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Borderenergetic Threshold Graphs

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Abstract

The energy of a graph is defined as the sum the absolute values of the eigenvalues of its adjacency matrix. A graph G on n vertices is said to be borderenergetic if its energy equals the energy of the complete graph K_n. Using the spectra of threshold graphs, a family of non-regular and non-integral borderenergetic graphs is obtained.

1 Introduction

All graphs in this paper are simple and undirected. The energy $E(G)$ of a graph G is defined as [4, 9]

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the adjacency matrix of G. There are many results on energy and its applications in chemistry, see references in [9].

It is well known that the complete graph K_n is determined by its adjacency spectrum. It was originally believed that the complete graph K_n has maximum energy

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among all graphs on n vertices. However this conjecture is false. There are many graphs with energies greater than the energy of the complete graph and these graphs are called hyperenergetic. Hou and Gutman [6] established a method for constructing hyperenergetic line graphs with any number of vertices (nine or more) and relatively few edges. It may be an interesting problem to find graphs with the same energy as the complete graph K_n, which has energy $E(K_n) = 2n - 2$.

The earliest example of $E(G) = E(K_n)$ is a graph on 9 vertices, that was reported in [6]. This is the line graph of a 3-regular graph on 6 vertices, see Fig. 1.

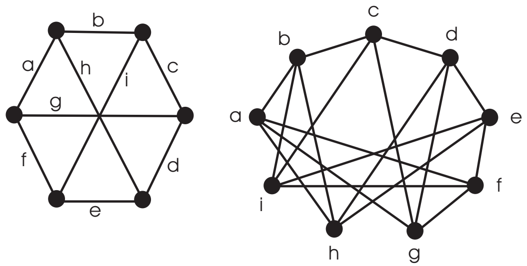

Fig. 1. A 3-regular graph on 6 vertices and its line graph. The eigenvalues of the line graph are $\{4, 1, 1, 1, 1, -2, -2, -2, -2\}$.

Recently, Gong, Li, Xu, Gutman and Furtula [3] studied the graphs with the same energy as a complete graph. They put forward the concept of borderenergetic graphs. A graph G on n vertices is said to be *borderenergetic* if its energy equals the energy of the complete graph K_n. In [3], it was shown that there exist borderenergetic graphs on order n for each integer $n \geq 7$, and all borderenergetic graphs with 7, 8, and 9 vertices were determined. In [10], Li, Wei and Gong determined all borderenergetic graphs with 10 vertices.

Recently, Jacobs, Trevisan and Tura [8] considered the eigenvalues and energies of threshold graphs. They showed that if $4|n$ and $n \geq 8$, then there is an n-vertex threshold graph equienergetic with the complete graph K_n. In addition, if $9|n$, then there are two n-vertex threshold graphs equienergetic to K_n and these are non-cospectral.

For each $n \geq 3$, they determined $n-1$ threshold graphs on n^2 vertices, pairwise non-cospectral and equienergetic to the complete graph K_{n^2}.

In this note, we prove that for each $n \geq 2$ and $p \geq 1$ ($p \geq 2$ if $n = 2$), there are $n-1$ threshold graphs on $p\,n^2$ vertices, pairwise non-cospectral and equienergetic with the complete graph $K_{p\,n^2}$. By this, we unify and generalize the above results of [8] and provide examples of non-integral and non-regular borderenergetic graphs.

2 Threshold graphs with same energy as the complete graph

Threshold graphs were first introduced in 1977 by Chvátal and Hammer [2]. The spectral properties of threshold graphs were studied in [1,7,8]. A graph G is threshold (or degree maximal graph) if and only if it can be obtained from a single vertex by iterating the operations of adding a new vertex that is either connected to no other vertex (an *isolated vertex*) or connected to every other vertex (a *cone vertex*). The sequence of these operations is called the *building sequence* of the respective threshold graph.

In view of this, we may represent a threshold graph on n vertices using a binary sequence $b = b_1 b_2 \dots b_n$. Here b_i is 0 if the vertex v_i was added as an isolated vertex, and b_i is 1 if v_i was added as a cone vertex. In our representation, b_1 is always zero. We write 0^s (resp. 1^s) if there are s repeated 0's (resp. 1's) in the building sequence. For example, we write $0^2 1^2 0 1^3$ for 00110111.

