Clar Sets and Maximum Forcing Numbers of Hexagonal Systems *

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Abstract

Let $H$ be a hexagonal system with a perfect matching. Xu et al. discovered that the maximum forcing number of $H$ equals its Clar number. In this article we obtain a result: for any resonant set $K$ of a peri-condensed hexagonal system $H$ consisting of disjoint hexagons not meeting the boundary of $H$, if the subgraph obtained from $H$ by deleting $K$ and the boundary of $H$ has a perfect matching or is empty, then the Clar number of $H$ is at least $|K| + 2$. This fact improves the previous corresponding result due to Zheng and Chen. Based on the result, we prove that for each perfect matching $M$ of $H$ with the maximum forcing number, there exists a Clar set consisting of disjoint $M$-alternating hexagons of $H$.

1 Introduction

A hexagonal system, also called benzenoid system, is a 2-connected finite plane graph whose every interior face is bounded by a regular hexagon of side length one [17]. It can also be formed by a cycle with its interior in the infinite hexagonal lattice on the plane (graphene) [4]. A perfect matching of a hexagonal system $H$ is a set of disjoint edges covering all vertices of $H$. This concept coincides with that of a Kekulé structure in organic chemistry. Since a hexagonal system with at least one perfect matching may be viewed as the carbon-skeleton of a benzenoid hydrocarbon molecule, various topological

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properties of hexagonal systems have been extensively studied. The interested reader may refer to [2–4] with references therein.

A basic concept of Clar’s aromatic sextet theory is that of Clar number, which can measure the stability of polycyclic benzenoid hydrocarbons [2]. According to Clar’s theory, within a series of isometric benzenoid hydrocarbons, the one with larger Clar number is more stable [2].

Let $M$ be a perfect matching of a graph $G$. A cycle $C$ (resp. a path $P$) of $G$ is said to be $M$-alternating if the edges of $C$ (resp. $P$) appear alternately in and off $M$. For a subgraph $F$ of $G$, let $G - F$ denote the graph obtained from $G$ by deleting all the vertices of $F$ together with their incident edges.

Let $H$ be a hexagonal system with a perfect matching. A set $K$ of disjoint hexagons of $H$ is called a resonant set (or cover) if there is a perfect matching $M$ of $H$ such that all hexagons in $K$ are $M$-alternating. It is obvious that $K$ is a resonant set of $H$ if and only if $H - K$ either has a perfect matching or is an empty graph. In particular, $\emptyset$ is regarded as a perfect matching of an empty graph (no vertices). A resonant set is maximum if its cardinality is maximum. A maximum resonant set of $H$ is also called a Clar set or Clar formula, and its size is called the Clar number of $H$, denoted by $Cl(H)$. For the relevant researches on the Clar number and Clar formula, please see [5, 7, 9, 12, 18, 19, 22].

In 1985, Zheng and Chen [26] gave an important property for a maximum resonant set of a hexagonal system as follows.

**Theorem 1.1.** [26] Let $H$ be a hexagonal system and $K$ a maximum resonant set of $H$. Then $H - K$ has a unique perfect matching.

The proof of Theorem 1.1 is based on the following Lemma 1.2.

**Lemma 1.2.** [26] Let $H$ be a peri-condensed hexagonal system, $K$ a resonant set of internal hexagons and $\partial(H)$ the boundary of the exterior face of $H$. If $H - K - \partial(H)$ has a perfect matching, then $K$ is not a maximum resonant set.

A hexagonal system $H$ is said to be fully benzenoid if a maximum resonant set of $H$ includes all vertices. Gutman and Salem showed [6] that a fully benzenoid has a unique maximum resonant set.

The innate degree of freedom of a Kekulé structure was defined by Randić and Klein [13] as the minimum number of double bonds which simultaneously belong to the given
kekulé structure and to no other one, nowadays it is named “forcing number” by Harary et al. [10].

