MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **74** (2015) 399-410

ISSN 0340 - 6253

On Minimum Matching Energy of Graphs

Wen–Huan Wang[1,*], **Wasin So**[2 †]

[1] *Department of Mathematics, Shanghai University, Shanghai 200444, China*

[2] *Department of Mathematics, San José State University, San José, California 95192-0103, USA.*

(Received November 24, 2014)

Abstract

New transformations which preserve the order and size of a graph, but decrease its matching energy are proposed to compare the matching energy between two graphs. Let $\Psi_{n,m}$ be the set of the (n,m)-graphs, where n and m are the numbers of vertices and edges, respectively. With the aid of the new method, the graphs with the minimum matching energy in $\Psi_{n,m}$ are deduced for three cases, namely unicyclic and bipartite unicyclic graphs ($m = n$), bicyclic graphs ($m = n + 1$), and cactus graphs, where a cactus is a graph with all of its blocks being either edges or cycles.

1 Introduction

Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A k-matching of G is a union of k independent edges in G. We denote $m(G, k)$ the number of k-matchings in G. It is consistent to define $m(G, 0) = 1$ and $m(G, k) = 0$ for $k < 0$. The following recursive formula is very important in studying k-matching numbers [1]:

$$m(G, k) = m(G - e, k) + m(G - u - v, k - 1),$$

where $e = uv \in E(G)$. Recently, Gutman and Wagner [2] define the matching energy of G using the matching polynomial, which is given by

$$\alpha(G) = \sum_{k\geq 0}(-1)^k m(G, k)x^{n-2k},$$

*Corresponding author, whwang@shu.edu.cn
†wasin.so@sjsu.edu

where $n = |V(G)|$ is the order of G. Note that all the zeros of $\alpha(G)$ are real and they are symmetrical about 0, see Godsil and Gutman [3].

Definition 1.1. Let G be a simple graph of order n. Then the matching energy of G is defined as

$$ME(G) = \sum_{1 \leq i \leq n} |\alpha_i|,$$

where α_1, α_2, $\cdots$, and α_n denote the zeros of $\alpha(G)$.

Gutman and Wagner [2] pointed out that $ME(G)$ is a quantity of relevance for chemical applications which can be traced back to the 1970s. For more details about the matching energy, one can refer to [2].

Example 1.2. (i) Let G_1 be the connected graph of order 4 obtained by appending a pendant edge to a cycle of order 3. Then $m(G_1, 1) = 4$, $m(G_1, 2) = 1$ and $m(G_1, k) = 0$ for $k \geq 3$, hence

$$\alpha(G_1) = x^4 - 4x^2 + 1$$

has 4 zeros $\pm\sqrt{2 \pm \sqrt{3}}$, and so $ME(G_1) = 2\sqrt{6}$.

(ii) Let G_2 be a cycle of order 4. Then $m(G_2, 1) = 4$, $m(G_2, 2) = 2$ and $m(G_2, k) = 0$ for $k \geq 3$, hence

$$\alpha(G_2) = x^4 - 4x^2 + 2$$

has 4 zeros $\pm\sqrt{2 \pm \sqrt{2}}$, and so $ME(G_2) = 2\sqrt{2(2 + \sqrt{2})}$.

Note that the sum of the zeros of $\alpha(G)$ is zero. Similar to the Coulson integral formula for the energy of a graph, see [4], we have an equivalent definition for the matching energy of a graph G:

$$ME(G) = \frac{2}{\pi} \int_0^{+\infty} \frac{1}{x^2} \ln \Big[\sum_{k \geq 0} m(G, k) x^{2k} \Big] \mathrm{d}x.$$

This equivalent definition makes comparison of matching energies easier by using matching numbers directly. Let G_1 and G_2 be two graphs. If $m(G_1, k) \leqslant m(G_2, k)$ holds for all $k \geqslant 0$, we denote $G_1 \preccurlyeq G_2$. Furthermore, if $G_1 \preccurlyeq G_2$ and there exists at least one k such that $m(G_1, k) < m(G_2, k)$, we have $G_1 \prec G_2$. If $m(G_1, k) = m(G_2, k)$ holds for all $k \geqslant 0$, we write $G_1 \sim G_2$. It follows easily from the definitions that

$$G_1 \preccurlyeq G_2 \Rightarrow ME(G_1) \leq ME(G_2),$$

$$G_1 \prec G_2 \Rightarrow ME(G_1) < ME(G_2),$$

$$G_1 \sim G_2 \Rightarrow ME(G_1) = ME(G_2),$$

and there are examples showing that all the converses are false, see [5, 6]. For the graphs G_1 and G_2 in Example 1.2, we have $G_1 \prec G_2$ and so $ME(G_1) < ME(G_2)$.

