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Some Results on Kirchhoff Index and Degree–Kirchhoff Index

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Abstract

In this paper, we give some relations between the Kirchhoff and degree–Kirchhoff index of a connected graph, and obtain some formulas for these indices. Using these results, we obtain expressions and bounds for Kirchhoff indices of some composite graphs, which extend some work of Yang and Klein. We give formulas for the Kirchhoff index, Laplacian–energy–like invariant and Laplacian Estrada index of the line graph of a semiregular graph, and obtain a formula for the Kirchhoff index of the t-para-line graph of a regular graph.

1 Introduction

All graphs in this paper are simple and undirected, and all connected graphs in this paper have at least two vertices. Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G, respectively. The resistance distance is a distance function on graphs introduced by Klein and Randić [20]. For two vertices i, j in a connected graph G, the *resistance distance* between i and j, denoted by $r_{ij}(G)$, is defined to be the effective resistance between them when unit resistors are placed on every edge of G. The *Kirchhoff index* of G, denoted by $Kf(G)$, is the sum of resistance distances between all pairs of vertices of G, i.e.,

$$Kf(G) = \sum_{\{i,j\} \subseteq V(G)} r_{ij}(G) \ .$$

In [5], Chen and Zhang defined the *multiplicative degree–Kirchhoff index* as

$$Kf^*(G) = \sum_{\{i,j\} \subseteq V(G)} d_i d_j r_{ij}(G)\,,$$

where d_i denotes the degree of the vertex i. In [15], Gutman et al. defined the *additive degree–Kirchhoff index* as

$$Kf^+(G) = \sum_{\{i,j\} \subseteq V(G)} (d_i + d_j) r_{ij}(G)\ .$$

The Kirchhoff index and degree–Kirchhoff index are investigated extensively in mathematical and chemical literatures [3,9,10,12,23,30-32,35-38]. It is of interest to study the Kirchhoff index of graph operations, such as corona [35], join [35], line graph [14,30], total graph [33], subdivision [14, 26, 31], triangulation [28, 32], semi total point graph [7] etc.

$P_3 \diamond P_2$ $\qquad$ $P_3 \underline{\diamond} P_2$

Fig.1. The graphs $P_3 \diamond P_2$ and $P_3 \underline{\diamond} P_2$

Edge corona is a graph operation introduced by Hou and Shiu [18]. For two disjoint graphs G_1 and G_2, the *edge corona* $G_1 \diamond G_2$ is the graph obtained by taking one copy of G_1 and $|E(G_1)|$ copies of G_2, and then joining two end-vertices of the i-th edge of G_1 to every vertex in the i-th copy of G_2 $(i = 1, \ldots, |E(G_1)|)$. Let $G_1 \underline{\diamond} G_2$ denote the graph obtained from $G_1 \diamond G_2$ by deleting all edges belong to $E(G_1)$. For example, the graphs $P_3 \diamond P_2$ and $P_3 \underline{\diamond} P_2$ (P_n is the path of order n) are shown in Fig.1. If $G_2 = K_1$ is an isolated vertex, then $G_1 \diamond K_1$ is the triangulation [32] of G_1, and $G_1 \underline{\diamond} K_1$ is the subdivision [31] of G_1. Kirchhoff indices of the subdivision and triangulation of a graph are studied in [14, 26, 28, 31, 32].

The *para-line graph* of a graph G, denoted by $C(G)$, is defined as the line graph of the subdivision graph $G \underline{\diamond} K_1$ [24,25,30]. Para-line graphs are also called clique-inserted graphs in [34]. Kirchhoff index of the para-line graph of a regular graph is studied in [24,30]. We define the line graph of $G \underline{\diamond} \overline{K_t}$ as the *t-para-line graph* of G, where $\overline{K_t}$ the complement of the complete graph K_t. Clearly, $C(G)$ is the 1-para-line graph of G.

For a graph G, let A_G denote the adjacency matrix of G, and let D_G denote the diagonal matrix of vertex degrees of G. The matrices $L_G = D_G - A_G$ and $Q_G = D_G + A_G$

are called the *Laplacian matrix* and *signless Laplacian matrix* of G, respectively. Let $\mu_1(G) \geqslant \mu_2(G) \geqslant \cdots \geqslant \mu_n(G) = 0$ $(n = |V(G)|)$ denote the eigenvalues of L_G. Liu and Liu [22] defined the *Laplacian–energy–like invariant* of G as

$$LEL(G) = \sum_{i=1}^{n} \sqrt{\mu_i(G)} \ .$$

The LEL is an energy like invariant [17]. In [11], the Laplacian Estrada index of G was defined as

$$LEE(G) = \sum_{i=1}^{n} e^{\mu_i(G)} \ .$$

The Kirchhoff index, LEL and LEE of the line graph of a regular graph are studied in [6, 11, 14, 22, 27, 30].

This paper is organized as follows. In Section 2, some auxiliary lemmas are given. In Section 3, we give some relations between the Kirchhoff and degree–Kirchhoff index of a connected graph, and obtain some formulas for these indices. In Section 4, we obtain expressions and bounds for Kirchhoff indices of $G_1 \underline{\diamond} G_2$ and $G_1 \diamond G_2$, which extend some results in [31, 32]. In Section 5, we give formulas for the Kirchhoff index, LEL and LEE of the line graph of a semiregular graph, and obtain a formula for the Kirchhoff index of the t-para-line graph of a regular graph.