For a graph G, by $n_0(G)$ and n_{-1} we denote the multiplicities of the eigenvalues 0 and -1, respectively. The following results provide formulas for $n_0(G)$ and n_{-1}.

Theorem 1. [1] *Let G be a connected threshold graph with building sequence $b = 0^{s_1} 1^{t_1} \dots 0^{s_k} 1^{t_k}$, where the s_i's and t_i's are positive integers. Then*

$$n_0(G) = \sum_{i=1}^{k} (s_i - 1)\,.$$

Theorem 2. [8] *Let G be a connected threshold graph with building sequence $b = 0^{s_1}1^{t_1}\ldots 0^{s_k}1^{t_k}$, where the s_i's and t_i's are positive integers. Then*

$$n_{-1}(G) = \begin{cases} \sum_{i=1}^{k}(t_i - 1) & \text{if } s_1 > 1 \\ \\ 1 + \sum_{i=1}^{k}(t_i - 1) & \text{if } s_1 = 1. \end{cases}$$

We first prove that there are no borderenergetic threshold graphs of the form 0^k1^j or $0^p1^q0^t$.

Proposition 3. *There are no borderenergetic threshold graphs 0^k1^j with $k > 1$.*

Proof. The independent set of size k and clique of size j form an equitable partition of the adjacency matrix with quotient matrix $B = \begin{bmatrix} 0 & j \\ k & j-1 \end{bmatrix}$, whose eigenvalues satisfy $x^2 - (j-1)x - jk = 0$. Since in the threshold graph 0^k1^j, 0 is an eigenvalue with multiplicity $k-1$, and -1 is an eigenvalue with multiplicity $j-1$, the energy of the threshold graph 0^k1^j is

$$\begin{aligned} E(G) &= j-1+\frac{1}{2}\left[(j-1)+\sqrt{(j-1)^2+4kj}\right]+\frac{1}{2}\left[-(j-1)+\sqrt{(j-1)^2+4kj}\right] \\ &= j-1+\sqrt{(j-1)^2+4kj}\,. \end{aligned}$$

If we assume that $j - 1 + \sqrt{(j-1)^2 + 4kj} = 2(k + j - 1)$, then we get $k = 1$, which yields a contradiction with $k > 1$. □

Proposition 4. *There are no borderenergetic threshold graphs $0^p1^q0^t$.*

Proof. The independent set of size p, the clique of size q and the independent set of size t form an equitable partition with matrix $B = \begin{bmatrix} 0 & q & 0 \\ p & q-1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, whose eigenvalues are 0 and $\frac{1}{2}\left(q - 1 \pm \sqrt{(q-1)^2 + 4pq}\right)$. Therefore the energy of the graph $0^p1^q0^t$ is

$$E(G) = q - 1 + \sqrt{(q-1)^2 + 4pq}\,.$$

Suppose that $q-1+\sqrt{(q-1)^2 + 4pq} = 2(p+q+t-1)$. Then by simple calculation we find that there is no solution to this equation. Therefore, the proposition holds. □

We now give a family of threshold graphs $01^s0^k1^j$ which are borderenergetic. In order to do this, we need following results from [8]

Lemma 5. *For positive integers s, k, and j, the characteristic polynomial of the threshold graph $01^s0^k1^j$ is, to within a sign,*

$$x^{k-1}(x+1)^{s+j-1}\left(x^3-(s+j-1)x^2-(s+jk+j)x+kjs\right).$$

Lemma 6. *Let b and c be positive real numbers, and let x^2-bx+c have real roots λ_1, λ_2. Then $|\lambda_1|+|\lambda_2|=b$.*

The following is the main result in this section.

