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Effects on the Energy and Estrada Indices by Adding Edges Among Pendent Vertices

Oscar Rojo

Department of Mathematics, Universidad Católica del Norte, Antofagasta, Chile

orojo@ucn.cl

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Abstract

Let G be a simple undirected connected graph possessing a vertex v to which $s > 1$ pendent vertices are attached. Let H be a graph of order s. Let $G(H)$ be the graph obtained from G by adding the edges of H among the pendent vertices attached to v. Let $M(R)$ be the Laplacian matrix or the signless Laplacian matrix or the adjacency matrix of a graph R. If the all $1-$ vector of order s is an eigenvector of $M(H)$, it is proved that $M(G(H))$ is orthogonally similar to a 2×2 block diagonal matrix in which one of the blocks is a diagonal matrix. This result is used to study the effects on the Energy and Estrada indices when edges are added on the pendent vertices of a given graph.

1 Introduction

Let G be a simple undirected graph on n vertices. Let $D(G)$ be the diagonal matrix of order n whose $(i,i)-$entry is the degree of the $i-th$ vertex of G and let $A\left(G\right)$ be the adjacency matrix of G. The matrices $L(G) = D(G) - A(G)$ and $L^{+}\left(G\right) = D\left(G\right) + A\left(G\right)$ are the Laplacian and signless Laplacian matrix of G, respectively. The matrices $L(G)$ and $L^{+}(G)$ are both positive semidefinite and $(0, \mathbf{1}_n)$ is an eigenpair of $L\left(G\right)$ where $\mathbf{1}_n$ is the all-1 vector of size n. Fiedler [14] proved that G is a connected graph if and only if the second smallest eigenvalue of $L(G)$ is positive. This eigenvalue is called the algebraic connectivity of G and it is denoted by $a\left(G\right)$. Moreover, it is known that for any bipartite graph G, the characteristic polynomials of $L(G)$ and $L^{+}(G)$ coincide [3].

A pendent vertex is a vertex of degree 1. We mention below some already known results concerning the effects on some graph invariants by adding edges among the pendent vertices.

Theorem 1. *[25], Corollary 4.2. Let G be a connected graph on n vertices. Suppose that $v_1, v_2, \ldots, v_s$ are s pendent vertices of G adjacent to a common vertex v. Let $\widetilde{G}$ be a graph obtained from G by adding any t, $0 \leq t \leq \frac{s(s-1)}{2}$, edges among $v_1, v_2, \ldots, v_s$. If $a(G) \neq 1$ then $a(\widetilde{G}) = a(G)$.*

Theorem 2. *[15], Theorem 2.3. Let G be a connected graph on n vertices. Suppose that $v_1, v_2, \ldots, v_s$ are s pendent vertices of G adjacent to a common vertex v. Let $\widetilde{G}$ be a graph obtained from G by adding any t, $0 \leq t \leq \frac{s(s-1)}{2}$, edges among $v_1, v_2, \ldots, v_s$. Then the largest Laplacian eigenvalue of G is also the largest Laplacian eigenvalue of $\widetilde{G}$.*

Definition 1. *Let G be a connected graph of order n possessing a vertex v to which $s > 1$ pendent vertices are attached. Let H be a graph of order s. Then $G(H)$ denotes the graph obtained from G and H identifying the vertices of H with the pendent vertices attached to v.*

In [26], the above results as generalized as follows.

Theorem 3. *Let $G(H)$ as in Definition 1.*

(i) If $\mu \neq 0$ and $\mu \neq 1$ is a Laplacian eigenvalue of G then μ is a Laplacian eigenvalue of $G(H)$, and

(ii) if μ is a Laplacian eigenvalue of H, $\mu \neq 0$ or $\mu = 0$ with an eigenvector orthogonal to $\mathbf{1}_s$, then $1 + \mu$ is a Laplacian eigenvalue of $G(H)$.