A theme in the study of matching energy is to determine graphs achieving the extremal matching energy in a class of graphs. Observe that if H is a subgraph of G then $H \preccurlyeq G$, and so $ME(H) \leq ME(G)$. It follows that

$$ME(E_n) \leq ME(G) \leq ME(K_n)$$

for any graph G of order n, where E_n is the empty graph of order n (i.e. edgeless graph), and K_n is the complete graph of order n [2]. The connected unicyclic [7], bicyclic [8], and tricyclic [9] graphs with the maximum matchings, namely the newly proposed maximum matching energy, were determined in the 1980s. Recently, Gutman and Wagner [2] obtained the connected unicyclic graphs with the minimum and the maximum matching energies. Ji et al. [10] determined the connected bicyclic graphs with the minimum and the maximum matching energies. Li and Yan [11] characterized the connected graph with given connectivity κ (resp. chromatic number χ) achieving the maximum matching energy. In this paper, we focus on the minimum matching energy.

Let $\Psi_{n,m}$ be the set of connected graphs of order n and size m. In particular, for $m = n, n+1, n+2$, $\Psi_{n,m}$ are the sets of connected unicyclic graphs, connected bicyclic graphs, and connected tricyclic graphs, respectively. Let $\Phi_{n,m}$ be the set of connected cactus graphs of order n and size m, where a cactus is a graph with all of its blocks being either edges or cycles. Obviously, the graphs in $\Phi_{n,m}$ have $m-n+1$ edge-disjoint cycles and $\Phi_{n,m} \subset \Psi_{n,m}$. In particular, $\Phi_{n,n-1}$ and $\Phi_{n,n}$ are the sets of trees and of connected unicyclic graphs, respectively.

Denote S_n (as shown in Fig. 1(a)) the star graph, S_n^k the graph obtained from S_n by joining k pairs of independent pendant vertices to form k edge-disjoint triangles, and S_n^+ the graph obtained from S_n by joining a pendant vertex to two other pendant vertices. Gutman and Wagner [2] noted that S_n (resp., S_n^1) has the minimum matching energy in $\Psi_{n,n-1}$ (resp., $\Psi_{n,n}$). Ji et al. [10] showed that S_n^+ has the minimum matching energy in

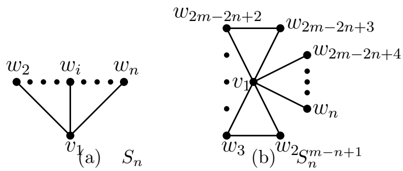


Figure 1: S_n and S_n^{m-n+1}

$\Psi_{n,n+1}$. We propose a uniform and simpler approach to prove these known results using graph transforms. Among $\Psi_{n,m}$, Ji and Ma [12] and Xu et al. [13] independently obtained the graph with the minimum matching energy when $n \leq m \leq 2(n-2)$; So and Wang [14] derived the graphs with the minimum matching energy when $n-1 \leq m \leq 2n-3$ and $n(n-1)/2-(n-2) \leq m \leq n(n-1)/2$. However the minimum matching energy (and the graphs achieving such minimum) in $\Psi_{n,m}$ is still unknown for $2n-2 \leq m \leq n(n-1)/2-(n-1)$. Nonetheless, we use this approach to deduce a new result: S_n^{m-n+1} (as shown in Fig. 1(b)) has the minimum matching energy in $\Phi_{n,m}$.

The rest of the paper is organized as follows. In Section 2, we introduce some new graph transforms that preserve order, size, but decrease matching energy. Some immediate consequences are observed. Then we use them to determine the graph achieving the minimum matching energy in the set of connected cactus graphs in Section 3. The results for connected unicyclic graphs and connected bipartite unicyclic graphs are proved in Section 4 while those for connected bicyclic graphs in Section 5.