2 Preliminaries

The $\{1\}$-*inverse* of a matrix M is a matrix X such that $MXM = M$. If M is singular, then it has infinite $\{1\}$-inverses [2, 4, 26]. We use $M^{(1)}$ to denote any $\{1\}$-inverse of M, and let $(M)_{ij}$ denote the (i,j)-entry of M. For a square matrix M, the *group inverse* of M, denoted by $M^{\#}$, is the unique matrix X such that $MXM = M$, $XMX = X$ and $MX = XM$. If M is real symmetric, then $M^{\#}$ exists and $M^{\#}$ is a symmetric $\{1\}$-inverse of M. Actually, $M^{\#}$ is equal to the Moore-Penrose inverse of M if M is real symmetric [4].

Lemma 2.1. [2, 4] *Let G be a connected graph. Then*

$$r_{ij}(G) = (L_G^{(1)})_{ii} + (L_G^{(1)})_{jj} - (L_G^{(1)})_{ij} - (L_G^{(1)})_{ji} = (L_G^{\#})_{ii} + (L_G^{\#})_{jj} - 2(L_G^{\#})_{ij} \ .$$

Lemma 2.2. [13, 31] *Let G be a connected graph of order n. Then*

$$\sum_{uv \in E(G)} r_{uv}(G) = n - 1 \ .$$

For a square matrix M, let $\mathrm{tr}(M)$ denote the trace of M. Let $\mathrm{j} = (1, 1, \ldots, 1)^\top$ denote an all-ones column vector.

Lemma 2.3. [26] *Let G be a connected graph of order n. Then*

$$Kf(G) = n\mathrm{tr}(L_G^{(1)}) - \mathrm{j}^\top L_G^{(1)} \mathrm{j} = n\mathrm{tr}(L_G^{\#}) \ .$$

Lemma 2.4. [16,38] *Let G be a connected graph of order n. Then*

$$Kf(G) = n \sum_{i=1}^{n-1} \frac{1}{\mu_i(G)} \ .$$

For a connected graph G with n vertices and m edges, its normalized Laplacian matrix is $\mathcal{L}_G = D_G^{-\frac{1}{2}} L_G D_G^{-\frac{1}{2}}$. Chen and Zhang [5] proved that $Kf^*(G) = 2m \sum_{i=1}^{n-1} \frac{1}{\lambda_i}$, where $\lambda_1, \ldots, \lambda_{n-1}$ are nonzero eigenvalues of $\mathcal{L}_G$. We can obtain the following lemma from [37, Proposition 4].

Lemma 2.5. *Let G be a connected graph with n vertices and m edges, and let Δ and δ be the maximum and minimum degree of G, respectively. Then*

$$\frac{2m\delta}{n} Kf(G) \leqslant Kf^*(G) \leqslant \frac{2m\Delta}{n} Kf(G) \,,$$

equalities in both sides hold if and only if G is regular.

Lemma 2.6. [4] *Let S be a real symmetric matrix such that $S\mathrm{j} = 0$. Then $S^{\#}\mathrm{j} = 0, \mathrm{j}^\top S^{\#} = 0$.*

Lemma 2.7. [21] *Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix with one positive eigenvalue and $n-1$ negative eigenvalues. For a positive vector $x \in \mathbb{R}^n$ and an arbitrary vector $y \in \mathbb{R}^n$, we have*

$$(x^\top A y)^2 \geqslant (x^\top A x)(y^\top A y) \,,$$

with equality if and only if $y = \lambda x$ for some constant λ.

Lemma 2.8. [26] *Let $M = \begin{pmatrix} A & B \\ B^\top & C \end{pmatrix}$ be a real symmetric matrix, and A is nonsingular. Then $N = \begin{pmatrix} A^{-1} + A^{-1} B S^{\#} B^\top A^{-1} & -A^{-1} B S^{\#} \\ -S^{\#} B^\top A^{-1} & S^{\#} \end{pmatrix}$ is a symmetric $\{1\}$-inverse of M, where $S = C - B^\top A^{-1} B$.*

3 Kirchhoff and degree–Kirchhoff index of a graph

For a connected graph G of order n, the *resistance matrix* of G is defined as $R_G = (r_{ij}(G))_{n\times n}$ (see [1, 29]).

Theorem 3.1. *Let G be a connected graph. Then*

$$Kf^+(G) \geqslant 2\sqrt{Kf(G)Kf^*(G)}\ ,$$

with equality if and only if G is regular.

Proof. Let $\mathrm{j} = (1, 1, \dots, 1)^\top$, $\pi = (d_1, \dots, d_n)^\top$, where $d_1, \dots, d_n$ is the degree sequence of G. Then

$$\begin{aligned} \mathrm{j}^\top R_G \mathrm{j} &= \sum_{i,j=1}^{n} r_{ij}(G) = 2Kf(G),\ \pi^\top R_G \pi = \sum_{i,j=1}^{n} d_i d_j r_{ij}(G) = 2Kf^*(G)\,, \\ \mathrm{j}^\top R_G \pi &= \sum_{i,j=1}^{n} d_j r_{ij}(G) = \sum_{\{i,j\}\subseteq V(G)} (d_i + d_j) r_{ij}(G) = Kf^+(G)\ . \end{aligned}$$