Definition 2. *Let G be a connected graph of order n possessing vertices v_i, $1 \leq i \leq r$, to which $s_i > 1$ pendent vertices are attached. For $1 \leq i \leq r$, let H_i be a graph order s_i. Then $G(H_1, \ldots, H_r)$ denotes the graph obtained from G and the graphs $H_1, \ldots, H_r$ identifying the vertices of H_i with the s_i pendent vertices attached to v_i.*

Remark 1. *The graph $G(H_1, H_2, \ldots, H_r)$ can be constructed as follows:*

- *the graph $G_1 = G(H_1)$ is obtained from G and H_1 identifying the vertices of H_1 with the pendent vertices attached to v_1, and*
- *for $i = 2, \ldots, r$, the graph $G_i = G(H_1, \ldots, H_i)$ is obtained from $G_{i-1} = G(H_1, \ldots, H_{i-1})$ and H_i identifying the vertices of H_i with the pendent vertices attached to v_i.*

At this point, we recall that for a connected graph G of order n the Laplacian-energy-like invariant of G is

$$LEL(G) = \sum_{i=1}^{n-1} \sqrt{\mu_i(G)},$$

the Kirchhoff index of G is

$$Kf(G) = n \sum_{i=1}^{n-1} \frac{1}{\mu_i(G)}$$

and the Laplacian Estrada index of G is

$$LEE(G) = \sum_{i=1}^{n} e^{\mu_i(G)}$$

where

$$\mu_1(G) \geq \mu_2(G) \geq \ldots \geq \mu_{n-1}(G) \geq \mu_n(G) = 0$$

are the Laplacian eigenvalues of G.

Some results on $LEL(G)$ are given in [23], [27] and [4], on $Kf(G)$ in [28], [29], [32], [33] and [34], and on $LEE(G)$ in [13], [30] and [31].

The effects on the Laplacian-Energy-Like invariant and on the Kirchhoff index by adding edges among the pendent vertices were studied in [26] and the corresponding results are

Theorem 4. *Let $G(H_1, \ldots, H_r)$ as in Definition 2. For $1 \leq i \leq r$, let*

$$\mu_1(H_i) \geq \mu_2(H_i) \geq \ldots \geq \mu_{s_i-1}(H_i) \geq \mu_{s_i}(H_i) = 0$$

be the Laplacian eigenvalues of H_i. Then

$$LEL(G(H_1, \ldots, H_r)) - LEL(G) = \sum_{i=1}^{r} \sum_{j=1}^{s_i} \sqrt{1 + \mu_j(H_i)} - \sum_{i=1}^{r} s_i$$

and

$$Kf(G(H_1, \ldots, H_r)) - Kf(G) = n(\sum_{i=1}^{r} \sum_{j=1}^{s_i} \frac{1}{1 + \mu_j(H_i)} - \sum_{i=1}^{r} s_i).$$

We recall that the energy of G is

$$E(G) = \sum_{i=1}^{n} |\lambda_i(G)|$$

and the Estrada index of G is

$$EE(G) = \sum_{i=1}^{n} e^{\lambda_i(G)}$$

where

$$\lambda_1(G) \leq \lambda_2(G) \leq \ldots \leq \lambda_{n-1}(G) \leq \lambda_n(G)$$

are the eigenvalues of $A(G)$.

Some results on $EE(G)$ are in [6] - [12].

The notion of the energy of a graph was introduced by Gutman in 1978 as the sum of the absolute values of its adjacency eigenvalues, it is studied in Chemistry and used to approximate the total π-electron energy of a molecule [16, 17]. There are many contributions on the energy of a graph and still is a subject of research. A few of these contributions are [18]- [20]. More on the chemical background of graph energy can be found in Chapter 2 of the book Graph Energy by X. Li, Y. Shi and I. Gutman [22]. This book provides a comprehensive survey of all results and common proof methods on graph energies together with a complete reference section.