2 Main tools

In this section, we study the consequences of some graph transforms which preserve the order and size, but decrease the matching numbers and so the matching energy. Lemma 2.1 is a special case of Lemma 5 in [7]. From Lemma 2.1, we get Corollaries 2.2–2.4 directly. Lemmas 2.5 and 2.7 and Corollaries 2.6 and 2.8 are new results for the comparison of the matching numbers between two graphs with the same order and size.

Lemma 2.1. [7] Let G be a connected graph and $uv \in E(G)$. Let G' be the graph obtained from G by first identifying u with v, and then attaching a pendant vertex to the common vertex $u(v)$. If $d_G(u), d_G(v) \geq 2$ and $N(u) \cap N(v) = \emptyset$, then $G' \prec G$, where $d_G(u)$ is the degree of u of G and $N(u)$ is the set of neighbors of u.

Corollary 2.2. If G has the minimum matching energy in $\Psi_{n,m}$ and C_l is a cycle of G which is edge-disjoint with other cycles of G, then $l = 3$.

Proof: Suppose $l \geq 4$. By applying Lemma 2.1 to G on an edge of C_l, we obtain a new graph G' having a cycle C_{l-1} such that $G' \in \Psi_{n,m}$ and $G' \prec G$. This contradicts the minimality of G. ■

Corollary 2.3. If G has the minimum matching energy in $\Psi_{n,m}$ then a cut-edge of G must be a pendant edge.

Proof: Suppose that G has a cut-edge $e = uv$ which is not a pendant edge. Hence u and v are two vertices of degree at least 2 with $N(v) \cap N(u) = \emptyset$. Hence, by Lemma 2.1, there is a graph $G' \in \Psi_{n,m}$ such that $G' \prec G$, i.e., G does not have the minimum matching energy in $\Psi_{n,m}$, a contradiction. ■

Corollary 2.4. [2] S_n has the minimum matching energy in $\Psi_{n,n-1}$.

Proof: Let $G_0^\star \in \Psi_{n,n-1}$ has the minimum matching energy in $\Psi_{n,n-1}$. Since $G_0^\star$ is a tree, all edges are cut-edges. By Corollary 2.3, all edges of $G_0^\star$ are pendant edges, and so $G_0^\star = S_n$. ■

Lemma 2.5. Let G, H' and H'' be three disjoint connected graphs, $u, v \in V(G)$, $u' \in V(H')$, and $u'' \in V(H'')$, where $|V(H')|, |V(H'')| \geq 2$. Let $G_{u,v}$, G_u and G_v be three graphs constructed from G, H' and H'' as follows:

(i) $G_{u,v}$ is obtained by identifying u with u', and v with u'',

(ii) G_u is obtained by identifying u with both u' and u'', and

(iii) G_v is obtained by identifying v with both u' and u''.

If $G - u \cong G - v$, then $G_u \cong G_v$ and $G_u \prec G_{u,v}$.

Proof. Let $d_{H'}(u') = x$ and $d_{H''}(u'') = y$. As $|V(H')|, |V(H'')| \geq 2$, we have $x, y \geq 1$. In H' (resp., H''), the vertices which are adjacent to u' (resp., u'') are denoted by $w'_1, \cdots, w'_x$ (resp., $w''_1, \cdots, w''_y$). Let $e'_i = u'w'_i$ with $1 \leq i \leq x$ and $e''_j = u''w''_j$ with $1 \leq j \leq y$. The k-matchings of $G_{u,v}$ can be divided into four groups according to whether e'_i and e''_j are contained or not. The first group does not contain e'_i nor e''_j; the second group contains e'_i but not e''_j; the third group contains e''_j but not e'_i; and the fourth group contains both e'_i and e''_j. Accordingly, the number of the k-matchings of $G_{u,v}$ in the t-th group is denoted by n_t, where $1 \leq t \leq 4$.