Let $M$ be a perfect matching of a graph $G$. A forcing set $S$ of $M$ is a subset of $M$ such that $S$ is contained in no other perfect matchings of $G$. The forcing number of $M$, denoted by $f(G, M)$, is the smallest cardinality over all forcing sets of $M$. The maximum (resp. minimum) forcing number of $G$ is the maximum (resp. minimum) value of forcing numbers of all perfect matchings of $G$, denoted by $F(G)$ (resp. $f(G)$). For the relevant researches on the matching forcing problem, we refer to [1,11,23–25].

For planar bipartite graphs, Pachter and Kim revealed a minimax fact that connects the forcing number of a perfect matching and its alternating cycles as follows.

**Theorem 1.3.** [16] Let $M$ be a perfect matching of a plane bipartite graph $G$. Then $f(G, M) = c(M)$, where $c(M)$ is the maximum number of disjoint $M$-alternating cycles of $G$.

For a hexagonal system $H$ with a perfect matching $M$, let $h(M)$ denote the maximum number of disjoint $M$-alternating hexagons of $H$. Theorem 1.3 implies $f(H, M) = c(M) \geq h(M)$. Second equality does not hold alway. Let us see an example in Fig. 1. The bold edges of Coronene form a perfect matching $M'$ whose forcing number equals 2, but the graph has only one $M'$-alternating hexagon. However, Xu et al. [20] obtained the following result by finding a perfect matching $M$ of $H$ so that $F(H) = f(H, M) = h(M)$.

**Theorem 1.4.** [20] Let $H$ be a hexagonal system with perfect matchings. Then $Cl(H) = F(H)$.

![Figure 1: Coronene.](image-url)

In this article, we show that for every perfect matching $M$ of a hexagonal system $H$ with the maximum forcing number, i.e. $F(H) = f(H, M)$, there exist $F(H)$ disjoint $M$-alternating hexagons in $H$. That is, $f(H, M) = h(M)$. To prove this, we mainly improve
Lemma 1.2 to obtain Lemma 2.1 in Section 2. Based on this crucial lemma, in Section 3 we describe clearly structure properties for a maximum set of disjoint $M$-alternating cycles of $H$ for any perfect matchings $M$ with the maximum forcing number, then we give a proof of this main result.

2 A crucial lemma

In this section, all hexagonal systems considered are placed in the plane so that an edge-direction is vertical. A peak (resp. valley) of a hexagonal system is a vertex whose neighbors are below (resp. above) it. For convenience, the vertices of a hexagonal system are colored with white and black such that any pair of adjacent vertices receive different colors and the peaks are black.

Let $H$ be a hexagonal system. The boundary of $H$ means the boundary of the infinite face, denoted by $\partial(H)$. An edge on the boundary of $H$ is a boundary edge. A hexagon of $H$ is called an external hexagon if it contains a boundary edge and an internal hexagon otherwise. $H$ is said to be cata-condensed if all vertices lie on its boundary and pericondensed otherwise.

We state a crucial lemma as follows.

**Lemma 2.1.** Let $H$ be a pericondensed hexagonal system and $K$ a resonant set consisting of internal hexagons of $H$. Suppose $H - K - \partial(H)$ has a perfect matching or is an empty graph. Then $\text{Cl}(H) \geq |K| + 2$.

In order to prove the lemma, we need some further terminology and a known result.

Let $M$ be a perfect matching of a hexagonal system $H$. An edge of $H$ is called an $M$-double edge if it belongs to $M$ and an $M$-single edge otherwise. An $M$-alternating cycle $C$ of $H$ is said to be proper if each edge of $C$ in $M$ goes from white end-vertex to black end-vertex along the clockwise direction of $C$.

The symmetric difference of two finite sets $A$ and $B$ is defined as $A \oplus B := (A \cup B) - (A \cap B)$. Given a perfect matching $M$ of a hexagonal system $H$. If $C$ is an $M$-alternating cycle (or hexagon) of $H$, then the symmetric difference $M \oplus C$ is another perfect matching of $H$ and $C$ is an $(M \oplus C)$-alternating cycle of $H$. Here $C$ may be viewed as its edge-set. Let $P$ be a set of some hexagons of $H$ and let $F$ be a subgraph of $H$. The set of the common hexagons of $P$ and $F$ is denoted by $P \cap F$. 
A hexagonal system $H$ is called a *linear chain* if the centers of all hexagons lie on a straight line. Zhang et al. obtained the following result in [21, Theorem 4].