Theorem 7.
(1) *For each $n \geq 3$ and $p \geq 1$, there exist $n-1$ pairwise non-cospectral borderenergetic threshold graphs on pn^2 vertices.*
(2) *Let $p \geq 2$. Then the threshold graph $G = 01^p0^{p-1}1^{2p}$ and K_{4p} are non-cospectral and equienergetic.*

Proof. We claim that the graphs $G_i = 01^{p(n-i)^2}0^{p(n-i)i-1}1^{pin}$, for $1 \leq i \leq n-1$ satisfy the theorem. It is easy to see that each G_i has order pn^2. Since $p(n-i)i-1>0$ for $n \geq 3$, $p \geq 1$, and for $n=2$, $p \geq 2$, it follows that $G_i \neq 01^{pn^2-1}$. From Lemma 5, we see that the characteristic polynomial of G_i is, modulo the sign,

$$x^{p(n-i)i-2} \cdot (x+1)^{p(n-i)^2+pin-1} \cdot q(x) \tag{1}$$

where $q(x)$ is the cubic polynomial

$$\begin{aligned} q(x) = x^3 &- [p(n-i)^2+pin-1]x^2 - [p(n-i)^2+pin(p(n-i)i-1)+pin]x \\ &+ p(n-i)^2(p(n-i)i-1)pin\,. \end{aligned}$$

Although tedious, it can be verified that $q(x)$ can be factored as

$$\left[x+p(n-i)i\right]\left[x^2-(pn^2-1)x+pn(pin-pi^2-1)(n-i)\right]. \tag{2}$$

We claim that any two graphs G_i are non-cospectral. Suppose by contradiction that G_i and G_j are cospectral. Then their corresponding polynomials $q(x)$ must have the same roots. Note that the roots from the linear terms are both negative, and the roots from the quadratic terms are both positive. Therefore the two terms must be equal, and we have $p(n-i)i = p(n-j)j$ which implies $i + j = n$. The quadratic terms have the same roots. Their leading and middle terms agree, their constant terms must also. That is

$$pn(n-i)(pin - pi^2 - 1) = pn(n-j)(pjn - pj^2 - 1)\,.$$

Dividing by pn and using $i+j = n$, we obtain $j(pin-pi^2-1) = i(pjn-pj^2-1)$ which implies that $j - i = pij(j-i)$. This means that $i = j$ or $pij = 1$, a contradiction.

Finally, we calculate $E(G_i)$ from (1) and (2), using Lemma 6. This yields

$$E(G_i) = p(n-i)^2 + pin - 1 + p(n-i)i + pn^2 - 1 = 2(pn^2 - 1) = E(K_{pn^2})\,.$$

□

Let $n = 3, p = m$. Then we have:

Corollary 8. *(Theorem 9 in* [8]*) For* $m \geq 1$*, the threshold graphs* $01^{4m}0^{2m-1}1^{3m}$ *and* $01^{m}0^{2m-1}1^{6m}$ *of order* n *are non-cospectral and both borderenergetic.*

Let $p = 1$. Then we have:

Corollary 9. *(Theorem 10 in* [8]*) For each* $n \geq 3$*, there exist* $n-1$ *threshold graphs on* n^2 *vertices, pairwise non-cospectral and borderenergetic.*

By the case of $n = 2$, let $p = 2m$ and $p = 2m+1$. Then we have:

Corollary 10. *(Theorem 7 in* [8]*) For* $m \geq 1$*, the threshold graphs* $01^{2m}0^{2m-1}1^{4m}$ *are borderenergetic.*

Corollary 11. *(Theorem 8 in* [8]*) For* $m \geq 1$*, the threshold graphs* $01^{2m+1}0^{2m}1^{4m+2}$ *are borderenergetic.*

Remark 12. *Let* p, q, r *be non-negative integers and let* $p+q = 2$*. It has been shown that the graphs* $\overline{pC_4 \cup qC_6 \cup rC_3}$ *are regular and integral borderenergetic graphs. In the general case, the borderenergetic threshold graphs in this note are non-regular and non-integral.*

Remark 13. *In the Appendix, we list some borderenergetic threshold graphs, found by means of a computer search. Table 1 lists all borderenergetic threshold graphs with* n *vertices,* $n \leq 23$*. Table 2 lists all borderenergetic threshold graphs of the form* $0^p1^q0^s1^t$ *with* n *vertices,* $n = p + q + s + t \leq 100$*.*

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3 Appendix

Table 1. All non-complete connected borderenergetic threshold graphs with $n \leq 23$ vertices.