It is known [29] that R_G has one positive eigenvalue and $n-1$ negative eigenvalues. By Lemma 2.7, we have

$$(Kf^+(G))^2 \geqslant 4Kf(G)Kf^*(G)\,,$$

with equality if and only if G is regular. ■

Theorem 3.2. *Let G be a connected graph with n vertices and m edges. Then*

$$\begin{aligned} Kf^*(G) &= 2m\mathrm{tr}(D_G L_G^{(1)}) - \pi^\top L_G^{(1)} \pi = 2m\mathrm{tr}(D_G L_G^{\#}) - \pi^\top L_G^{\#} \pi\,, \\ Kf^+(G) &= n\mathrm{tr}(D_G L_G^{\#}) + \frac{2m}{n} Kf(G)\,, \end{aligned}$$

where D_G is the diagonal matrix of vertex degrees of G, $\pi = (d_1, \dots, d_n)^\top$ is the column vector of the degree sequence of G.

Proof. By Lemma 2.1, we have

$$\begin{aligned} Kf^*(G) &= \frac{1}{2} \sum_{i,j=1}^{n} d_i d_j [(L_G^{(1)})_{ii} + (L_G^{(1)})_{jj} - (L_G^{(1)})_{ij} - (L_G^{(1)})_{ji}] \\ &= \frac{1}{2} \sum_{i=1}^{n} d_i \sum_{j=1}^{n} (d_j (L_G^{(1)})_{ii} + d_j (L_G^{(1)})_{jj}) - \sum_{i,j=1}^{n} d_i d_j (L_G^{(1)})_{ij} \\ &= \frac{1}{2} \sum_{i=1}^{n} d_i [2m (L_G^{(1)})_{ii} + \mathrm{tr}(D_G L_G^{(1)})] - \pi^\top L_G^{(1)} \pi \\ &= 2m\mathrm{tr}(D_G L_G^{(1)}) - \pi^\top L_G^{(1)} \pi\ . \end{aligned}$$

Since $L_G^\#$ is a $\{1\}$-inverse of L_G, we also have $Kf^*(G) = 2m\mathrm{tr}(D_G L_G^\#) - \pi^\top L_G^\# \pi$. By Lemma 2.1, we get

$$\begin{aligned} Kf^+(G) &= \frac{1}{2}\sum_{i,j=1}^{n}(d_i + d_j)[(L_G^\#)_{ii} + (L_G^\#)_{jj} - 2(L_G^\#)_{ij}] \\ &= \frac{1}{2}\sum_{i,j=1}^{n}(d_i + d_j)[(L_G^\#)_{ii} + (L_G^\#)_{jj}] - \sum_{i,j=1}^{n}(d_i + d_j)(L_G^\#)_{ij} \ . \end{aligned}$$

Since $L_G^\#$ is real symmetric and $L_G \mathrm{j} = 0$, by Lemma 2.6, all row sums and column sums of $L_G^\#$ are zero. Hence $\sum_{i,j=1}^{n}(d_i + d_j)(L_G^\#)_{ij} = 0$ and

$$Kf^+(G) = \frac{1}{2}\sum_{i,j=1}^{n}(d_i + d_j)[(L_G^\#)_{ii} + (L_G^\#)_{jj}] = n\mathrm{tr}(D_G L_G^\#) + 2m\mathrm{tr}(L_G^\#) \ .$$

By Lemma 2.3, we have $Kf^+(G) = n\mathrm{tr}(D_G L_G^\#) + \frac{2m}{n}Kf(G)$. ■

Let $\Delta(G)$ and $\delta(G)$ denote the maximum and minimum degree of graph G, respectively.

Corollary 3.3. *Let G be a connected graph with n vertices and m edges. Then*

$$(\delta(G) + \frac{2m}{n})Kf(G) \leqslant Kf^+(G) \leqslant (\Delta(G) + \frac{2m}{n})Kf(G)\,,$$

equalities in both sides hold if and only if G is regular.

Proof. By Theorem 3.2, we have $Kf^+(G) = n\mathrm{tr}(D_G L_G^\#) + \frac{2m}{n}Kf(G)$. From [19, Proposition 2.2], we know that all diagonal entries of $L_G^\#$ are positive. By Lemma 2.3, we have

$$(\delta(G) + \frac{2m}{n})Kf(G) \leqslant Kf^+(G) \leqslant (\Delta(G) + \frac{2m}{n})Kf(G)\,,$$

equalities in both sides hold if and only if G is regular. ■

Let G be a connected graph, and its Laplacian matrix is partitioned as $L_G = \begin{pmatrix} L_1 & L_2 \\ L_2^\top & L_3 \end{pmatrix}$ (L_1 is square). Since the Schur complement $S = L_3 - L_2^\top L_1^{-1} L_2$ is symmetric, $S^\#$ exists and is symmetric.