We recall now that the signless Laplacian Estrada index of G is defined as

$$SLEE(G) = \sum_{i=1}^{n} e^{\mu_i^+(G)}$$

where

$$\mu_1^+(G) \leq \mu_2^+(G) \leq \ldots \leq \mu_{n-1}^+(G) \leq \mu_n^+(G)$$

are the signless Laplacian eigenvalues of G. Results on $SLEE(G)$ can be found in [1], [2], [5] and [21].

Let $M(R)$ be the Laplacian matrix or the signless Laplacian or the adjacency matrix of a graph R.

In Section 2, we prove that if the all $1-$vector of order s is an eigenvector of $M(H)$ then $M(G(H))$ is orthogonally similar to a 2×2 block diagonal matrix in which one of the blocks is a diagonal matrix. In Section 3, we apply this result to study the effect on the Laplacian Estrada index. Section 4 is devoted to study the effects on the energy and on the Estrada index. Finally, in Section 5, we study the effect on the signless Laplacian Estrada index. In Sections 4 and 5, it is assumed that the edges added among the pendent vertices of G are the edges of regular graphs.

In this paper, the zero matrix and the identity matrix of the appropriate orders are denoted by 0 and I, respectively. Furthermore, I_m is the identity matrix of order m, $\det(A)$ and $tr(A)$ are the determinant and trace of a square matrix A, respectively, and A^T is the transpose of A.

2 A result on graphs constructed by adding edges on pendent vertices

Consider $G(H)$ as in Definition 1. Let $M(R)$ be the Laplacian matrix or the signless Laplacian or the adjacency matrix of a graph R. Throughout this paper

$$\sigma = \begin{cases} -1 & \text{if } M \text{ is the Laplacian matrix,} \\ 1 & \text{if } M \text{ is the signless Laplacian or the adjacency matrix} \end{cases}$$

and

$$\delta = \begin{cases} 0 & \text{if } M \text{ is the adjacency matrix} \\ 1 & \text{if } M \text{ is the Laplacian or the signless Laplacian matrix} \end{cases}$$

In this section, we assume that

$$\mu_1(M(H)), \mu_2(M(H)), \ldots, \mu_s(M(H))$$

are the eigenvalues of $M(H)$ and that

$$\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_{s-1}}, \mathbf{x_s} = \frac{1}{\sqrt{s}}\mathbf{1}_s$$

is an orthonormal basis of eigenvectors of $M(H)$ in which, for $i = 1, \ldots, s$,

$$M(H)\mathbf{x_i} = \mu_i(M(H))\mathbf{x_i}. \tag{1}$$

In particular

$$M(H)\mathbf{1}_s = \mu_s(M(H))\mathbf{1}_s. \tag{2}$$

We observe that (2) holds when $M(H) = L(H)$ and when $M(H) = L^+(G)$ or $M(H) = A(H)$ if H is a regular graph.

Let

$$X = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_{s-1} & \frac{1}{\sqrt{s}}\mathbf{1}_s \end{bmatrix}$$

where the columns of X are the eigenvectors of $M(H)$ as in (1).

Let

$$S = \begin{bmatrix} X & & \\ & 1 & \\ & & I_{n-s-1} \end{bmatrix}.$$

Clearly, X and S are both orthonormal matrices.

The graphs G and $G(H)$ have the same set of vertices. We recall that G is a graph possessing a vertex v to which s pendent vertices are attached. We label the vertices of G as follows: the labels $1, 2, \ldots, s$ are for the pendents attached to v, the label $s+1$ is for the vertex v and the labels $s+2, \ldots, n$ are for the remaining vertices of G. Let d_i be the degree of the vertex i of G. In particular, the degree of v is denoted by $d(v)$. For the given labeling, $M(G)$ and $M(G(H))$ become

$$M(G)=\begin{bmatrix} \delta I_s & \sigma \mathbf{1}_s & 0 \\ \sigma \mathbf{1}_s^T & \delta d(v) & \mathbf{a}^T \\ 0 & \mathbf{a} & B \end{bmatrix} \tag{3}$$