**Theorem 2.2.** [21] A hexagonal system $H$ has $Cl(H) = 1$ if and only if $H$ is a linear chain.

For a cycle $C$ of a hexagonal system $H$, let $I[C]$ denote the subgraph of $H$ formed by $C$ together with its interior.

**Proof of Lemma 2.1.** By the assumption, we can choose a cycle $C$ of $H$ satisfying that

(i) the graph $I[C]$ is a peri-condensed hexagonal system, and

(ii) $C$ is disjoint with each member of $K$ and $H - K - C$ has a perfect matching, and $I[C]$ contains as few hexagons as possible subject to (i) and (ii). Set $H' := I[C]$ and $K' := K \cap H'$.

**Claim 1.** For any resonant set $K_0$ of $H'$, $K_0 \cup (K \setminus K')$ is a resonant set of $H$.

*Proof.* Since $H - K - C$ has a perfect matching, $H$ has a perfect matching $M_0$ such that each member in $K \cup \{C\}$ is $M_0$-alternating. So the restriction of $M_0$ on $H - H'$ is a perfect matching of $H - H'$, denoted by $M_c$. Let $M'_0$ be a perfect matching of $H'$ such that each member in $K_0$ is $M'_0$-alternating. Let $M' := M'_0 \cup M_c$. Then $M'$ is a perfect matching of $H$ such that each member in $K_0 \cup (K \setminus K')$ is an $M'$-alternating hexagon. ■

From Claim 1 it suffices to prove that $Cl(H') \geq |K'| + 2$. If $K' = \emptyset$, by Theorem 2.2 we have that $Cl(H') \geq |K'| + 2$. From now on suppose that $K' \neq \emptyset$. Without loss of generality, let $M$ be a perfect matching of $H'$ such that the boundary $C$ of $H'$ and each member in $K'$ are proper $M$-alternating cycles. We have the following claim.

**Claim 2.** $H'$ has no external hexagons that are proper $M$-alternating.

*Proof.* Suppose to the contrary that an external hexagon $h$ of $H'$ is proper $M$-alternating. Then $M \oplus h$ is a perfect matching of $H'$, and each component of $C \oplus h$ is a proper $(M \oplus h)$-alternating cycle. Since any two proper $M$-alternating hexagons of $H'$ are disjoint, $h$ is disjoint with each member of $K'$. Since $K' \neq \emptyset$, $C \oplus h$ has a component as a cycle $C'$ which satisfies the above conditions (i) and (ii). But $I[C']$ has fewer hexagons than $I[C]$, contradicting the choice for $C$. Hence Claim 2 holds. ■
Along the boundary $C$ of $H'$, we will find two substructures of $H'$ in its left-top corner and left-bottom corner as Figs. 3 and 4, respectively, as follows.

A $b$-chain of hexagonal system $H'$ is a maximal horizontal linear chain consisting of the consecutive external hexagons when traversing (counter)clockwise the boundary $\partial(H')$. A $b$-chain is called high (resp. low) if all hexagons adjacent to it are below (resp. above) it. For example, in Fig. 2 $D_0, D_1, D_2, G_1, G_2, \ldots, G_9, G'_1, D_5, D_6$ and $D_7$ are $b$-chains. In particular, $D_0, D_1, D_2$ and $G_1$ are high $b$-chains, while $G'_1, D_5$ and $D_6$ are low $b$-chains. But $G_2, G_3, \ldots, G_9$ and $D_7$ are neither high nor low $b$-chains.