Theorem 3.4. *Let $L_G = \begin{pmatrix} L_1 & L_2 \\ L_2^\top & L_3 \end{pmatrix}$ (L_1 is square) be the Laplacian matrix of a connected graph G of order n, and let $S = L_3 - L_2^\top L_1^{-1} L_2$, $T = L_1^{-1} + L_1^{-1} L_2 S^\# L_2^\top L_1^{-1}$. Then*

$$Kf(G) = n\mathrm{tr}(T) + n\mathrm{tr}(S^\#) - \mathrm{j}^\top T \mathrm{j} \ .$$

Proof. Let $N = \begin{pmatrix} T & -L_1^{-1}L_2S^{\#} \\ -S^{\#}L_2^{\top}L_1^{-1} & S^{\#} \end{pmatrix}$. From Lemma 2.8, we know that N is a symmetric $\{1\}$-inverse of L_G. By Lemma 2.3, we get

$$Kf(G) = n\mathrm{tr}(N) - \mathrm{j}^{\top}N\mathrm{j} = n\mathrm{tr}(T) + n\mathrm{tr}(S^{\#}) - \mathrm{j}^{\top}T\mathrm{j} - \mathrm{j}^{\top}S^{\#}\mathrm{j} + 2\mathrm{j}^{\top}L_1^{-1}L_2S^{\#}\mathrm{j}.$$

By $L_G\mathrm{j} = 0$, we get $L_1\mathrm{j} + L_2\mathrm{j} = 0$, $L_2^{\top}\mathrm{j} + L_3\mathrm{j} = 0$. Hence $S\mathrm{j} = L_3\mathrm{j} - L_2^{\top}L_1^{-1}L_2\mathrm{j} = L_3\mathrm{j} + L_2^{\top}L_1^{-1}L_1\mathrm{j} = 0$. By Lemma 2.6, we have $S^{\#}\mathrm{j} = 0$. Hence

$$Kf(G) = n\mathrm{tr}(T) + n\mathrm{tr}(S^{\#}) - \mathrm{j}^{\top}T\mathrm{j}\ .$$

■

4 Kirchhoff indices of $G_1 \diamond G_2$ and $G_1 \underline{\diamond} G_2$

For two matrices $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{p\times q}$, the Kronecker product $A \otimes B$ is the $mp \times nq$ matrix obtained from A by replacing each entry a_{ij} by $a_{ij}B$. If A and B are square, then $\mathrm{tr}(A\otimes B) = \mathrm{tr}(A)\mathrm{tr}(B)$. For matrices A, B, C and D such that products AC and BD exist, we have $(A\otimes B)(C\otimes D) = AC\otimes BD$. It is known that $(A\otimes B)^{\top} = A^{\top}\otimes B^{\top}$, and if A and B are nonsingular, then $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

Let I_n denote the identity matrix of order n, and let j_n denote an all-ones column vector of dimension n. The adjacency matrix of the edge corona $G_1 \diamond G_2$ can be written as (see [18])

$$A_{G_1\diamond G_2} = \begin{pmatrix} I_{m_1} \otimes A_{G_2} & B^{\top} \otimes \mathrm{j}_{n_2} \\ B \otimes \mathrm{j}_{n_2}^{\top} & A_{G_1} \end{pmatrix}, \tag{4.1}$$

where B is the vertex-edge incidence matrix of G_1, $m_1 = |E(G_1)|$, $n_2 = |V(G_2)|$. Clearly, the adjacency matrix of $G_1\underline{\diamond}G_2$ can be written as

$$A_{G_1\underline{\diamond}G_2} = \begin{pmatrix} I_{m_1} \otimes A_{G_2} & B^{\top} \otimes \mathrm{j}_{n_2} \\ B \otimes \mathrm{j}_{n_2}^{\top} & 0 \end{pmatrix}. \tag{4.2}$$

Theorem 4.1. *Let G_1 be a connected graph with n_1 vertices and m_1 edges, and let G_2 be a graph with n_2 vertices. Then*

$$\begin{aligned} Kf(G_1\underline{\diamond}G_2) &= \frac{2}{n_2}Kf(G_1) + Kf^{+}(G_1) + \frac{n_2}{2}Kf^{*}(G_1) - \frac{(n_1+m_1n_2)(n_1-1)+m_1n_2}{2} \\ &+ \sum_{i=1}^{n_2}\frac{m_1(n_1+m_1n_2)}{\mu_i(G_2)+2}\ . \end{aligned}$$

Proof. From equation (4.2), we know that the Laplacian matrix of $G_1\underline{\diamond}G_2$ has the following form

$$L_{G_1\underline{\diamond}G_2} = \begin{pmatrix} I_{m_1} \otimes (L_{G_2} + 2I_{n_2}) & -B^{\top} \otimes \mathrm{j}_{n_2} \\ -B \otimes \mathrm{j}_{n_2}^{\top} & n_2D_{G_1} \end{pmatrix},$$

where $B \in \mathbb{R}^{n_1 \times m_1}$ is the vertex-edge incidence matrix of G_1. The matrix S defined in Theorem 3.4 is

$$\begin{aligned} S &= n_2 D_{G_1} - (B \otimes \mathrm{j}_{n_2}^\top)(I_{m_1} \otimes (L_{G_2} + 2I_{n_2})^{-1})(B^\top \otimes \mathrm{j}_{n_2}) \\ &= n_2 D_{G_1} - (B \otimes \mathrm{j}_{n_2}^\top)(B^\top \otimes \frac{1}{2}\mathrm{j}_{n_2}) \\ &= n_2 D_{G_1} - \frac{n_2}{2} B B^\top = n_2 D_{G_1} - \frac{n_2}{2}(D_{G_1} + A_{G_1}) = \frac{n_2}{2} L_{G_1} . \end{aligned}$$