and

$$M(G(H))=\begin{bmatrix} \delta I_s+M(H) & \sigma \mathbf{1}_s & 0 \\ \sigma \mathbf{1}_s^T & \delta d(v) & \mathbf{a}^T \\ 0 & \mathbf{a} & B \end{bmatrix} \tag{4}$$

for some common column vector $\mathbf{a}$ of size $n-s-1$ and some common square matrix B of order $n-s-1$. The vector $\mathbf{a}$ and the matrix B are both independent of H. The diagonal entries of B are δd_i, $s+2 \le i \le n$. The components of $\mathbf{a}$ and the off-diagonal entries of B are σ if the corresponding vertices of G are adjacent and 0 otherwise.

Theorem 5. *Let $G(H)$ as in Definition 1. Then*

$$S^T M(G(H))S=\begin{bmatrix} \begin{bmatrix} \delta+\mu_1(M(H)) & & \\ & \ddots & \\ & & \delta+\mu_{s-1}(M(H)) \end{bmatrix} & \\ & \begin{bmatrix} \delta+\mu_s(M(H)) & \sigma\sqrt{s} & 0 \\ \sigma\sqrt{s} & \delta d(v) & \mathbf{a}^T \\ 0 & \mathbf{a} & B \end{bmatrix} \end{bmatrix} \tag{5}$$

and

$$S^T M(G)S=\begin{bmatrix} \begin{bmatrix} \delta & & \\ & \ddots & \\ & & \delta \end{bmatrix} & \\ & \begin{bmatrix} \delta & \sigma\sqrt{s} & 0 \\ \sigma\sqrt{s} & \delta d(v) & \mathbf{a}^T \\ 0 & \mathbf{a} & B \end{bmatrix} \end{bmatrix} \tag{6}$$

The vector $\mathbf{a}$ *and the matrix* B *are as above mentioned.*

Proof. We use (4) to compute

$$\begin{aligned} & S^T M(G(H))S \\ = & \begin{bmatrix} X^T & & \\ & 1 & \\ & & I_{n-s-1} \end{bmatrix} \begin{bmatrix} \delta I_s+M(H) & \sigma \mathbf{1}_s & 0 \\ \sigma \mathbf{1}_s^T & \delta d(v) & \mathbf{a}^T \\ 0 & \mathbf{a} & B \end{bmatrix} \begin{bmatrix} X & & \\ & 1 & \\ & & I_{n-s-1} \end{bmatrix} \\ = & \begin{bmatrix} \delta I_s+X^T M(H)X & \sigma X^T \mathbf{1}_s & 0 \\ \sigma \mathbf{1}_s^T X & \delta d(v) & \mathbf{a}^T \\ 0 & \mathbf{a} & B \end{bmatrix} \\ = & \begin{bmatrix} \begin{bmatrix} \delta+\mu_1(M(H)) & & & \\ & \ddots & & \\ & & \delta+\mu_{s-1}(M(H)) & \\ & & & \delta+\mu_s(M(H)) \end{bmatrix} & \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \sigma\sqrt{s} \end{bmatrix} & \\ \begin{bmatrix} 0 & \cdots & 0 & \sigma\sqrt{s} \end{bmatrix} & \delta d(v) & \mathbf{a}^T \\ 0 & \mathbf{a} & B \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} \begin{bmatrix} \delta + \mu_1(M(H)) & & \\ & \ddots & \\ & & \delta + \mu_{s-1}(M(H)) \end{bmatrix} & \\ & \begin{bmatrix} \delta + \mu_s(M(H)) & \sigma\sqrt{s} & 0 \\ \sigma\sqrt{s} & \delta d(v) & \mathbf{a}^T \\ 0 & \mathbf{a}^T & B \end{bmatrix} \end{bmatrix}.$$

Thus (5) is obtained. Similarly, the computation of the product $S^T M(G) S$ with $M(G)$ as in (3) yields to (6). ■