Choose a high $b$-chain and a low $b$-chain of $H'$. They are distinct. Otherwise $H'$ itself is a linear chain, contradicting that $H'$ is peri-condensed. From the high $b$-chain to the low $b$-chain along the boundary $\partial(H')$ counterclockwise, we pass through a sequence of consecutive $b$-chains. In this process, let $G_1$ be the last high $b$-chain and let $G'_1$ be the first low $b$-chain after $G_1$. Clearly, there is no other high $b$-chain and low $b$-chain between $G_1$ and $G'_1$. That is, those $b$-chains between $G_1$ and $G'_1$ descend monotonously.

From high $b$-chain $G_1$ we have a sequence of consecutive $b$-chains $G_1, G_2, \ldots, G_m$ with the following properties: (1) for each $1 \leq i < m$, $G_{i+1}$ is next to $G_i$, and the left end hexagon of $G_{i+1}$ lies on the lower left side of $G_i$, (2) either $G_m$ is just the low $b$-chain $G'_1$ or $G_{m+1}$ is the $b$-chain next to $G_m$ such that $G_{m+1}$ has no hexagon lies on the lower left side of $G_m$. Let $G$ be a hexagonal chain of $H'$ consisting of $b$-chains $G_1, G_2, \ldots, G_m$. Then $G$ is a ladder-shape hexagonal chain.

Similarly, from low $b$-chain $G'_1$ we have a sequence of consecutive $b$-chains $G'_1, G'_2, \ldots, G'_s$.
with the following properties: (1) for each $1 \leq j < s$, $G'_j$ is next to $G'_{j+1}$, and the left end hexagon of $G'_{j+1}$ lies on the higher left side of $G'_j$, (2) either $G'_s$ is just the high b-chain $G_1$ or $G'_s$ is next to the b-chain $G'_{s+1}$ such that $G'_{s+1}$ has no hexagon lies on the higher left side of $G'_m$. Let $G'$ be a hexagonal chain of $H'$ consisting of b-chains $G'_1, G'_2, \ldots, G'_s$.

For example, given a high b-chain $D_1$ and a low b-chain $D_3$ in Fig. 2, we can get two required hexagonal chains $G = G_1 \cup G_2 \cup G_3 \cup G_4$ and $G' = G_9 \cup G'_1$.

**Claim 3.** Either $G$ and $G'$ are disjoint or the last b-chain $G_m$ in $G$ coincides with the first b-chain $G'_s$ in $G'$.

To analyze the substructures $G$ and $G'$ of $H'$, as [26] we label the hexagons of $G$ and some edges as follows (see Fig. 3): let $S_{i,j}$, $1 \leq i \leq m$ and $1 \leq j \leq n(i)$, be the hexagons of b-chain $G_i$ as Fig. 3, neither $A$ nor $A'$ is contained in $H'$. Denote by $e_{i,j}$ be the boundary edge of $H'$ which is parallel to $e_{1,1}$ and belongs to $S_{i,j}$, $1 \leq i \leq m$ and $1 \leq j \leq n(i)$, and denote the other boundary edges in $S_{1,1}$ and $S_{m,n(m)}$ by $a, a', e_0, e'_0$ respectively, as shown in Fig. 3.

Since the boundary $C$ of $H'$ is a proper $M$-alternating cycle, all the edges $e_0, e'_0, e_{i,j}$, $1 \leq i \leq m$, $1 \leq j \leq n(i)$, are $M$-double edges. So we can draw a ladder-shape broke line segment $L_1 = P_0P_1 \cdots P_{q+1}(q \geq 1)$ satisfying the following conditions.

**A1** The endpoints $P_0$ and $P_{q+1}$ of $L_1$ are the midpoints of the edges $a$ and $a'$, respectively. $P_i$ ($1 \leq i \leq q$) is the center of a hexagon $S_i$ of $H'$, $P_iP_{i+1}$ ($0 \leq i \leq q$) is orthogonal to one of the three edge directions, and $P_{i+1}$ ($0 \leq i \leq q$) lies on the lower left side or the left side of $P_i$ according as $i$ is even or odd (see Fig. 3). $L_1$ only passes through hexagons of $H'$. Clearly, the graph consisting of the hexagons intersected by $L_1$ is a hexagonal chain, denoted by $H_1$.