Hence $S^\# = \frac{2}{n_2} L_{G_1}^\#$. The matrix T defined in Theorem 3.4 is

$$\begin{aligned} T &= I_{m_1} \otimes (L_{G_2} + 2I_{n_2})^{-1} + (I_{m_1} \otimes (L_{G_2} + 2I_{n_2})^{-1})(B^\top \otimes \mathrm{j}_{n_2}) S^\# (B \otimes \mathrm{j}_{n_2}^\top) \\ &\quad (I_{m_1} \otimes (L_{G_2} + 2I_{n_2})^{-1}) \\ &= I_{m_1} \otimes (L_{G_2} + 2I_{n_2})^{-1} + (B^\top \otimes \frac{1}{2}\mathrm{j}_{n_2}) S^\# (B \otimes \frac{1}{2}\mathrm{j}_{n_2}^\top) \\ &= I_{m_1} \otimes (L_{G_2} + 2I_{n_2})^{-1} + \frac{1}{2n_2} B^\top L_{G_1}^\# B \otimes J_{n_2} , \end{aligned}$$

where J_{n_2} is the all-ones matrix of order n_2. Since $S^\# = \frac{2}{n_2} L_{G_1}^\#$, by Theorem 3.4 and Lemma 2.3, we have

$$Kf(G_1 \underline{\diamond} G_2) = \frac{2(n_1 + m_1 n_2)}{n_1 n_2} Kf(G_1) + (n_1 + m_1 n_2)\mathrm{tr}(T) - \mathrm{j}^\top T \mathrm{j} . \tag{4.3}$$

Let $\pi = (d_1, \ldots, d_{n_1})^\top$ be the column vector of the degree sequence of G_1. By computation, we have

$$\mathrm{j}^\top T \mathrm{j} = m_1 \mathrm{j}^\top (L_{G_2} + 2I_{n_2})^{-1} \mathrm{j} + \frac{n_2^2}{2n_2} \pi^\top L_{G_1}^\# \pi = \frac{m_1 n_2}{2} + \frac{n_2}{2} \pi^\top L_{G_1}^\# \pi , \tag{4.4}$$

$$\begin{aligned} \mathrm{tr}(T) &= m_1 \mathrm{tr}[(L_{G_2} + 2I_{n_2})^{-1}] + \frac{n_2}{2n_2} \mathrm{tr}(B^\top L_{G_1}^\# B) \\ &= \sum_{i=1}^{n_2} \frac{m_1}{\mu_i(G_2) + 2} + \frac{1}{2} \sum_{ij \in E(G_1)} [(L_{G_1}^\#)_{ii} + (L_{G_1}^\#)_{jj} + 2(L_{G_1}^\#)_{ij}] . \end{aligned}$$

By Lemmas 2.1 and 2.2, we have

$$\begin{aligned} \mathrm{tr}(T) &= \sum_{i=1}^{n_2} \frac{m_1}{\mu_i(G_2) + 2} + \frac{1}{2} \sum_{ij \in E(G_1)} [2(L_{G_1}^\#)_{ii} + 2(L_{G_1}^\#)_{jj} - r_{ij}(G_1)] \\ &= \sum_{i=1}^{n_2} \frac{m_1}{\mu_i(G_2) + 2} + \mathrm{tr}(D_{G_1} L_{G_1}^\#) - \frac{n_1 - 1}{2} . \end{aligned}$$

From Eqs. (4.3), (4.4) and the above equation, we have

$$\begin{aligned} Kf(G_1 \underline{\diamond} G_2) &= \frac{2(n_1 + m_1 n_2)}{n_1 n_2} Kf(G_1) + \sum_{i=1}^{n_2} \frac{m_1(n_1 + m_1 n_2)}{\mu_i(G_2) + 2} + (n_1 + m_1 n_2)\mathrm{tr}(D_{G_1} L_{G_1}^\#) \\ &\quad - \frac{(n_1 + m_1 n_2)(n_1 - 1) + m_1 n_2}{2} - \frac{n_2}{2} \pi^\top L_{G_1}^\# \pi . \end{aligned}$$

By Theorem 3.2, we have $\pi^\top L^{\#}_{G_1}\pi = 2m_1\mathrm{tr}(D_{G_1}L^{\#}_{G_1}) - Kf^*(G_1)$ and $n_1\mathrm{tr}(D_{G_1}L^{\#}_{G_1}) = Kf^+(G_1) - \frac{2m_1}{n_1}Kf(G_1)$. Hence

$$\begin{aligned} Kf(G_1\underline{\diamond}G_2) &= \frac{2(n_1+m_1n_2)}{n_1n_2}Kf(G_1) + n_1\mathrm{tr}(D_{G_1}L^{\#}_{G_1}) \\ &+ \frac{n_2}{2}Kf^*(G_1) + \sum_{i=1}^{n_2}\frac{m_1(n_1+m_1n_2)}{\mu_i(G_2)+2} - \frac{(n_1+m_1n_2)(n_1-1)+m_1n_2}{2} \\ &= \frac{2}{n_2}Kf(G_1) + Kf^+(G_1) + \frac{n_2}{2}Kf^*(G_1) - \frac{(n_1+m_1n_2)(n_1-1)+m_1n_2}{2} \\ &+ \sum_{i=1}^{n_2}\frac{m_1(n_1+m_1n_2)}{\mu_i(G_2)+2}\ . \end{aligned}$$

■

Remark 4.1. If $G_2 = K_1$ is an isolated vertex in Theorem 4.1, then we can obtain Theorem 2.3 in [31].

