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Some Interplay of the Three Kirchhoffian Indices

José Luis Palacios
Electrical and Computer Engineering Department, The University of New Mexico, Albuquerque, NM 87131, USA
jpalacios@unm.edu

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Abstract

We review some known facts of the three Kirchhoffian descriptors, and find new relations among them. Specifically we find new lower and upper bounds for the additive degree-Kirchhoff index based on other bounds known for the multiplicative degree-Kirchhoff index, and a new identity relating the three indices.

1 Introduction

Let $G = (V, E)$ be a finite simple connected graph with vertex set $V = \{1, 2, \dots, n\}$ and degrees d_i for $1 \leq i \leq n$, with δ and Δ the smallest and largest such degrees, respectively. There is a family of resistive - or Kirchhoffian - descriptors $R^f(G)$ with the general formula

$$R^f(G) = \sum_{i<j} f(i,j) R_{i,j}, \tag{1}$$

where R_{ij} is the effective resistance between vertices i and j and $f(i,j)$ is some real function of the vertices. Among the more popular of these descriptors one should mention the Kirchhoff index $R(G)$, the multiplicative degree-Kirchhoff index $R^*(G)$ and the additive degree-Kirchhoff index $R^+(G)$, defined by (1) when taking $f(i,j) = 1$, $f(i,j) = d_i d_j$ and $f(i,j) = d_i + d_j$, respectively, and introduced in [7], [4] and [8] respectively. There exists an abundant literature on $R(G)$, of which we mention [9], [13], [11], [14], [21], [15], [2] for a variety of approaches, and a growing number of references for the other two indices (see [16], [6], [3], [12], [10] and their references).

Here is a first look at the relative orders of these indices:

Proposition 1 *For any G we have*

$$R^+(G) \leq 2(n-1)R(G), \tag{2}$$

$$R^*(G) \leq (n-1)^2 R(G), \tag{3}$$

$$R^*(G) \leq \frac{n-1}{2} R^+(G). \tag{4}$$

Proof. Only (4) needs proof:

$$R^*(G) = \sum_{i<j} d_i d_j R_{ij} \leq \frac{n-1}{2} \sum_{i<j} d_i d_j \left(\frac{1}{d_i} + \frac{1}{d_j}\right) R_{ij}$$

$$= \frac{n-1}{2} \sum_{i<j} (d_i + d_j) R_{ij} = \frac{n-1}{2} R^+(G) \bullet$$

These bounds may seem trivial, but they give the correct orders of the three indices, showing, for instance, that there are no graphs with $R(G)$ of order n^2, $R^+(G)$ of order n^4 and $R^*(G)$ of order n^5; likewise, there are no graphs such that $R(G)$, $R^+(G)$ and $R^*(G)$ have orders n, n^2 and n^4, respectively, etc.

Let us denote by K_n, P_n and S_n the complete graph, the linear path graph and the star graph on n vertices, respectively. The extremal values for the Kirchhoffian indices are shown in the following inequalities, valid for all G:

$$n-1 = R(K_n) \leq R(G) \leq R(P_n) = \frac{n^3-n}{6} \tag{5}$$

$$2n^2 - 5n + 3 = R^*(S_n) \leq R^*(G) \leq \frac{n(n+1)(n-1)^3}{6} \tag{6}$$

$$2(n-1)^2 = R^+(K_n) \leq R^+(G) \leq \frac{n(n+1)(n-1)^2}{3}. \tag{7}$$

The left inequality of (5) was shown in [11], the right in [14]. In the cases of (6) and (7), the left inequalities were found in [20] and in [19], respectively. The right hand side inequalities are a simple consequence of (5) and the naive bounds (2) and (3). These right hand side inequalities of (6) and (7) give the correct order of the maxima, since in [19] it was shown that $R^*(G) \sim \frac{2}{243}n^5$, when G is the $(1/3, 1/3, 1/3)$-barbell graph, consisting of two copies of $K_{n/3}$ united by a $P_{n/3}$, and this same graph was shown in [16] to attain the value $R^+(G) \sim \frac{2}{27}n^4$. The obvious conjectures are that the balanced barbell graph maximizes both descriptors $R^+(G)$ and $R^*(G)$.

Notice that the inequality (4) implies that every time a new upper bound is found for $R^+(G)$ that improves the right inequality in (7) there is a chance that this will imply a better upper bound for $R^*(G)$ that improves (6). Thus for instance, in [12] it is shown that

$$R^+(G) \leq \frac{1}{8}(n-1)^3(n+14), \tag{8}$$

which is the best known upper bound for $R^+(G)$, and this implies

$$R^*(G) \leq \frac{1}{16}(n-1)^4(n+14),$$

which improves the right inequality in (6). However, the best known upper bound for the multiplicative degree-Kirchhoff index at this time seems to be

$$R^*(G) \leq \frac{1}{54}(n^5 + 50n^3 - 164n^2 + 165n - 52), \tag{9}$$

for $n \geq 49$ and

$$R^*(G) \leq (n-1)^4, \tag{10}$$

for $n \leq 48$, as can be seen in [3].