3 Application to the Laplacian Estrada index

Let $G(H)$ as in Definition 1. Here we consider $M(G) = L(G)$. Then $\delta = 1$ and $\sigma = -1$. Applying Theorem 5, we obtain

Theorem 6. *Let $G(H)$ as in Definition 1. Let*

$$\mu_1(H) \geq \mu_2(H) \geq \ldots \geq \mu_s(H) = 0$$

be the Laplacian eigenvalues of H. Then

$$S^T L(G(H)) S = \begin{bmatrix} \begin{bmatrix} 1 + \mu_1(H) & & \\ & \ddots & \\ & & 1 + \mu_{s-1}(H) \end{bmatrix} & \\ & \begin{bmatrix} 1 & -\sqrt{s} & 0 \\ -\sqrt{s} & d(v) & \mathbf{a}^T \\ 0 & \mathbf{a} & B \end{bmatrix} \end{bmatrix}$$

and

$$S^T L(G) S = \begin{bmatrix} \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} & \\ & \begin{bmatrix} 1 & -\sqrt{s} & 0 \\ -\sqrt{s} & d(v) & \mathbf{a}^T \\ 0 & \mathbf{a} & B \end{bmatrix} \end{bmatrix}$$

for some common column vector $\mathbf{a}$ of size $n - s - 1$ and some common square matrix B of order $n - s - 1$ both independent of H. The diagonal entries of B are the degrees of the vertices i, $s + 2 \leq i \leq n$. The components of $\mathbf{a}$ and the off-diagonal entries of B are -1 if the corresponding vertices of G are adjacent and 0 otherwise.

Corollary 1. *Let $G(H)$ as in Definition 1. Then*

$$\det(\lambda I - L(G(H))) = \lambda P_H(\lambda) R(\lambda) \tag{7}$$

and

$$\det(\lambda I - L(G)) = \lambda(\lambda-1)^{s-1}R(\lambda) \tag{8}$$

where

$$P_H(\lambda) = \prod_{i=1}^{s-1}(\lambda - (1+\mu_i(H))) \tag{9}$$

and $R(\lambda)$ is a polynomial of degree $n-s$ such that $R(0) \neq 0$.

Proof. From Theorem 6, the characteristic polynomial of $L(G(H))$ is

$$\det(\lambda I - L(G(H))) = P_H(\lambda)S(\lambda) \tag{10}$$

with $P_H(\lambda)$ is as in (9) and the characteristic polynomial of $L(G)$ is

$$\det(\lambda I - L(G)) = (\lambda-1)^{s-1}S(\lambda) \tag{11}$$

where $S(\lambda)$ is the characteristic polynomial of the matrix

$$\begin{bmatrix} 1 & -\sqrt{s} & 0 \\ -\sqrt{s} & d(v) & \mathbf{a}^T \\ 0 & \mathbf{a} & B \end{bmatrix}.$$

Since $G(H)$ and G are connected graphs, 0 is a simple eigenvalue of $L(G(H))$ and $L(G)$. Hence $S(\lambda) = \lambda R(\lambda)$ where $R(\lambda)$ is a polynomial of degree $n-s$ such that $R(0) \neq 0$. Replacing in (10) and (11), we obtain (7) and (8). ∎

Theorem 7. *Let $G(H)$ as in Definition 1. Then*

$$LEE(G(H)) - LEE(G) = (LEE(H) - s)e. \tag{12}$$

Proof. Let

$$\mu_1(H) \geq \mu_2(H) \geq \ldots \geq \mu_{s-1}(H) \geq \mu_s(H) = 0$$

be the Laplacian eigenvalues of H. As we already mentioned, the Laplacian Estrada index of a graph G is

$$LEE(G) = \textstyle\sum_{i=1}^{n} e^{\mu_i(G)}.$$

We apply Corollary 1. From (7) and (8), we obtain

$$LEE(G(H)) = 1 + \sum_{j=1}^{s-1} e^{1+\mu_j(H)} + \sum_{\mu:R(\mu)=0} e^{\mu}. \tag{13}$$