Let $H' - u' = A_0$, $H' - u' - w'_i = A_i$ with $1 \leq i \leq x$, $H'' - u'' = B_0$, and $H'' - u'' - w''_j = B_j$ with $1 \leq j \leq y$. We get

$$n_1 = m(G \cup A_0 \cup B_0, k),$$

$$n_2 = \Sigma_{1 \leq i \leq x} m((G-u) \cup A_i \cup B_0, k-1),$$

$$n_3 = \Sigma_{1 \leq j \leq y} m((G-v) \cup A_0 \cup B_j, k-1),$$

$$n_4 = \Sigma_{1 \leq i \leq x}(\Sigma_{1 \leq j \leq y} m((G-u-v) \cup A_i \cup B_j, k-2)).$$

Therefore, we obtain

$$m(G_{u,v}, k) = n_1 + n_2 + n_3 + n_4.$$

For $k = 2$, we have $n_4 \geq 1$ since H' and H'' have at least one edge.

Similarly, the k-matchings of G_u can be divided into three groups according to whether e'_i and e''_j are contained or not, where $1 \leq i \leq x$ and $1 \leq j \leq y$. The first group does not contain e'_i nor e''_j; the second group contains e'_i but not e''_j; and the last group contains e''_j but not e'_i. We have

$$m(G_u, k) = n_1 + n_2 + n_5,$$

where $n_5 = \Sigma_{1 \leq j \leq y} m((G-u) \cup A_0 \cup B_j, k-1)$. As $G - u \cong G - v$, we have $n_3 = n_5$. Therefore, $G_u \cong G_v$ and $G_u \prec G_{u,v}$. ■

Corollary 2.6. Let G be a connected graph, $u, v \in V(G)$. Let $G^{s,t}_{u,v}$, G^{s+t}_u and G^{s+t}_v be three graphs constructed from G as follows:

(i) $G^{s,t}_{u,v}$ is obtained by attaching s and t pendant edges to u and v respectively,

(ii) G^{s+t}_u is obtained by attaching $s+t$ pendant edges to u, and

(iii) G^{s+t}_v is obtained by attaching $s+t$ pendant edges to v.

If $G - u \cong G - v$, then $G^{s+t}_u \cong G^{s+t}_v$ and $G^{s+t}_u \prec G^{s,t}_{u,v}$, where s and t are positive integers.

Proof. Corollary 2.6 directly follows from Lemma 2.5. ■

Lemma 2.7. Let G be a connected graph and $v_s, v_{s-1}, v_{s-2} \in V(G)$, where $d_G(v_s) = 1$, $d_G(v_{s-2}) \geq 2$, and v_s and v_{s-2} are neighbors of v_{s-1}. If $v_t \in V(G)$ and $v_t \neq v_s, v_{s-1}, v_{s-2}$, then $G + v_t v_{s-2} \prec G + v_t v_s$.

Proof. Note that

$$m(G + v_t v_{s-2}, k) = m(G, k) + m(G - v_t - v_{s-2}, k - 1), \quad (1)$$

$$m(G + v_t v_s, k) = m(G, k) + m(G - v_t - v_s, k - 1). \quad (2)$$

Let $G - v_t - v_s - v_{s-1} - v_{s-2} = H$. We have

$$m(G - v_t - v_{s-2}, k - 1) = m(G - v_t - v_{s-2} - v_s v_{s-1}, k - 1) + m(H, k - 2), \quad (3)$$

$$m(G - v_t - v_s, k - 1) = m(G - v_t - v_s - v_{s-1} v_{s-2}, k - 1) + m(H, k - 2). \quad (4)$$

As $d_G(v_{s-2}) \geq 2$, $G - v_t - v_s - v_{s-1} v_{s-2}$ contains $G - v_t - v_{s-2} - v_s v_{s-1}$ as its proper subgraph. Therefore, we get

$$m(G - v_t - v_{s-2} - v_s v_{s-1}, k - 1) \leq m(G - v_t - v_s - v_{s-1} v_{s-2}, k - 1). \quad (5)$$

Furthermore, as $k = 2$, the inequality in (5) is strict since $d_G(v_{s-2}) \geq 2$. Substitution of (3)–(5) into (1) and (2) yields $m(G + v_t v_{s-2}, k) \leq m(G + v_t v_s, k)$ for all $k \geq 0$ and the inequality is strict for $k = 2$. Hence, we have $G + v_t v_{s-2} \prec G + v_t v_s$. ■

Corollary 2.8. Let G be a connected graph with two cycles C_a and C_b. Let the vertices of C_a and C_b be labeled clockwise by $u_1, u_2, \cdots, u_a$ and $u'_1, u'_2 \cdots, u'_b$, respectively. If C_a and C_b share a common vertex (say $u_1 = u'_1$) and $d_G(u_2) = 2$, then the graph $G - u_3 u_2 + u_3 u'_b \prec G$.