From Lemma 2.5, Corollary 3.3 and Theorem 4.1, we can obtain bounds of $Kf(G_1\underline{\diamond}G_2)$ as follows.

Proposition 4.2. *Let G_1 and G_2 be two graphs satisfying conditions in Theorem 4.1, and let*

$$\begin{aligned} c_1 &= \frac{(n_2\delta(G_1)+2)(n_1+m_1n_2)}{n_1n_2},\ c_2 = \frac{(n_2\Delta(G_1)+2)(n_1+m_1n_2)}{n_1n_2}, \\ c_3 &= -\frac{(n_1+m_1n_2)(n_1-1)+m_1n_2}{2} + \sum_{i=1}^{n_2}\frac{m_1(n_1+m_1n_2)}{\mu_i(G_2)+2}\ . \end{aligned}$$

Then

$$c_1Kf(G_1) + c_3 \leqslant Kf(G_1\underline{\diamond}G_2) \leqslant c_2Kf(G_1) + c_3\ .$$

Equalities in both sides hold if and only if G_1 is regular.

We can obtain the following result from Proposition 4.2.

Corollary 4.3. *Let G be a connected graph with n vertices and m edges, and let*

$$c_1 = \frac{(\delta(G)+2)(m+n)}{n},\ c_2 = \frac{(\Delta(G)+2)(m+n)}{n}\ .$$

Then

$$c_1Kf(G) + \frac{m^2-n^2+n}{2} \leqslant Kf(G\underline{\diamond}K_1) \leqslant c_2Kf(G) + \frac{m^2-n^2+n}{2}\ .$$

Equalities in both sides hold if and only if G is regular.

Remark 4.2. The bounds in Corollary 4.3 are better than the bounds given in [31, Proposition 2.5].

Theorem 4.4. *Let G_1 be a connected graph with n_1 vertices and m_1 edges, and let G_2 be a graph with n_2 vertices. Then*

$$\begin{aligned} Kf(G_1 \diamond G_2) &= \frac{2}{n_2+2}Kf(G_1) + \frac{n_2}{n_2+2}[Kf^+(G_1) + \frac{n_2}{2}Kf^*(G_1)] \\ &+ \sum_{i=1}^{n_2} \frac{m_1(n_1+m_1n_2)}{\mu_i(G_2)+2} - \frac{n_2(n_1^2-n_1+m_1n_1n_2+2m_1)}{2(n_2+2)} \, . \end{aligned}$$

Proof. From equation (4.1), we know that the Laplacian matrix of $G_1 \diamond G_2$ has the following form

$$L_{G_1 \diamond G_2} = \begin{pmatrix} I_{m_1} \otimes (L_{G_2} + 2I_{n_2}) & -B^\top \otimes \mathrm{j}_{n_2} \\ -B \otimes \mathrm{j}_{n_2}^\top & L_{G_1} + n_2 D_{G_1} \end{pmatrix},$$

where B is the vertex-edge incidence matrix of G_1. Similar with the proof of Theorem 4.1, we know that matrices S and T defined in Theorem 3.4 are

$$\begin{aligned} S &= L_{G_1} + n_2 D_{G_1} - (B \otimes \mathrm{j}_{n_2}^\top)(I_{m_1} \otimes (L_{G_2} + 2I_{n_2})^{-1})(B^\top \otimes \mathrm{j}_{n_2}) \\ &= L_{G_1} + n_2 D_{G_1} - \frac{n_2}{2} BB^\top = \frac{n_2+2}{2} L_{G_1} \, , \\ T &= I_{m_1} \otimes (L_{G_2} + 2I_{n_2})^{-1} + (I_{m_1} \otimes (L_{G_2} + 2I_{n_2})^{-1})(B^\top \otimes \mathrm{j}_{n_2}) S^{\#} (B \otimes \mathrm{j}_{n_2}^\top) \\ &\quad (I_{m_1} \otimes (L_{G_2} + 2I_{n_2})^{-1}) \\ &= I_{m_1} \otimes (L_{G_2} + 2I_{n_2})^{-1} + \frac{1}{2(n_2+2)} B^\top L_{G_1}^{\#} B \otimes J_{n_2} \, , \end{aligned}$$

where J_{n_2} is the all-ones matrix of order n_2. Since $S^{\#} = \frac{2}{n_2+2} L_{G_1}^{\#}$, by Theorem 3.4 and Lemma 2.3, we have

$$Kf(G_1 \diamond G_2) = \frac{2(n_1+m_1n_2)}{n_1(n_2+2)} Kf(G_1) + (n_1+m_1n_2)\mathrm{tr}(T) - e^\top T e \, . \tag{4.5}$$

Let $\pi = (d_1, \ldots, d_{n_1})^\top$ be the column vector of the degree sequence of G_1. Similar with the proof of Theorem 4.1, we can get

$$e^\top T e = \frac{m_1 n_2}{2} + \frac{n_2^2}{2(n_2+2)} \pi^\top L_{G_1}^{\#} \pi \, , \tag{4.6}$$

$$\mathrm{tr}(T) = \sum_{i=1}^{n_2} \frac{m_1}{\mu_i(G_2)+2} + \frac{n_2}{n_2+2} \mathrm{tr}(D_{G_1} L_{G_1}^{\#}) - \frac{n_2(n_1-1)}{2(n_2+2)} \, . \tag{4.7}$$