and

$$LEE(G) = 1 + (s-1)e + \sum_{\mu:R(\mu)=0} e^{\mu}. \tag{14}$$

Subtracting (14) from (13), we get

$$LEE(G(H)) - LEE(G) = \sum_{j=1}^{s} e^{1+\mu_j(H)} - se = (LEE(H) - s)e.$$

Thus (12) is obtained. ■

Theorem 8. *Let $G(H_1, \ldots, H_r)$ as in Definition 2. Then*

$$LEE(G(H_1, \ldots, H_r)) - LEE(G) = e \sum_{i=1}^{r} (LEE(H_i) - s_i).$$

Proof. We recall that $G_r = G(H_1, \ldots, H_r)$ can be constructed as in Remark 1. By repeated application of Theorem 7, we have

$$LEE(G_1) - LEE(G) = (LEE(H_1) - s_1)e,$$

$$LEE(G_2) - LEE(G_1) = (LEE(H_2) - s_2)e,$$

$$\vdots$$

$$LEE(G_{r-1}) - LEE(G_{r-2}) = (LEE(H_{r-1}) - s_{r-1})e$$

and

$$LEE(G_r) - LEE(G_{r-1}) = (LEE(H_r) - s_r)e.$$

Add these equalities together and the result follows. ■

4 Application to the energy and Estrada index

Let M be an $m \times n$ complex matrix. Let $q = \min\{m, n\}$. Let

$$\sigma_1(B) \geq \sigma_2(B) \geq ... \geq \sigma_q(B)$$

be the singular values of M. Nikiforov [24] defines the energy of M, denoted by $E(M)$, as

$$E(M) = \sum_{j=1}^{q} \sigma_j(B).$$

Since $A(G)$ is a real symmetric matrix, its singular values are the modulus of its eigenvalues. Then $E(G) = E(A(G))$.

At this point, we recall that given a natural number k such that $1 \leq k \leq n$, the Ky Fan $k-$ norm of a matrix X of order $n \times n$ is defined to be the sum of the k largest singular values of X, that is,

$$\|X\|_k = \sum_{i=1}^{k} \sigma_i(X).$$

In particular, $\|X\|_n = E(X)$.

Let $G(H)$ as in Definition 1. In this section, $M(G) = A(G)$. Then $\delta = 0$ and $\sigma = 1$. Applying Theorem 5, we obtain

Theorem 9. *Let H be a regular of degree h. Let $G(H)$ as in Definition 1. Let*

$$\lambda_1(H) \le \lambda_2(H) \le \ldots \le \lambda_s(H) = h$$

be the adjacency eigenvalues of H. Then

$$S^T A(G(H)) S = \begin{bmatrix} \begin{bmatrix} \lambda_1(H) & & \\ & \ddots & \\ & & \lambda_{s-1}(H) \end{bmatrix} & \\ & \begin{bmatrix} h & \sqrt{s} & 0 \\ \sqrt{s} & 0 & \mathbf{a}^T \\ 0 & \mathbf{a} & B \end{bmatrix} \end{bmatrix} \tag{15}$$

and

$$S^T A(G) S = \begin{bmatrix} \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix} & \\ & \begin{bmatrix} 0 & \sqrt{s} & 0 \\ \sqrt{s} & 0 & \mathbf{a}^T \\ 0 & \mathbf{a} & B \end{bmatrix} \end{bmatrix} \tag{16}$$

for some common column vector $\mathbf{a}$ of size $n-s-1$ and some common square matrix B of order $n-s-1$ both independent of H. The diagonal entries of B are 0. The components of $\mathbf{a}$ and the off-diagonal entries of B are 1 if the corresponding vertices of G are adjacent and 0 otherwise.