In [16] we found the following equation relating the two degree-Kirchhoff indices:

$$R^+(G) = \frac{n}{2|E|}R^*(G) + \sum_j \sum_i \pi_i E_i T_j, \tag{11}$$

where $\pi = (\pi_i)_{1 \leq i \leq n}$ is the stationary distribution of the random walk on G, which can be given explicitly as $\pi_i = \dfrac{d_i}{2|E|}$, and where $E_i T_j$ denotes the expected hitting time of the vertex j by the walk on G started at the vertex i (see [1] for all matters regarding random walks on graphs). We argued in [16] that both summands in (11) can be bounded below by $(n-1)^2$ and therefore we found that

$$R^+(G) \geq \frac{n}{2|E|}R^*(G) + (n-1)^2 \geq 2(n-1)^2, \tag{12}$$

a bound that is attained by the complete graph K_n.

In what follows we will further exploit the equation (11) in order to find a set of inequalities linking the three Kirchhoffian descriptors simultaneously, and more specialized upper and lower bounds for the additive degree-Kirchhoff index.

2 The three descriptors together

We look at the equation (11) and notice that we can extract from it the following bounds involving the three Kirchhoffian descriptors

Proposition 2 *For any G we have*

$$\frac{n}{2|E|}R^*(G)+\delta R(G)\leq R^+(G)\leq \frac{n}{2|E|}R^*(G)+\Delta R(G), \tag{13}$$

where δ and Δ are the smallest and the largest degrees of G, respectively.

Proof. Replacing $\pi_i=\frac{d_i}{2|E|}$ in (11) we observe that

$$\sum_j\sum_i \pi_i E_i T_j = \frac{1}{2|E|}\sum_j\sum_i d_i E_i T_j \leq \frac{\Delta}{2|E|}\sum_j\sum_i E_i T_j.$$

But now we recall that $\frac{1}{2|E|}\sum_j\sum_i E_iT_j$ is precisely the hitting time expression for $R(G)$ (see [14]) and the upper bound of (13) is shown. The lower bounds proceeds similarly •

Remarks. Another interesting relation involving all three Kirchhoffian descriptors can be found in [5]. Notice that for G d-regular, the inequalities in (13) become equalities and all terms are equal to $2dR(G)$.

3 Lower bounds for $R^+(G)$

For the lower bounds, rather than (13) we will use the left inequality in (12). Let $\sigma = \sqrt{\frac{2}{n}\sum_{(i,j)\in E}\frac{1}{d_id_j}}$. Using prior results for $R^*(G)$ we get the following

Proposition 3 *For any G we have*

$$R^+(G)\geq \frac{4|E|}{\Delta}\left(n-2+\frac{1}{\Delta+1}\right), \tag{14}$$

$$R^+(G)\geq n\left(n-2+\frac{1}{\Delta+1}\right)+(n-1)^2, \tag{15}$$

$$R^+(G)\geq \frac{4|E|}{\Delta}\left[\frac{1}{1+\frac{\sigma}{\sqrt{n-1}}}+\frac{(n-2)^2}{n-1-\frac{\sigma}{\sqrt{n-1}}}\right], \tag{16}$$

and

$$R^+(G)\geq n\left[\frac{1}{1+\frac{\sigma}{\sqrt{n-1}}}+\frac{(n-2)^2}{n-1-\frac{\sigma}{\sqrt{n-1}}}\right]+(n-1)^2. \tag{17}$$

Both equalities in (14) and (15) are attained by the complete graph K_n.

If G is bipartite then

$$R^+(G)\geq \frac{n(2n-3)}{2}+(n-1)^2, \tag{18}$$

and

$$R^+(G)\geq \frac{2|E|}{\Delta}(2n-3). \tag{19}$$

Proof. The following bound was shown in [20]:

$$R^*(G) \geq 2|E|\left(n-2+\frac{1}{\Delta+1}\right). \tag{20}$$

It is clear that (20) and the obvious refinement of (4):

$$R^*(G) \leq \frac{\Delta}{2}R^+(G) \tag{21}$$

prove (14). Also, (20) and (12) prove (15). In [19] it was shown that for bipartite graphs

$$R^*(G) \geq |E|(2n-3). \tag{22}$$

It is immediate that (12) and (22) prove (18), and (22) with (21) prove (19). Finally, it was shown in [2] that

$$R^*(G) \geq 2|E|\left[\frac{1}{1+\frac{\sigma}{\sqrt{n-1}}}+\frac{(n-2)^2}{n-1-\frac{\sigma}{\sqrt{n-1}}}\right]. \tag{23}$$