n	All connected threshold graphs with energy 2n-2
8	01^201^4
9	0101^6, 01^401^3
12	$01^30^21^6$
15	$0^31^501^6$, $01^60101^201^2$
16	010^21^{12}, $01^40^31^8$, $01^90^21^4$
17	$01^2010^21^30^21^5$
18	$01^20^31^{12}$, $01^80^31^6$
19	$0^41^401^201^201^4$, $0^210^21^2010101^201^4$, $0^21^90101^201^2$, $0101^20101^20^21^501$, $01^701^201^30^21^2$
20	$0^2101^301^2010^2101^4$, $0101^20^21^20101^30^21^3$, $0101^301^40^3101^4$, $01^20^31^3010101^201^3$, $01^201010^2101^701$, $01^50^41^{10}$
21	$010^210^31^201^501^4$, $010^21010^31^401^6$, $010^21^50^21^20101^201^2$, $01^201^20^31^{10}01$, $01^201^30^210^21^20^21^5$
22	$0^410^21^201^601^5$, $0^4101^301^301^20101^3$, $0^41^201^30^2101^201^5$, $0^310101^50^31^7$, $0^21^301010^21^201^201^301$, $010^3101^40101^201^201^2$, $010^21^201^201^20^21^201^301$, $01010^21^20^21^201^40^21^3$, $0101^2010101010^31^6$, $0101^30^21^20^21^20^21^201^3$, $01^2010^2101^30101^201^201$, $01^30^21^401^20^21^201^201$, $01^3010^210101^2010^21^4$, $01^50101^50^3101^3$,
23	$0^310^2101^301^201^601$, $0^31010^2101^20^21^301^5$, $0^31^20101^201^20101^20^21^3$, $0^210^31^201^20101^301^201^2$, $0^210^2101^20101^301^20^21^3$, $0^210^21^{10}0^21^301^2$, $0^2101^201^30^310^21^7$, $0^21^30^21010^21^901$, $0^21^301^2010101^40^21^201$, $010^3101^701^30^21^3$, $010^31^3010^31^{10}$, $010^21010^2101^20^21^401^3$, $0101^20^3101^501^20^21^3$, $0101^20101^20101010^21^3$, $0101^3010^210101^50101$, $0101^301^20^3101^40^21^3$, $0101^40^410101^301^4$, $0101^50^31^40^31^5$, $0101^701^40^2101^201$, $0101^901^40^31^3$, $01^20^31^201^7010^21^3$, $01^201010^41^50101^4$, $01^201^201^301^301^3010^21$, $01^201^40^2101^201010^21^3$, $01^3010^21^301^401^30^21$, $01^30101^70101^30^21$, $01^40^210^3101^701^2$, $01^5010^21010^21^601$, $01^60101^40101^30^21$, $01^601^2010^210^2101^4$

Table 2. All connected borderenergetic threshold graphs $0^p1^q0^s1^t$ with $n \leq 100$ vertices.