Proof. Let $H = G - u_2 u_3$. As $d_G(u_2) = 2$, we have $d_H(u_2) = 1$. Obviously, in H, $d_H(u'_b) \geq 2$ and u_2 and u'_b are neighbors of u_1. By Lemma 2.7, we get $H + u_3 u'_b \prec H + u_3 u_2 = G$. ■

In Corollary 2.8, it should be noted that $G - u_3 u_2 + u_3 u'_b$ has two cycles which share a common edge $u_1 u'_b = u'_1 u'_b$ while G has two cycles which share a common vertex. Therefore, for such two cyclic graphs, Lemma 2.7 and Corollary 2.8 provide us with straightforward methods to compare their matching numbers. For example, one can see the proof of Theorem 5.2 in Section 5. We will use Lemmas 2.1 and 2.5 and Corollaries 2.2, 2.3, 2.6, and 2.8 to derive our results in Sections 3–5.

3 Cactus Graphs

In this section, we characterize that S_n^{m-n+1} is the graph with the minimum matching energy in $\Phi_{n,m}$. In $\Phi_{n,m}$, it is interesting that Lu et al. [15] deduced that S_n^{m-n+1} is the graph with the minimum Randić index; Liu and Lu [16] obtained that S_n^{m-n+1} is the extremal cactus for the Wiener index, the Merrifield-Simmons index, the Hosoya index, and the spectral radius by a unified approach.

Lemma 3.1. Let G_0 be the graph with the minimum matching energy in $\Phi_{n,m}$.

(i) All the cut-edges of G_0 are pendant edges.

(ii) If C_l is an induced cycle of G_0 then $l = 3$.

(iii) For any C_3 of G_0, only one vertex at C_3 is attached by a graph.

Proof. (i) For G_0, we suppose that there exists a cut-edge uv that is not a pendant edge. Namely, $d_{G_0}(u), d_{G_0}(v) \geq 2$. Applying Lemma 2.1 to G_0 on uv, we get a new graph G_0' such that $G_0' \in \Phi_{n,m}$ and $G_0' \prec G_0$. This contradicts the minimality of G_0.

(ii) Since $G_0 \in \Phi_{n,m}$, all the cycles of G_0 are mutually edge-disjoint. By Corollary 2.2, we get that all cycles of G_0 have girth 3.

(iii) Choose an arbitrary C_3 of G_0 and let u and v be two vertices at C_3. In G_0, we suppose that u and v are respectively attached by graphs H' and H'', where $|V(H')|, |V(H'')| \geq 2$. We denote by u' and u'' the vertices of H' and H'' which are identified with u and v, respectively. Let G be the graph obtained from G_0 by deleting the vertices in $V(H') \cup V(H'') \setminus \{u', u''\}$. Obviously, $G - u \cong G - v$ and $G_{u,v}$ in Lemma 2.5 is G_0. By applying Lemma 2.5, we get two graphs $G_u, G_v \in \Phi_{n,m}$ such that $G_u \cong G_v$ and $G_u \prec G_{u,v} = G_0$. This contradicts the minimality of G_0. ■

By Lemma 3.1, we can directly obtain $G_0 = S_n^{m-n+1}$. Thus, we get Theorem 3.2 as follows.

Theorem 3.2. For $G \in \Phi_{n,m}$, $ME(S_n^{m-n+1}) \leq ME(G)$, where the equality holds if and only if $G = S_n^{m-n+1}$.

Remark. If a graph G has no even cycle, then any two odd cycles of G are mutually edge-disjoint [17]. Thus, the set of graphs without an even cycle is a subset of $\Phi_{n,m}$. For the significance of graphs without an even cycle, one can refer to [18]. Therefore, by Theorem 3.2, we get that S_n^{m-n+1} is the graph with the minimum matching energy among graphs without an even cycle.