**A2** All the edges intersected by $L_1$ are $M$-single edges, all the $M$-double edges which are located in the region above $L_1$ are parallel to $e_{1,1}$ (see Fig. 3).

Note that there exists such a broke line segment such that it only passes through hexagons $S_{i,j}$, $1 \leq i \leq m$ and $1 \leq j \leq n(i)$. Among all those broke line segments, we can select one, also denoted by $L_1$, such that there are the maximum number of $M$-double edges above it.

Symmetrically we treat substructure $G'$ of $H'$ as follows. Let $T_{i,j}$, $1 \leq i \leq s$ and $1 \leq j \leq t(i)$, be the hexagons of b-chain $G'_i$, neither hexagon $B$ nor hexagon $B'$ is
Figure 3: The hexagonal chain $G$ on the left-top corner of $H'$ (bold edges are $M$-double edges, $m = 6$, $n(1)=3$, $n(2)=1$, $n(3)=3$, $n(4)=2$, $n(5)=2$. And $A, A' \notin H'$.)

contained in $H'$ as Fig. 4. Let $f_{k,\ell}$, $1 \leq k \leq s$ and $1 \leq \ell \leq t(k)$, be a series of boundary edges on this structure as indicated in Fig. 4. Since the boundary of $H'$ is a proper $M$-alternating cycle, we can see that all the edges $f_0, f'_0, f_{k,\ell}$, $1 \leq k \leq s$ and $1 \leq \ell \leq t(k)$, are $M$-double edges (see Fig. 4).

Figure 4: The hexagonal chain $G'$ on the left-bottom corner of $H'$ (bold edges are $M$-double edges, $s = 4$, $t(1)=3$, $t(2)=1$, $t(3)=3$, $t(4)=1$. And $B, B' \notin H'$.)

Like $L_1$, we also draw a ladder-shape broke line segment $L_2 = Q_0Q_1 \cdots Q_{r+1}(r \geq 1)$ as indicated in Fig. 4 so that the part below $L_2$ has as many $M$-double edges parallel to $f_{1,1}$ as possible. Let $Q_i$ ($1 \leq i \leq r$) be the center of a hexagon $T_i$ of $H'$. Let $H_2$ be the hexagonal chain consisting of the hexagons intersected by $L_2$.

Clearly, both $L_1$ and $L_2$ have an odd number of turning points. We now have the
following claim.

**Claim 4.** The boundary of $H_1$ (resp. $H_2$) is a proper $M$-alternating cycle and $m \geq 2$ (resp. $s \geq 2$).

**Proof.** We only consider $H_1$ (the other case is almost the same). Let $d_i$ be the edge of $S_{1,i}$ opposite to $e_{1,i}$, $1 \leq i \leq n(1)$ (see Fig. 3). By Claim 2, $S_{1,1}$ is not an $M$-alternating hexagon. It implies that all edges $d_2, \ldots, d_{n(1)}$ are $M$-double edges. Hence, $S_{2,1}$ is a hexagon of $H'$ and $m \geq 2$.

Let $P_1$ be the path induced by those vertices of $H_1$ which are just upon $L_1$. By the choice of $L_1$, we can see that $P_1$ is an $M$-alternating path with two end edges in $M$. Let $P_2$ be the path induced by those vertices of $H_1$ which are just below $L_1$. It suffices to show that $P_2$ is also an $M$-alternating path with two end edges in $M$.

Let $w_1 (= e'_0), w_2, \ldots, w_{\ell_2}$ be a series of parallel edges on the bottom of $H_1$ and let $h_1 (= e_0), h_2, \ldots, h_{\ell_1}$ be a series of vertical edges of $H_1$ on the right of $P_0P_1$ (see Fig. 5).