From Eqs. (4.5), (4.6) and (4.7), we have

$$\begin{aligned} Kf(G_1\diamond G_2) &= \frac{2(n_1+m_1n_2)}{n_1(n_2+2)}Kf(G_1)+\sum_{i=1}^{n_2}\frac{m_1(n_1+m_1n_2)}{\mu_i(G_2)+2}\\ &+ \frac{n_2(n_1+m_1n_2)}{n_2+2}\mathrm{tr}(D_{G_1}L^{\#}_{G_1})-\frac{n_2(n_1^2-n_1+m_1n_1n_2+2m_1)}{2(n_2+2)}\\ &- \frac{n_2^2}{2(n_2+2)}\pi^\top L^{\#}_{G_1}\pi\ . \end{aligned}$$

By Theorem 3.2, we have $\pi^\top L^{\#}_{G_1}\pi = 2m_1\mathrm{tr}(D_{G_1}L^{\#}_{G_1}) - Kf^*(G_1)$ and $n_1\mathrm{tr}(D_{G_1}L^{\#}_{G_1}) = Kf^+(G_1) - \frac{2m_1}{n_1}Kf(G_1)$. Hence

$$\begin{aligned} Kf(G_1\diamond G_2) &= \frac{2(n_1+m_1n_2)}{n_1(n_2+2)}Kf(G_1)+\frac{n_1n_2}{n_2+2}\mathrm{tr}(D_{G_1}L^{\#}_{G_1})+\frac{n_2^2}{2(n_2+2)}Kf^*(G_1)\\ &- \frac{n_2(n_1^2-n_1+m_1n_1n_2+2m_1)}{2(n_2+2)}+\sum_{i=1}^{n_2}\frac{m_1(n_1+m_1n_2)}{\mu_i(G_2)+2}\\ &= \frac{2}{n_2+2}Kf(G_1)+\frac{n_2}{n_2+2}[Kf^+(G_1)+\frac{n_2}{2}Kf^*(G_1)]\\ &+ \sum_{i=1}^{n_2}\frac{m_1(n_1+m_1n_2)}{\mu_i(G_2)+2}-\frac{n_2(n_1^2-n_1+m_1n_1n_2+2m_1)}{2(n_2+2)}\ . \end{aligned}$$

■

Remark 4.3. If $G_2 = K_1$ is an isolated vertex in Theorem 4.4, then we can obtain Theorem 4.3 in [32].

5 Kirchhoff indices, LEL and LEE of line graphs of semiregular graphs

A graph G is called *semiregular* with parameters (n_1, n_2, r_1, r_2) if G is bipartite and $V(G)$ has a bipartition $V(G) = V_1 \cup V_2$ such that $|V_1| = n_1, |V_2| = n_2$ and vertices in the same colour class have the same degree (n_i vertices of degree r_i, $i = 1, 2$). Let $\phi_M(x)$ denote the characteristic polynomial of a square matrix M.

Lemma 5.1. [8] *Let G be a semiregular graph with parameters (n_1, n_2, r_1, r_2) $(n_1 \geqslant n_2)$. Then*

$$\phi_{Q_G}(x) = x(x-r_1-r_2)(x-r_1)^{n_1-n_2}\prod_{i=2}^{n_2}((x-r_1)(x-r_2)-\lambda_i^2)\,,$$

where $\lambda_1, \ldots, \lambda_{n_2}$ are the first n_2 largest eigenvalues of A_G.

Let $l(G)$ denote the line graph of a graph G. We can obtain the following lemma from [24, Lemma 3.3].

Lemma 5.2. *Let G be a semiregular graph with parameters (n_1, n_2, r_1, r_2). Then the eigenvalues of $L_{l(G)}$ are*

$$(r_1 + r_2)^{n_1 r_1 - n_1 - n_2}, r_1 + r_2 - \mu_1(G), \ldots, r_1 + r_2 - \mu_{n_1+n_2}(G)\,,$$

where the superscript denotes the multiplicity of the eigenvalue.

For a semiregular graph G, some bounds for $Kf(l(G))$ and $LEL(l(G))$ are given in [24]. Here we give formulas of $Kf(l(G)), LEL(l(G))$ and $LEE(l(G))$ as follows.

Theorem 5.3. *Let G be a semiregular graph with parameters (n_1, n_2, r_1, r_2) $(n_1 \geqslant n_2)$. Then*

$$\begin{aligned} LEL(l(G)) &= LEL(G) + (n_1 - n_2)(\sqrt{r_2} - \sqrt{r_1}) + (m - n)\sqrt{r_1 + r_2}\,, \\ LEE(l(G)) &= LEE(G) + (n_1 - n_2)(e^{r_2} - e^{r_1}) + (m - n)e^{r_1 + r_2}\,, \end{aligned}$$

where $m = n_1 r_1 = n_2 r_2$, $n = n_1 + n_2$. If G is connected, then

$$Kf(l(G)) = \frac{m}{n} Kf(G) + \frac{m(m-n)}{r_1 + r_2} - (n_1 - n_2)^2\,.$$

Proof. Suppose that $\lambda_1, \ldots, \lambda_{n_2}$ are the first n_2 largest eigenvalues of A_G. Since G is bipartite, L_G and Q_G have the same spectrum. From Lemma 5.1, we known that the eigenvalues of L_G are

