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A Note on the Additive Degree Kirchhoff Index

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Abstract

In this note we present an asymthotic proof of conjecture from [J. L. Palacios, Upper and Lower Bounds for the Additive Degree-Kirchhoff Index, *MATCH Commun. Math. Comput. Chem.* **70** (2013) 651–655] about the upper bound for additive degree Kirchhoff index, by reducing the problem to the distance degree invariant.

1 Introduction

Let $G = (V, E)$ be a connected undirected graph. For any two distinct vertices $u, v \in V$, the resistance distance between them, denoted by R_{uv}, is defined as the net effective resistance between nodes u and v in the electrical network constructed from G where each edge is identified as a unit resistor. The Kirchhoff index $R(G)$ was defined by Klein and Randić [7] as

$$R(G) = \sum_{u,v \in V} R_{uv}.$$

This concept is also linked to the fruitful area of random walks on graphs [8].

We denote the degree of a vertex v by $deg(v)$ and the distance between vertices $v \in V$ and $u \in V$ by $d(v, u)$. The degree distance topological index of G is defined as

$$D'(G) = \frac{1}{2} \sum_{u,v \in V} (deg(u) + deg(v))d(u, v).$$

The degree distance is a modification of the Wiener index $W(G)$, first introduced by Dobrynin and Kochetova [3]. Recently, a lot of research was focused on the mathematical properties of $D'(G)$ and on the upper and lower bounds (for example see [6, 10]).

The additive degree Kirchhoff index is motivated by the degree distance of a graph, and defined as:

$$R^+(G) = \sum_{u,v \in V} (deg(u) + deg(v)) R_{uv}.$$

Gutman et al. [4, 5] characterized n-vertex unicyclic graphs having minimum and second minimum additive degree Kirchhoff index. Bianchi et al. [1] exhibited various bounds for this index using vertex degrees and majorization techniques. In [11, 12], recursion formulas for different Kirchhoffian indices of the subdivision and triangulation of a graph G are calculated.

In [9] Palacios showed that for any graph G on n vertices holds

$$R^+(G) \leq \frac{n^4 - n^3 - n^2 + n}{3}.$$

In addition it was conjectured that the maximum of the additive degree Kirchhoff index $R^+(G)$ over all graphs on n vertices is attained by the $B(1/3, 1/3, 1/3)$ barbell graph which consists of two complete graphs on $n/3$ vertices united by a path of length $n/3$, and for which it holds $R^+(G) \sim \frac{2n^4}{27}$. In this note, we prove the above conjecture by just combining previous results on resistance distance and degree distance invariant.

2 Main result

Theorem 1 *Let G be a connected graph of order n. Then*

$$R^+(G) \leq \frac{2n^4}{27} + O(n^{7/2}).$$

Klein and Randić [7] proved that for all distinct pairs of vertices $u, v \in V$, it holds

$$d(u, v) \geq R_{uv},$$

with equality iff there is a unique path connecting vertices u and v. Therefore, for trees we have $2D'(T) = R^+(T)$ and all results that apply to degree distance invariant actually hold for the additive degree Kirchhoff index as well. For example, it holds [3]

$$R^+(T) = 2D'(T) = 4W(T) - n(n-1).$$

In 1999 Tomescu [10] conjectured that the maximum asymptotic value of the degree distance invariant $D'(G)$ for n-vertex graphs is exactly $\frac{n^4}{27}$. Ten years later, Dankelmann et al. [2] proved the following inequality for a graph G with n vertices and diameter d, and thus solved and generalized the above conjecture:

$$D'(G) \leq \frac{nd(n-d)^2}{4} + O(n^{7/2}).$$

Combining the above two results, we get

$$R^+(G) \leq 2D'(G) \leq \frac{nd(n-d)^2}{2} + O(n^{7/2}).$$

Simple calculus shows that the maximum is achieved for $d = \frac{n}{3}$.

It is very interesting that Palacios [9] formulated the same conjecture as Tomescu, just in terms of a different invariant and similarly provided a weaker upper bound. As pointed before, for the $(1/3, 1/3, 1/3)$ barbell graph we have

$$R^+(G) \sim 2D'(G) \sim \frac{2n^4}{27}.$$

This completes the proof.

Remark. However, by direct computation from [9] we have

$$R^+(B(1/3, 1/3, 1/3)) \sim \frac{2n^4}{27} + \frac{2n^3}{81},$$

which is stronger bound than the presented one because of the term $O(n^{7/2})$. This is also noted in [2] for the upper bound of degree distance, and gives opportunities for further optimizations.

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