For $q = 1$, by the condition (A2) and $\{e_0, e'_0\} \subseteq M$, it follows that $h_1, h_2, \ldots, h_{\ell_1}$ (resp. $w_1, w_2, \ldots, w_{\ell_2}$) are forced by $e_0$ (resp. $e'_0$) in turn and thus belong to $M$ (see Fig. 5(a)). Therefore, $P_2$ is an $M$-alternating path with two end edges in $M$.

![Figure 5: Illustration for Claim 4 in the proof of Lemma 2.1.](image-url)

Let $q \geq 3$. For even $i$, $2 \leq i \leq q - 1$, let $e''_i$ be the slant edge of $S_i$ below $L_1$. Let $e_i$ and $e'_i$ be the two edges of $H'$ which are adjacent to $e''_i$ and below $L_1$ (see Fig. 6(a)). Clearly, $e_i$ is parallel to $e_0$, and $e'_i$ is parallel to $e'_0$. We assert that $e''_i \notin M$. Otherwise, $e''_i$ is an $M$-double edge. Since $C$ is a proper $M$-alternating cycle, $e''_i$ does not lie on the boundary
\(C\) of \(H'\). Thus \(S'_1\) is a hexagon of \(H'\) (see Fig. 6(b)). Moreover, we can switch from \(L_1\) to a new brok line segment \(L'_1\) which passes through \(S'_1\) and satisfies the conditions (A1–A2) (see Fig. 6(b)). But the part above \(L'_1\) has more \(M\)-double edges than above \(L_1\), contradicting the choice for \(L_1\). Thus the assertion is true. From condition (A2), we can see that \(\{e_0, e'_0, e_2, e'_2, \ldots, e_{q-1}, e'_{q-1}\} \subseteq M\). It follows that \(P_2\) is an \(M\)-alternating path with two end edges in \(M\) (see Fig. 5(b)).

For odd \(i\), by Claim 4 \(S_i\) (1 \(\leq\) \(i\) \(\leq\) \(q\)) and \(T_i\) (1 \(\leq\) \(i\) \(\leq\) \(r\)) are all proper \(M\)-alternating hexagons, and the other hexagons of \(H_1\) and \(H_2\) are not \(M\)-alternating. For convenience, let \(S_0 := S_{1,1}, S_{q+1} := S_{m,n(m)}, T_0 := T_{1,1}\) and \(T_{r+1} := T_{s,t(s)}\). By Claim 2, we have that \(S_0 \neq S_1, S_{q+1} \neq S_q, T_0 \neq T_1\) and \(T_{r+1} \neq T_r\). Further, by Claim 4 we can see that each hexagon in \(K'\) either belongs to \(H_1 \cup H_2\) or is disjoint with \(H_1 \cup H_2\).

Let \(K_1 := \{S_0, S_2, \ldots, S_{q+1}\}\) and \(K_2 := \{T_0, T_2, \ldots, T_{r+1}\}\). To complete the proof of the lemma, there are two cases to be considered.

**Case 1.** \(H_1\) and \(H_2\) are disjoint (see Figs. 3 and 4).

It is straightforward to verify that \(H_i - K_i\) has a perfect matching, \(i = 1, 2\), so \(K_i\) is a resonant set of \(H_i\) and \(|K_i| \geq |H_i \cap K'| + 1\).

Let \(K'' := (K_1 \cup K_2) \cup (K' - K' \cap H_1 - K' \cap H_2)\). Similar to the proof of Claim 1, we have that \(K''\) is a resonant set of \(H'\) and \(|K''| \geq |K'| + 2\). Thus \(\text{Cl}(H') \geq |K'| + 2\).