$$0, r_1 + r_2, r_1^{n_1 - n_2}, \frac{r_1 + r_2 \pm \sqrt{(r_1 - r_2)^2 + 4\lambda_i^2}}{2}\,, \quad i = 2, \ldots, n_2\,, \tag{5.1}$$

where the superscript denotes the multiplicity of the eigenvalue. By Lemma 5.2, the eigenvalues of $L_{l(G)}$ are

$$0, (r_1 + r_2)^{n_1 r_1 - n_1 - n_2 + 1}, r_2^{n_1 - n_2}, \frac{r_1 + r_2 \pm \sqrt{(r_1 - r_2)^2 + 4\lambda_i^2}}{2}\,, \quad i = 2, \ldots, n_2\,. \tag{5.2}$$

From (5.1) and (5.2), we have

$$\begin{aligned} LEL(l(G)) &= LEL(G) + (n_1 - n_2)(\sqrt{r_2} - \sqrt{r_1}) + (m - n)\sqrt{r_1 + r_2}\,, \\ LEE(l(G)) &= LEE(G) + (n_1 - n_2)(e^{r_2} - e^{r_1}) + (m - n)e^{r_1 + r_2}\,, \end{aligned}$$

where $m = n_1 r_1 = n_2 r_2$, $n = n_1 + n_2$. If G is connected, then by (5.1), (5.2) and Lemma 2.4, we have

$$\begin{aligned} Kf(l(G)) &= \frac{m(m-n+1)}{r_1+r_2} + \frac{m(n_1-n_2)}{r_2} + \sum_{i=2}^{n_2} \frac{2m}{r_1+r_2+\sqrt{(r_1-r_2)^2+4\lambda_i^2}} \\ &+ \sum_{i=2}^{n_2} \frac{2m}{r_1+r_2-\sqrt{(r_1-r_2)^2+4\lambda_i^2}}, \\ Kf(G) &= \frac{n}{r_1+r_2} + \frac{n(n_1-n_2)}{r_1} + \sum_{i=2}^{n_2} \frac{2n}{r_1+r_2+\sqrt{(r_1-r_2)^2+4\lambda_i^2}} \\ &+ \sum_{i=2}^{n_2} \frac{2n}{r_1+r_2-\sqrt{(r_1-r_2)^2+4\lambda_i^2}} . \end{aligned}$$

From the above equations, we get

$$\begin{aligned} Kf(l(G)) &= \frac{m}{n} Kf(G) + \frac{m(m-n)}{r_1+r_2} + m(n_1-n_2)(\frac{1}{r_2} - \frac{1}{r_1}) \\ &= \frac{m}{n} Kf(G) + \frac{m(m-n)}{r_1+r_2} - (n_1-n_2)^2 . \end{aligned}$$

■

Let $C_t(G)$ denote the t-para-line graph of a graph G. We generalize Theorem 3.11 in [24] as follows.

Theorem 5.4. *Let G be a connected r-regular graph of order n. Then*

$$Kf(C_t(G)) = r(rt+2)Kf(G) + \frac{nrt(1-2n)}{rt+2} + n^2(rt-1) .$$

Proof. The number of edges of G is $m = \frac{nr}{2}$. Note that $G \underline{\diamond} \overline{K_t}$ is a semiregular graph with parameters $(n, mt, rt, 2)$, where $\overline{K_t}$ the complement of the complete graph K_t. Since $C_t(G)$ is the line graph of $G \underline{\diamond} \overline{K_t}$, by Theorem 5.3, we have

$$\begin{aligned} Kf(C_t(G)) &= \frac{nrt}{n+mt} Kf(G \underline{\diamond} \overline{K_t}) + \frac{nrt(nrt-n-mt)}{rt+2} - (n-mt)^2 \\ &= \frac{2rt}{rt+2} Kf(G \underline{\diamond} \overline{K_t}) + \frac{nrt(nrt-n-mt)}{rt+2} - (n-mt)^2 . \end{aligned}$$

By Theorem 4.1, we have

$$\begin{aligned} Kf(G \underline{\diamond} \overline{K_t}) &= \frac{2}{t} Kf(G) + Kf^+(G) + \frac{t}{2} Kf^*(G) - \frac{(n+mt)(n-1)+mt}{2} \\ &+ \frac{mt(n+mt)}{2} = \frac{(rt+2)^2}{2t} Kf(G) + \frac{m^2t^2-n^2+n}{2} . \end{aligned}$$

From the expressions of $Kf(C_t(G))$ and $Kf(G \underline{\diamond} \overline{K_t})$, we get

$$\begin{aligned} Kf(C_t(G)) &= r(rt+2)Kf(G) + \frac{rt(m^2t^2 - n^2 + n) + nrt(nrt - n - mt)}{rt+2} - (n-mt)^2 \\ &= r(rt+2)Kf(G) + \frac{nrt(nrt - 2n - mt + 1) - 2m^2t^2}{rt+2} - (n^2 - 2mnt) \end{aligned}$$

Since $m = \frac{nr}{2}$, we have

$$Kf(C_t(G)) = r(rt+2)Kf(G) + \frac{nrt(1-2n)}{rt+2} + n^2(rt-1) \ .$$

■

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