{ii}^2 + \mu_{ii}^2 - \ell^2)^{k-s} (-2\lambda_{ii}\,\mu_{ii})^s . \tag{7}$$

Since $\ell^2 = \lambda_{11}^2 + \mu_{11}^2$, we can write

$$tr(D^2 - \ell^2 I)^k - tr(E^2 - \ell^2 I)^k = 2 \sum_{s \text{ is odd}} \binom{k}{s} \sum_{i=2}^{n} (\lambda_{ii}^2 + \mu_{ii}^2 - \ell^2)^{k-s} (-2\lambda_{ii}\,\mu_{ii})^s . \tag{8}$$

Now, we indicate the sign of $tr(D^2-\ell^2 I)^k - tr(E^2-\ell^2 I)^k$, for natural number k . First note that for every $1 \le i \le n$, $\lambda_{ii}^2 + \mu_{ii}^2 - \ell^2 \le 0$. On the other hand, since G has no eigenvalue in the interval $(-1, 0)$ and $\mu_{ii} = -1 - \lambda_{ii}$, thus $\lambda_{ii}\mu_{ii} \le 0$, for $2 \le i \le n$. Therefore, if k is even, then $tr(E^2 - \ell^2 I)^k \ge tr(D^2 - \ell^2 I)^k$ and $tr(D^2 - \ell^2 I)^k \ge tr(E^2 - \ell^2 I)^k$, otherwise. Since $tr(A\overline{A}) = tr(\overline{A}A) = 0$, $tr(D^2 - \ell^2 I) = tr(E^2 - \ell^2 I)$. Using Theorem 3, completes the proof. □

For some similar results as Theorem 7 related to Laplacian and signless Laplacian eigenvalues of graphs see [1, 2]. Putting $\alpha = 1$ in the Theorem 7 we obtain the following result.

Corollary 8. *Let G be a $p-$ regular graph of order n and $p \ne \frac{n-1}{2}$. If G has no eigenvalue in the interval $(-1, 0)$, then the Haemers conjecture is true. In particular, if G is integral the conjecture is valid.*

Remark 9. *Let G be a $p-$ regular graph of order n. Suppose that $Spec(G) = \{\lambda_1, \ldots, \lambda_n\}$ and $Spec(\overline{G}) = \{\mu_1, \ldots, \mu_n\}$. As we state in the proof of Theorem 7, $\lambda_1 = p$, $\mu_1 = n-p-1$ and $\mu_i = -1 - \lambda_i$, for $2 \le i \le n$. Since $S(G) = A(\overline{G}) - A(G)$, $Spec(S(G)) = \{n - 2p - 1, -2\lambda_2 - 1, \ldots, -2\lambda_n - 1\}$. This shows that $0 \in Spec(S(G))$ if and only if $p = \frac{n-1}{2}$ (note that if $\lambda \in Spec(G)$ and λ is rational, then λ is integral). Thus if $p \ne \frac{n-1}{2}$, we can define $\Lambda_\alpha(S(G))$, for any α.*

Remark 10. *One can easily see that for every graph G of order n, $\Lambda_2(S(G)) = \Lambda_2(S(K_n))$ $= (n-1)^2 + (n-1) = n(n-1)$.*

4 The Seidel energy and the energy of graphs

In this section we investigate about the relation between the Seidel energy and the energy of graphs. We think that for every graph the Seidel energy is greater than the energy. More precisely we have the following conjecture.

Conjecture 11. *For every graph G, $\mathcal{E}(S(G)) \ge \mathcal{E}(G)$.*

In the following theorem we show that Haemers conjecture implies the above conjecture. Our key tool is a consequence of *singular-value* inequality which was proved by Fan [4].

Theorem 12. [4] *For every matrices $A, B \in SM_n(\mathbb{R})$,*

$$\mathcal{E}(A+B) \leq \mathcal{E}(A) + \mathcal{E}(B)\,.$$

Theorem 13. *Let G be a graph of order n. If $\mathcal{E}(S(G)) \geq \mathcal{E}(K_n)$, then $\mathcal{E}(S(G)) \geq \mathcal{E}(G)$.*

Proof. If $\mathcal{E}(G) \leq \mathcal{E}(K_n)$, then the theorem follows. Now suppose that $\mathcal{E}(G) \geq \mathcal{E}(K_n)$. Note that $S(G) = -(2A(G) - A(K_n))$. Thus by Theorem 12, $\mathcal{E}(S(G)) \geq \mathcal{E}(2A(G)) - \mathcal{E}(A(K_n))$. On the other hand $2\mathcal{E}(G) - \mathcal{E}(K_n) \geq \mathcal{E}(G)$. Therefore $\mathcal{E}(S(G)) \geq 2\mathcal{E}(G) - \mathcal{E}(K_n) \geq \mathcal{E}(G)$. □

Using Corollary 8 we obtain that Conjecture 11 is true for some regular graphs.