**Case 2.** \(H_1\) intersects \(H_2\).

By Claim 3 the last b-chain \(G_m\) in \(G\) coincides with the first b-chain \(G'_s\) in \(G'\). Hence \(S_{q+1} = T_{r+1}\). By Claim 4 both boundaries of \(H_1\) and \(H_2\) are proper \(M\)-alternating cycles. It follows that only segment \(P_qP_{q+1}\) of \(L_1\) is identical to segment \(Q_qQ_{r+1}\) of \(L_2\). Hence \(H_1 \cup H_2\) is a cata-condensed hexagonal system with exactly one branch hexagon \(S_q (= T_r)\) as Fig. 7, and its boundary is also a proper \(M\)-alternating cycle. So \(H_1\) and \(H_2\) have exactly one common \(M\)-alternating cycle. We also can see that \(K_1 \cup K_2\) is a resonant set of \(H_1 \cup H_2\), and \(|K_1 \cup K_2| \geq |K' \cap (H_1 \cup H_2)| + 2\). Let \(K'' := (K_1 \cup K_2) \cup (K' - K' \cap (H_1 \cup H_2))\). By Claim 1, we have that \(K''\) is a resonant set of \(H'\) and \(|K''| \geq |K'| + 2\). Thus \(\text{Cl}(H') \geq |K'| + 2\).

Now the entire proof of the lemma is complete. ■
3 Main results

We now state our main result as follows.

**Theorem 3.1.** Let $H$ be a hexagonal system with a perfect matching. For every perfect matching $M$ of $H$ such that $f(H, M) = F(H)$, there exist $F(H)$ disjoint $M$-alternating hexagons of $H$. 
By Theorem 1.3, there are $F(H)$ disjoint $M$-alternating cycles of $H$. It is well known that each $M$-alternating cycle of $H$ has an $M$-alternating hexagon in its interior [24]. In order to prove the above theorem, we only need to prove the following lemma.

Let $C$ be a set of disjoint cycles of a hexagonal system $H$. A member of $C$ is called *minimal* if it contains no other members of $C$ in its interior.

**Lemma 3.2.** Let $H$ be a hexagonal system. Let $M$ be a perfect matching of $H$ with the maximum forcing number and let $A$ be a maximum set of disjoint $M$-alternating cycles of $H$. Then for any two members in $A$ their interiors are disjoint, and for any $C \in A$, $I[C]$ is a linear chain.

**Proof.** Let $n := F(H) = f(H, M)$. By Theorem 1.3, $n = |A|$. Suppose to the contrary that there exist two cycles in $A$ so that their interiors have a containment relation. Then $A$ has a non-minimal member $C_0$ and its interior contains only minimal members of $A$.

Let $A_0$ denote the set of minimal members of $A$ whose interiors are contained in the interior of $C_0$. Then the restriction of $M$ on $I[C_0]$ is also a perfect matching of $I[C_0]$, denoted by $M_c$. Note that each $M$-alternating cycle has an $M$-alternating hexagon in its interior [24]. Then each cycle in $A_0$ can be replaced by an $M$-alternating hexagon, the set of these hexagons is a resonant set of $I[C_0]$, denoted by $K$. Clearly, $K$ is disjoint with $C_0$, $|K| = A_0$ and $I[C_0] - C_0 - K$ has a perfect matching. By Lemma 2.1, $I[C_0]$ has a resonant set $S$ such that $|S| \geq |K| + 2$. Let $M_0$ be a perfect matching of $I[C_0]$ such that all hexagons in $S$ are $M_0$-alternating. Let $M_1 := (M \setminus M_c) \cup M_0$ and $A' := S \cup (A - \{C_0\} - A_0)$. Then $M_1$ is a perfect matching of $H$ such that each member in $A'$ is an $M_1$-alternating cycle. Note that $|A'| \geq n + 1$. By Theorem 1.3, we have that $f(H, M_1) \geq n + 1$. This contradicts that the maximum forcing number of $H$ is $n$. Therefore, for any two members in $A$ their interiors are disjoint.

For any $C \in A$, we assert that the Clar number of $I[C]$ is 1. Otherwise, $I[C]$ has a resonant set $S'$ with $|S'| \geq 2$. Similar to the above discussion, we can obtain $n + 1$ disjoint cycles which are $M_2$-alternating with respect to some perfect matching $M_2$ of $H$. By Theorem 1.3, we have that $F(H) \geq n + 1$, a contradiction. Hence the assertion is true. By Theorem 2.2, for any $C \in A$, $I[C]$ is a linear chain.
References