Theorem 10. *Let H be a regular graph of degree h. Let $G(H)$ as in Definition 1. Then*

$$E(G(H)) - E(G) \le E(H).$$

Proof. We apply Theorem 9. From (15) and (16), using the fact that the singular values are invariant under unitary transformations, we obtain

$$E(G(H)) = E(A(G(H))) = \sum_{i=1}^{s-1} |\lambda_i(H)| + E(C) \tag{17}$$

where

$$C = \begin{bmatrix} h & \sqrt{s} & 0 \\ \sqrt{s} & 0 & \mathbf{a}^T \\ 0 & \mathbf{a} & B \end{bmatrix}$$

and

$$E(G) = E(A(G)) = E(D)$$

where

$$D = \begin{bmatrix} 0 & \sqrt{s} & 0 \\ \sqrt{s} & 0 & \mathbf{a}^T \\ 0 & \mathbf{a} & B \end{bmatrix}.$$

We have

$$C = D + F$$

where

$$F = \begin{bmatrix} h & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence $E(C) = \|C\|_{n-s+1} \leq \|D\|_{n-s+1} + \|F\|_{n-s+1} = E(D) + h = E(G) + h$. We use this inequality in (17) to obtain

$$E(G(H)) - E(G) \leq \sum_{i=1}^{s-1} |\lambda_i(H)| + h = \sum_{i=1}^{s} |\lambda_i(H)| = E(H).$$

The proof is complete. ■

Theorem 11. *For $i = 1, \ldots, r$, let H_i be a regular graph of degree h_i. Let $G(H_1, \ldots, H_r)$ as in Definition 2. Then*

$$E(G(H_1, \ldots, H_r)) - E(G) \leq \sum_{i=1}^{r} E(H_i).$$

Proof. The result follows easily from Remark 1 and repeated application of Theorem 10. ■

We search now for the effect on the Estrada index. Previously, we recall some well known facts:

- If A is a square matrix then

$$e^A = \textstyle\sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

- For a graph G

$$EE(G) = tr(e^{A(G)})$$

- If A and B are similar matrices then $tr(A) = tr(B)$.

Theorem 12. *Let H be a regular graph of degree h. Let $G(H)$ as in Definition 1. Then*

$$EE(G(H)) - EE(G) \geq EE(H) - (s-1). \tag{18}$$

Proof. Let C, D and F be as in the proof of Theorem 10. From Theorem 9, we get

$$EE(G(H)) = \sum_{i=1}^{s-1} e^{\lambda_i(H)} + tr(e^C) \tag{19}$$

and

$$EE(G) = s - 1 + tr(e^D). \tag{20}$$

From the series-expansion of e^C, we have

$$e^C = \sum_{k=0}^{\infty} \frac{1}{k!} C^k = \sum_{k=0}^{\infty} \frac{1}{k!}(D+F)^k = \sum_{k=0}^{\infty} \frac{1}{k!}(D^k + \ldots + F^k).$$

Since D and F are nonnegative matrices, it follows that

$$tr(e^C) \geq tr(\sum_{k=0}^{\infty} \frac{1}{k!} D^k) + tr(\sum_{k=0}^{\infty} \frac{1}{k!} F^k).$$

Hence

$$tr(e^C) \geq tr(e^D) + \sum_{k=0}^{\infty} \frac{1}{k!} h^k = tr(e^D) + e^h.$$

Using this inequality in (19), we get

$$EE(G(H)) \geq \sum_{i=1}^{s-1} e^{\lambda_i(H)} + tr(e^D) + e^h = EE(H) + tr(e^D). \tag{21}$$

Finally, from (20) and (21), the inequality (18) follows. ■

From Remark 1 and repeated application of Theorem 12, one can prove

Theorem 13. *For $i = 1, \ldots, r$, let H_i be a regular graph of degree h_i. Let $G(H_1, \ldots, H_r)$ as in Definition 2. Then*

$$EE(G(H_1, \ldots, H_r)) - EE(G) \geq \sum_{i=1}^{r} (EE(H_i) - s_i) + r.$$

5 Application to the signless Laplacian Estrada index

Let $G(H)$ as in Definition 1. In this section, $M(G) = L^+(G)$. Then $\delta = 1$ and $\sigma = 1$. Applying Theorem 5, we obtain