4 Unicyclic graphs

Let $\mathcal{U}_n = \Psi_{n,n}$ be the set of connected unicyclic graphs of order n. It should be noted that Theorem 4.1 as follows has been proved in Gutman and Wagner [2] by a different method. However, we provide a new and simpler method to derive the same result.

Theorem 4.1. For $G \in \mathcal{U}_n$ with $n \geq 3$, $ME(S_n^1) \leq ME(G)$ where the equality holds if and only if $G = S_n^1$.

Proof. Let G_0^* be the graph with the minimum matching energy in $\mathcal{U}_n$. By Corollaries 2.2 and 2.3, the cycle C_l of G_0^* has length $l = 3$ and all the cut-edges of G_0^* are pendant edges. Furthermore, by Corollary 2.6, we conclude that all the pendant edges of G_0^* are attached at the same vertex of C_3. Thus, $G_0^* = S_n^1$. ■

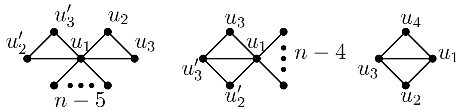
Furthermore, in Theorem 4.2, we use the methods proposed in this paper to obtain the graph with the minimum matching energy in $\mathcal{U}_n^+$, where $\mathcal{U}_n^+$ is the set of connected bipartite unicyclic graphs of order n.

Theorem 4.2. For $G \in \mathcal{U}_n^+$ with $n \geq 4$, $ME(C_4(n-4)) \leq ME(G)$ with the equality if and only if $G = C_4(n-4)$, where $C_4(n-4)$ is the graph obtained from C_4 by attaching $n-4$ pendant edges to a vertex of C_4.

Proof. Let G_0^+ be the graph with the minimum matching energy in $\mathcal{U}_n^+$. By the methods similar to Lemma 3.1 (i) and Corollary 2.2, all the cut-edges of G_0^+ are pendant edges and the induced cycle C_l of G_0^+ has length $l = 4$. Next, we prove that all the pendant edges of G_0^+ are attached at the same vertex of C_4. Let $C_4(n_1, n_2, n_3, n_4)$ be the unicyclic graph obtained from C_4 by attaching n_1, n_2, n_3, and n_4 pendant edges to u_1, u_2, u_3, and u_4, respectively, where $n_i \geq 0$ and $n_1 + n_2 + n_3 + n_4 = n - 4$. Note that the vertices of C_4 are labeled consecutively by u_1, u_2, u_3, and u_4. Therefore, by Corollary 2.6, we get $C_4(n_1, n_2, n_3, n_4) \preceq C_4(n_1, n_2 + n_4, n_3, 0) \preceq C_4(n_1 + n_3, n_2 + n_4, 0, 0) \preceq C_4(n_1 + n_2 + n_3 + n_4, 0, 0, 0)$. Thus, $G_0^+ = C_4(n-4)$. ■

5 Bicyclic graphs

Let $\mathcal{B}_n = \Psi_{n,n+1}$ be the set of connected bicyclic graphs of order n. For $G \in \mathcal{B}_n$, G has either two or three cycles. Let $\mathcal{B}_n = \mathcal{B}_n^1 \cup \mathcal{B}_n^2$, where $\mathcal{B}_n^1$ is the subset of $\mathcal{B}_n$ in which the



(a) $A_n^{3,3}$ (b) $E_n^{3,3}$ (c) $P(2,1,2)$

Figure 2: $A_n^{3,3}$, $E_n^{3,3}$, and $P(2,1,2)$

two cycles have no common edges, and $\mathcal{B}_n^2$ is the subset of $\mathcal{B}_n$ in which any two cycles have at least one common edge.

Let P_n be a path with n vertices. The vertices of P_n are labeled consecutively by $v_1, v_2, \cdots, v_n$.

We denote by $B(a,b,c)$ the bicyclic graph obtained by identifying a vertex of C_a with v_1 of P_c and identifying a vertex of C_b with v_c of P_c, where $a, b \geq 3$ and $c \geq 1$. Specifically, $B(a,b,1)$ is the bicyclic graph obtained by identifying u_1 of C_a with u_1' of C_b, where the vertices of C_b are labeled consecutively by $u_1', u_2', \cdots, u_b'$.