Corollary 14. *Let G be a $p-$ regular graph of order n and $p \neq \frac{n-1}{2}$. If G has no eigenvalue in the interval $(-1, 0)$, then $\mathcal{E}(S(G)) \geq \mathcal{E}(G)$. In particular, if G is integral, then $\mathcal{E}(S(G)) \geq \mathcal{E}(G)$.*

In sequel we want to apply the introduced method (see section 2) to compare the Seidel energy and the energy.

Theorem 15. *Let G be a $p-$ regular graph of order n (with adjacency matrix $A(G)$). Suppose that $p \leq \frac{n-2}{4}$. If every eigenvalue of G is non-zero, then the following hold:*

i) Let $0 < t < 2$ and G has no eigenvalue in the interval $(\frac{1}{t-2}, -\frac{1}{t+2})$. If $0 \leq \alpha \leq 2$, then $\Lambda_\alpha(S(G)) \geq \Lambda_\alpha(tA(G))$.

ii) Let $0 < t < 2$ and G has no eigenvalue in the interval $(\frac{1}{t-2}, -\frac{1}{t+2})$. If $2 \leq \alpha \leq 4$, then $\Lambda_\alpha(S(G)) \leq \Lambda_\alpha(tA(G))$.

iii) Let $t \geq \sqrt{\frac{n-1}{p}}$ (thus $t > 2$) and G has no eigenvalue in the interval $(-\frac{1}{t+2}, \frac{1}{t-2})$. If $0 \leq \alpha \leq 2$, then $\Lambda_\alpha(S(G)) \leq \Lambda_\alpha(tA(G))$.

iv) Let $t \geq \sqrt{\frac{n-1}{p}}$ (thus $t > 2$) and G has no eigenvalue in the interval $(-\frac{1}{t+2}, \frac{1}{t-2})$. If $2 \leq \alpha \leq 4$, then $\Lambda_\alpha(S(G)) \geq \Lambda_\alpha(tA(G))$.

Proof. Let $A = A(G)$ and $\overline{A} = A(\overline{G})$. Since $S(G) = -S(\overline{G})$, $\Lambda_\alpha(S(\overline{G})) = \Lambda_\alpha(S(G))$. Thus in the proof we use $S(\overline{G})$ instead of $S(G)$. We can write $S(\overline{G}) = (\frac{1+t}{2}A-\frac{1}{2}\overline{A})+(\frac{1-t}{2}A-\frac{1}{2}\overline{A})$ and $tA = (\frac{1+t}{2}A-\frac{1}{2}\overline{A})-(\frac{1-t}{2}A-\frac{1}{2}\overline{A})$. Let $X = \frac{1+t}{2}A-\frac{1}{2}\overline{A}$ and $Y = \frac{1-t}{2}A-\frac{1}{2}\overline{A}$. Thus $S(\overline{G})$ and tA are conjugate with respect to (X, Y). Let $D = S(\overline{G}) = X+Y$ and $E = tA = X-Y$. Similar to proof of Theorem 7, by Eq. (6) we obtain that

$$tr(D^2 - \ell^2 I)^k - tr(E^2 - \ell^2 I)^k = 2 \sum_{s \text{ is odd}} \binom{k}{s} \sum_{i=1}^{n} (x_{ii}^2 + y_{ii}^2 - \ell^2)^{k-s} (2x_{ii}\, y_{ii})^s\,, \quad (9)$$

where $Spec(X) = \{x_{11}, \ldots, x_{nn}\}$ and $Spec(Y) = \{y_{11}, \ldots, y_{nn}\}$.

Let $Spec(A) = \{\lambda_1, \ldots, \lambda_n\}$ and $Spec(\overline{A}) = \{\mu_1, \ldots, \mu_n\}$. Since $A\overline{A} = \overline{A}A$, similar to proof of Theorem 7 one can assume that $\lambda_1 + \mu_1 = n-1$ (thus $\lambda_1 = p$ and $\mu_1 = n-1-p$) and $\lambda_i + \mu_i = -1$, for $i = 2, \ldots, n$. In addition, $x_{ii} = \frac{1+t}{2}\lambda_i - \frac{1}{2}\mu_i$ and $y_{ii} = \frac{1-t}{2}\lambda_i - \frac{1}{2}\mu_i$, for $i = 1, \ldots, n$. Let $\ell = \sqrt{x_{11}^2 + y_{11}^2}$.