order n	All connected threshold graphs $0^p1^q0^s1^t$ with energy $2n-2$
8	$01^20 1^4$
9	0101^6, 01^401^3
12	$01^30^21^6$
15	$0^31^501^6$
16	010^21^{12}, $01^40^31^8$, $01^90^21^4$
18	$01^20^31^{12}$, $01^80^31^6$
20	$01^50^41^{10}$
24	$01^60^51^{12}$, $0^41^{12}0^21^6$
25	010^31^{20}, $01^40^51^{15}$, $01^90^51^{10}$, $01^{16}0^31^5$
27	$01^30^51^{18}$, $01^{12}0^51^9$
28	$01^70^61^{14}$, $0^61^701^{14}$
32	$01^20^51^{24}$, $01^80^71^{16}$, $01^{18}0^51^8$
35	$0^71^50^21^{21}$
36	010^41^{30}, $01^40^71^{24}$, $01^90^81^{18}$, $0^61^{10}0^51^{15}$, $01^{16}0^71^{12}$, $01^{25}0^41^6$
38	$0^71^{21}0^21^8$
40	$01^{10}0^91^{20}$
44	$01^{11}0^{10}1^{22}$, $0^51^{22}0^61^{11}$, $0^91^{22}0^21^{11}$
45	$01^50^91^{30}$, $01^{20}0^91^{15}$
48	$01^30^81^{36}$, $01^{12}0^{11}1^{24}$, $01^{27}0^81^{12}$
49	010^51^{42}, $01^40^91^{35}$, $01^90^{11}1^{28}$, $01^{16}0^{11}1^{21}$, $01^{25}0^91^{14}$, $01^{36}0^51^7$
50	$01^20^71^{40}$, $01^80^{11}1^{30}$, $01^{18}0^{11}1^{20}$, $01^{32}0^71^{10}$
51	$0^{10}1^{28}0^21^{11}$
52	$01^{13}0^{12}1^{26}$
54	$01^60^{11}1^{36}$, $01^{24}0^{11}1^{18}$
56	$01^{14}0^{13}1^{28}$
60	$01^{15}0^{14}1^{30}$
63	$01^70^{13}1^{42}$, $0^{12}1^70^21^{42}$, $01^{28}0^{13}1^{21}$
64	010^61^{56}, $01^40^{11}1^{48}$, $01^90^{14}1^{40}$, $01^{16}0^{15}1^{32}$, $01^{25}0^{14}1^{24}$, $01^{36}0^{11}1^{16}$, $01^{49}0^61^8$
68	$01^{17}0^{16}1^{34}$
70	$0^{15}1^{14}0^61^{35}$
72	$01^20^91^{60}$, $01^80^{15}1^{48}$, $01^{18}0^{17}1^{36}$, $01^{32}0^{15}1^{24}$, $0^{16}1^{18}0^21^{36}$, $01^{50}0^91^{12}$
75	$01^30^{11}1^{60}$, $01^{12}0^{17}1^{45}$, $0^71^70^{13}1^{48}$, $01^{27}0^{17}1^{30}$, $01^{48}0^{11}1^{15}$
76	$01^{19}0^{18}1^{38}$
78	$0^{11}1^{57}0^21^8$
79	$0^{17}1^{13}0^31^{46}$
80	$01^50^{14}1^{60}$, $01^{20}0^{19}1^{40}$, $01^{45}0^{14}1^{20}$

Table 2. (cont)

order n	All connected threshold graphs $0^p1^q0^s1^t$ with energy $2n-2$
81	010^71^{72}, $01^40^{13}1^{63}$, $01^90^{17}1^{54}$, $01^{16}0^{19}1^{45}$, $01^{25}0^{19}1^{36}$, $01^{36}0^{17}1^{27}$, $01^{49}0^{13}1^{18}$, $0^61^{53}0^91^{13}$, $01^{64}0^71^9$
83	$0^{16}1^90^61^{52}$
84	$01^{21}0^{20}1^{42}$
87	$0^{10}1^20^31^{72}$
88	$01^{22}0^{21}1^{44}$
90	$01^{10}0^{19}1^{60}$, $01^{40}0^{19}1^{30}$, $0^71^{61}0^91^{13}$
92	$01^{23}0^{22}1^{46}$, $0^71^{22}0^{20}1^{43}$
96	$01^60^{17}1^{72}$, $01^{24}0^{23}1^{48}$, $01^{54}0^{17}1^{24}$
97	$0^41^{31}0^{22}1^{40}$
98	$01^20^{11}1^{84}$, $01^80^{19}1^{70}$, $01^{18}0^{23}1^{56}$, $01^{32}0^{23}1^{42}$, $01^{50}0^{19}1^{28}$, $0^{14}1^{42}0^{14}1^{28}$, $0^{21}1^{42}0^71^{28}$, $01^{72}0^{11}1^{14}$
99	$01^{11}0^{21}1^{66}$, $01^{44}0^{21}1^{33}$
100	010^81^{90}, $01^40^{15}1^{80}$, $01^90^{20}1^{70}$, $01^{16}0^{23}1^{60}$, $01^{25}0^{24}1^{50}$, $01^{36}0^{23}1^{40}$, $01^{49}0^{20}1^{30}$, $01^{64}0^{15}1^{20}$, $01^{81}0^81^{10}$