We denote by $P(l,s,t)$ the bicyclic graph obtained by identifying u_1 and u_{l+1} of C_{l+t} with v_1 and v_{s+1} of P_{s+1}, respectively, where $l, t \geq 2$, $s \geq 1$ and $l, t \geq s$. Namely, in $P(l,s,t)$, u_1 and u_{l+1} of C_{l+t} are connected by a unique path P_{s+1} of length s.

For a graph $G \in \mathcal{B}_n$, the base of G, denoted by $\hat{G}$, is the minimal connected bicyclic subgraph of G. Namely, $\hat{G}$ is the bicyclic graph containing no pendant vertex and G can be obtained from $\hat{G}$ by attaching trees to some vertices of $\hat{G}$. Obviously, $\hat{G} = B(a,b,c)$ if $G \in \mathcal{B}_n^1$ and $\hat{G} = P(l,s,t)$ if $G \in \mathcal{B}_n^2$.

Let $A_n^{3,3}$ (resp., $E_n^{3,3}$) be the graph obtained from $B(3,3,1)$ (resp., $P(2,1,2)$) by attaching $n-5$ (resp., $n-4$) pendant edges to the vertex with the maximum degree of $B(3,3,1)$ (resp., $P(2,1,2)$). $A_n^{3,3}$, $E_n^{3,3}$ and $P(2,1,2)$ are shown in Fig. 2.

Hereinafter, we denote by H_0^1 and H_0^2 the graphs with the minimum matching energies in $\mathcal{B}_n^1$ and in $\mathcal{B}_n^2$, respectively.

Lemma 5.1. (i). $H_0^1 = A_n^{3,3}$ for $n \geq 5$.

(ii) $H_0^2 = E_n^{3,3}$ for $n \geq 5$.

Proof. (i) By Corollaries 2.2 and 2.3, we get $\hat{H}_0^1 = B(3,3,1)$. By the methods similar to the proof for Lemma 3.1 (iii), we obtain that for any C_3 of H_0^1, only one vertex at C_3 is attached by a graph. Therefore, $H_0^1 = A_n^{3,3}$.

(ii) Let $\hat{H}_0^2 = P(l,s,t)$. By Lemma 2.1, we get $l = t = 2$ and $s = 1$. Thus, we have $\hat{H}_0^2 = P(2,1,2)$. Let $G \in \mathcal{B}_n^2$ with $\hat{G} = P(2,1,2)$ (as shown in Fig. 2(c)). Let e be u_1u_3 of $P(2,1,2)$. We have

$$m(G,k) = m(G - u_1u_3, k) + m(G - u_1 - u_3, k-1).$$

Since $G - u_1u_3 \in \mathcal{U}_n^+$, by the proof of Theorem 4.2, for all $k \geq 1$, $m(G - u_1u_3, k)$ reaches its minimum number of k-matchings as $G - u_1u_3 = C_4(n-4)$, where $n \geq 5$. Furthermore, $m(G - u_1 - u_3, k-1)$ reaches its minimum number of k-matchings as $G - u_1 - u_3$ is a union of isolated vertices. Therefore, we have $H_0^2 = E_n^{3,3}$. ■

Theorem 5.2. For $G \in \mathcal{B}_n$ and $n \geq 4$, $ME(E_n^{3,3}) \leq ME(G)$ with the equality if and only if $G = E_n^{3,3}$.

Proof. For $n = 4$, $\mathcal{B}_n$ has only one graph $E_4^{3,3}$. Let $n \geq 5$. Since $E_n^{3,3} = A_n^{3,3} - u_3u_2 + u_3u_3'$, by Corollary 2.8, we have $E_n^{3,3} \prec A_n^{3,3}$. Furthermore, by Lemma 5.1, we obtain $ME(E_n^{3,3}) \leq ME(G)$ for all $G \in \mathcal{B}_n$. ■

It should be noted that Theorem 5.2 has been proved in [10] using a different method, namely the induction on n. The method used in this paper is simpler.

Acknowledgement: The first author thanks the support by the National Natural Science Foundation of China under Grant No. 11001166 and the Shanghai Leading Academic Discipline Project under Project No. S30104.

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