Clearly (12) and (23) show (17) and (21) and (23) show (16) •

Notice that the bound (17) is an increasing function of σ. Since $1 > \sigma \geq \dfrac{1}{\sqrt{n-1}}$, when we replace in (17) the smallest value of σ, we recover the rightmost bound in (12). The bounds (20) and (23) are not comparable, as was seen in [2]. Also, the bounds (12) (its left most inequality) and (21) are not comparable: for S_n, the former is quadratic and the latter is linear, whereas for the n-cycle the former is of order $\frac{n^3}{6}$ and the latter is of order $\frac{n^3}{3}$. This implies that the bounds in the previous proposition are not comparable.

4 Upper bounds for $R^+(G)$

For the upper bounds, we will use the right inequality in (13). Let $\mathbf{D} = diag(d_1, \ldots, d_n)$ be the diagonal matrix containing the degrees of the vertices in its diagonal and let $\mathbf{A}$ be the incidence matrix of G. Then the matrix $\mathbf{P} = \mathbf{D}^{-1}\mathbf{A}$ is the transition probability matrix of the random walk on G. Let

$$1 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_n \leq -1$$

be the eigenvalues of $\mathbf{P}$. The main results are contained in the next

Proposition 4 *For any G we have*

$$R^+(G) \leq n\left(1+\frac{\Delta}{\delta}\right)\left(\frac{n-k-2}{1-\lambda_2}+\frac{k}{2}+\frac{1}{\theta}\right), \tag{24}$$

where $k = \left\lfloor \frac{\lambda_2(n-1)+1}{\lambda_2+1} \right\rfloor$ *and* $\theta = \lambda_2(n-k-2) - k + 2$.

For G bipartite we have

$$R^+(G) \leq n\left(1+\frac{\Delta}{\delta}\right)\left(\frac{n-k-3}{1-\lambda_2}+\frac{k+1}{2}+\frac{1}{\theta}\right), \tag{25}$$

where $k = \left\lfloor \frac{\lambda_2(n-2)}{\lambda_2+1} \right\rfloor$ *and* $\theta = \lambda_2(n-k-3) - k + 1$.

Proof. We write $R^*(G)$ in terms of the λ_is:

$$R^*(G) = 2|E| \sum_{i\geq 2} \frac{1}{1-\lambda_i},$$

and we bound $R(G)$ with the same sum:

$$R(G) \leq \frac{n}{\delta} \sum_{i\geq 2} \frac{1}{1-\lambda_i},$$

as was shown in [19]. Putting these facts together in (13), we obtain

$$R^+(G) \leq n\left(1+\frac{\Delta}{\delta}\right) \sum_{i\geq 2} \frac{1}{1-\lambda_i}.$$

At this point we could bound the sum with the obvious upper bound

$$\frac{n-1}{1-\lambda_2},$$

obtaining the compact upper bound in terms of the *spectral gap* $1-\lambda_2$:

$$R^+(G) \leq \frac{n(n-1)}{1-\lambda_2}\left(1+\frac{\Delta}{\delta}\right), \tag{26}$$

for any G, and the tighter bound

$$R^+(G) \leq n\left(1+\frac{\Delta}{\delta}\right)\left(\frac{n-2}{1-\lambda_2}+\frac{1}{2}\right), \tag{27}$$

for bipartite G, using the fact that for such graphs $\lambda_n = -1$.

Alternatively, we can use majorization in order to minimize the sum $\sum_{i\geq 2} \frac{1}{1-\lambda_i}$, as we did in [2], and obtain the expressions in (24) and (25), finishing the proof •

Notice, for instance, that if $\lambda_2 = 0$, then $k = 1$ and $\theta = 1$ for general G, and $k = 0$ and $\theta = 1$ for bipartite G, so that (24), (25) and (27) all become

$$R^+(G) \leq n\left(n-\frac{3}{2}\right)\left(1+\frac{\Delta}{\delta}\right). \tag{28}$$

For a specific example, consider the complete bipartite graph $K_{r,s}$ with $r < s$. Then (28) yields

$$R^+(G) \leq (r+s)(r+s-\frac{3}{2})(1+\frac{s}{r}).$$

If $r = s = n$, the bound turns out to be $8n^2 - 6n$, which is the exact value of $R^+(K_{n,n})$. This exact value is also captured by the lower bound (19).

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