Theorem 14. *Let H be a regular of degree h. Let $G(H)$ as in Definition 1. Then*

$$S^T L^+(G(H))S = \begin{bmatrix} \begin{bmatrix} 1+\mu_1^+(H) & & \\ & \ddots & \\ & & 1+\mu_{s-1}^+(H) \end{bmatrix} & \\ & \begin{bmatrix} 1+2h & \sqrt{s} & 0 \\ \sqrt{s} & d(v) & \mathbf{a}^T \\ 0 & \mathbf{a} & B \end{bmatrix} \end{bmatrix}$$

and

$$S^T L^+(G)S = \begin{bmatrix} \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} & \\ & \begin{bmatrix} 1 & \sqrt{s} & 0 \\ \sqrt{s} & d(v) & \mathbf{a}^T \\ 0 & \mathbf{a} & B \end{bmatrix} \end{bmatrix}$$

for some common column vector $\mathbf{a}$ of size $n-s-1$ and some common square matrix B of order $n-s-1$ both independent of H. The diagonal entries of B are the degrees of the vertices i, $s+2 \leq i \leq n$. The components of $\mathbf{a}$ and the off-diagonal entries of B are 1 if the corresponding vertices of G are adjacent and 0 otherwise.

Theorem 15. *Let H be a regular graph of degree h. Let $G(H)$ as in Definition 1. Then*

$$SLEE(G(H)) - SLEE(G) \geq eSLEE(H) - e^{2h}(e-1) - (s-1)e.$$

Proof. From Theorem 14, we get

$$SLEE(G(H)) = \sum_{i=1}^{s-1} e^{1+\mu_i^+(H)} + tr(e^C) \tag{22}$$

and

$$SLEE(G) = (s-1)e + tr(e^D) \tag{23}$$

where

$$C = \begin{bmatrix} 1+2h & \sqrt{s} & 0 \\ \sqrt{s} & d(v) & \mathbf{a}^T \\ 0 & \mathbf{a} & B \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1 & \sqrt{s} & 0 \\ \sqrt{s} & d(v) & \mathbf{a}^T \\ 0 & \mathbf{a} & B \end{bmatrix}$$

We have $C = D + F$ where $F = \begin{bmatrix} 2h & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. From the series-expansion of e^C and using the fact that D and F are nonnegative matrices, we get

$$tr(e^C) \geq tr(\sum_{k=0}^{\infty} \frac{1}{k!} D^k) + tr(\sum_{k=0}^{\infty} \frac{1}{k!} F^k).$$

Hence

$$tr(e^C) \geq tr(e^D) + \sum_{k=0}^{\infty} \frac{1}{k!}(2h)^k = tr(e^D) + e^{2h}.$$

Using this inequality in (22), we get

$$SLEE(G(H)) \geq \sum_{i=1}^{s-1} e^{1+\mu_i^+(H)} + tr(e^D) + e^{2h}.$$

Therefore

$$SLEE(G(H)) \geq e \sum_{i=1}^{s} e^{\mu_i^+(H)} - e^{1+2h} + tr(e^D) + e^{2h}.$$

Then

$$SLEE(G(H)) \geq eSLEE(H) - e^{2h}(e-1) + tr(e^D). \tag{24}$$

Finally, from (24) and (23), Theorem (15) follows. ■

From Remark 1 and repeated application of Theorem 15, one can prove

Theorem 16. *For $i = 1, \ldots, r$, let H_i be a regular graph of degree h_i. Let $G(H_1, \ldots, H_r)$ as in Definition 2. Then*

$$SLEE(G(H_1, \ldots, H_r)) - SLEE(G) \geq e \sum_{i=1}^{r} SLEE(H_i) - (e-1) \sum_{i=1}^{r} e^{2h_i} - e \sum_{i=1}^{r} (s_i - 1).$$

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