Suppose that $i \geq 2$. One can see that $x_{ii}y_{ii} = \frac{(4-t^2)\lambda_i^2+4\lambda_i+1}{4}$. Consider the polynomial $(4-t^2)\lambda_i^2 + 4\lambda_i + 1$. We have the following cases:

i) If $t = 2$, then $x_{ii}y_{ii} = \frac{4\lambda_i+1}{4}$, for $i = 2, \ldots, n$.

ii) If $t > 2$, then the roots of $(4-t^2)\lambda_i^2 + 4\lambda_i + 1$ is $\frac{1}{t-2}$ and $\frac{1}{t+2}$. Thus $x_{ii}y_{ii} > 0$ for $\lambda_i \in (-\frac{1}{t+2}, \frac{1}{t-2})$ and $x_{ii}y_{ii} \leq 0$, otherwise.

iii) If $0 < t < 2$, then $x_{ii}y_{ii} < 0$ for $\lambda_i \in (\frac{1}{t-2}, -\frac{1}{t+2})$ and $x_{ii}y_{ii} \geq 0$, otherwise.

One can easily see that $x_{11}^2 + y_{11}^2 = \ell^2 = \frac{t^2+4}{2}p^2 + \frac{(n-1)^2}{2} - 2p(n-1)$ and for $i \geq 2$, $x_{ii}^2 + y_{ii}^2 = \frac{(t^2+4)\lambda_i^2+4\lambda_i+1}{2}$. We claim the following:

1) For $1 \leq i \leq n$, $x_{ii}^2 + y_{ii}^2 - \ell^2 \leq 0$.

2) For $1 \leq i \leq n$, $\ell^2 \geq \frac{t^2\lambda_i^2}{2}$. In addition $\ell^2 \geq \frac{(n-2p-1)^2}{2}$ and $\ell^2 \geq \frac{(2\lambda_i+1)^2}{2}$. In the other words for every $\lambda \in Spec(D) \cup Spec(E)$, $\ell \geq \frac{|\lambda|}{\sqrt{2}}$.

3) $tr(D^2 - \ell^2 I) \geq tr(E^2 - \ell^2 I)$ for $|t| \leq \sqrt{\frac{n-1}{p}}$ and $tr(D^2 - \ell^2 I) \leq tr(E^2 - \ell^2 I)$, otherwise.

Now, we prove the claim.

1') Let $2 \leq i \leq n$. Thus $x_{ii}^2 + y_{ii}^2 - \ell^2 = \frac{t^2+4}{2}(\lambda_i^2 - p^2) - (\frac{(n-1)^2}{2} - 2p(n-1) - \frac{4\lambda_i+1}{2})$. Since $p \leq \frac{n-2}{4}$, and $|\lambda_i| \leq p$, thus $\frac{(n-1)^2}{2} - 2p(n-1) \geq \frac{4p+1}{2} \geq \frac{4\lambda_i+1}{2}$. On the other hand $\frac{t^2+4}{2}(\lambda_i^2 - p^2) \leq 0$. So the first part of the claim is proved.

2') Let $1 \leq i \leq n$. Since $p \leq \frac{n-2}{4}$ and $|\lambda_i| \leq p$, we obtain $\ell^2 \geq \frac{t^2+4}{2}p^2 \geq \frac{t^2}{2}\lambda_i^2$. To complete the proof of this part, since $n-2p-1 \geq 2p+1 \geq |2\lambda_i+1|$, it is sufficient to show that $\ell^2 \geq \frac{(n-2p-1)^2}{2}$. Using the equality $\frac{(n-2p-1)^2}{2} = \frac{(n-1)^2-4p(n-1)+4p^2}{2}$, completes the proof.

3') $tr(D^2-\ell^2 I) - tr(E^2-\ell^2 I) = tr(D^2-E^2) = 4tr(XY)$. On the other hand $XY = \frac{1-t^2}{4}A^2 - \frac{1}{2}A\overline{A} + \frac{1}{4}\overline{A}^2$. Since $tr(A\overline{A}) = 0$, $tr(A^2) = \sum_{i=1}^{n}\lambda_i^2 = 2m = np$ (m is the number of edges of G) and $tr(\overline{A}^2) = n(n-1-p)$, we conclude that $tr(D^2-\ell^2 I) - tr(E^2-\ell^2 I) = n(n-1-t^2 p)$. This shows that $tr(D^2-\ell^2 I) \geq tr(E^2-\ell^2 I)$ if and only if $|t| \leq \sqrt{\frac{n-1}{p}}$.

Using Eq. (9) to obtain the sign of $tr(D^2-\ell^2 I)^k - tr(E^2-\ell^2 I)^k$ for even and odd k. Now, by applying Theorem 3 the proof is complete. □

As a direct consequence of the above theorem we conclude that for some families of regular graphs the Conjecture 11 is valid. More precisely, if we put $t=\alpha=1$ in the first part of Theorem 15, we obtain the following.

Theorem 16. *Let G be a $p-$ regular graph of order n and $p \leq \frac{n-2}{4}$. If G has no eigenvalue in the interval $(-1,-\frac{1}{3})$ and every eigenvalue of G is non-zero, then $\mathcal{E}(S(G)) \geq \mathcal{E}(G)$.*

5 Graphs with small eigenvalues

In this section we study the graphs which have small eigenvalues. We say that a graph G is *small graph* if G has a non-zero eigenvalue in the interval $(-1,1)$. It is easy to see that if $x \in [0,2\pi]$, then $0 < |\cos x| < \frac{1}{2}$ if and only if $\frac{\pi}{3} < x < \frac{2\pi}{3}$ or $\frac{4\pi}{3} < x < \frac{5\pi}{3}$ and $x \neq \frac{\pi}{2}, \frac{3\pi}{2}$. Since $Spec(C_n) = \{2\cos(\frac{2k\pi}{n}), k = 0, \ldots, n-1\}$. Thus C_n is small if and only if there exists a natural number k such that $k \neq \frac{n}{4}$ and $\frac{n}{6} < k < \frac{n}{3}$ or $k \neq \frac{3n}{4}$ and $\frac{4n}{6} < k < \frac{5n}{6}$. It is not hard to see that for every natural number $n \geq 10$ and $n \neq 12$, there exists a natural number k with $k \neq \frac{n}{4}$ and $\frac{n}{6} < k < \frac{n}{3}$. This shows that C_n is small for every $n \geq 10$ and $n \neq 12$. It is easy to check that among all cycles with at most 12 vertices, only the cycles C_5, C_7, C_9, C_{10} and C_{11} are small. Therefore we proved the following.

Theorem 17. *The cycle C_n is small if and only if $n \geq 5$ and $n \neq 6, 8$ and 12.*

Similarly one can obtain the following result.

Theorem 18. *The cycle C_n has an eigenvalue in the interval $(-1,0)$ if and only if $n \in \{7,10,11\}$ or $n \geq 13$.*

Small graphs appeared in the previous sections. We are interested to obtain regular small graphs.

Remark 19. *Let G_1 and G_2 be two graphs of order n_1 and n_2, respectively. Let Spec(G_1) $= \{\lambda_1, \ldots, \lambda_{n_1}\}$ and Spec$(G_2) = \{\mu_1, \ldots, \mu_{n_2}\}$. It is well known that the spectrum of the Cartesian product of G_1 and G_2, $G_1 \times G_2$, is $\{\lambda_i + \mu_j, i = 1, \ldots, n_1, \text{ and } j = 1, \ldots, n_2\}$. On the other hand if G_1 and G_2 are k_1 and k_2- regular , respectively, then $G_1 \times G_2$ is $k_1 + k_2-$ regular. Therefore if G is small, then $G \times K_2$ is small, too. Since C_n is small (for some n), we conclude that for every $k \geq 2$, there exists an infinite family of small $k-$ regular graphs. Similarly, one can construct another family of small graphs by the direct product of graphs.*

We say G is *integral* if all eigenvalues of G are integer. The following remark shows that there are infinite families of graphs that are not small.

Remark 20. *If G is integral, then G has no eigenvalue in the interval $(-1,0)$. For example $C_4 \times K_2 \times \cdots \times K_2$ is a regular integral graph.*

Problem 21. *Without using any operation on graphs such as Cartesian product or direct product, for every $p \geq 3$ introduce an infinite family of $p-$ regular small graphs.*

We say that a graph G is *self–complementary* if G and $\overline{G}$ are isomorphic. For example C_5 is a self–complementary graph. We note that if G is a self–complementary $p-$ regular of order n, then $p = \frac{n-1}{2}$ and p is even. We finish the paper by the following question.

Problem 22. *Find all self–complementary regular small graphs.*

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