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Pseudo–Radioactive Decomposition Through a Generalized Shannon's Recomposition Theorem

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Abstract

In [Guirao et al., J. Math. Chem. 50(2) (2012), 374–378] and [Bustos et al., J. Math. Chem., (2013), DOI:10.1007/s10910-013-0285-x] were stated that an asymptotic generalization of the classical Shannon–Whittaker–Kotel'nikov was a useful tool to study the dynamics of pseudo–radioactive product following behaviors of the form $e^{-\lambda t^2}$ and $e^{\cos(\pi t)}$. Inspired in both papers the objective of this work is to show that chemical products following a decomposition dynamics determined by maps of the form $e^{\text{sinc}\,(x-t)}$ can be recomposed through samples using this tool.

1 Introduction and statement of the main result

The decomposition of pseudo–radioactive products, studied in [5], follows a pseudo–exponential decomposition dynamics. In many cases the experts only have access to a limited measures of the product and it is very interesting to have a tool which allows to recomposed the dynamical systems modeling the dynamics of the chemical product analyzed. This is the situation of products which dynamics can be adjusted by maps of the form $e^{-\lambda t^2}$ and $e^{\cos(\pi t)}$, see [5] and [4] respectively.

The key point for doing this is the use of a generalization of the Shannon-Whittaker-Kotel'nikov Theorem (see, for instance, [7] and [8]) for a non-banded limited maps of $L^2(\mathbb{R})$ (i.e., for Paley–Wiener signals). Recall that a central result of the signal theory is the Shannon-Whittaker-Kotel'nikov Theorem (see [7] or [8]) based on the normalized cardinal sinus map $\operatorname{sinc}(t)$ defined by

$$\operatorname{sinc}(t) = \begin{cases} 1 & \text{if } t = 0, \\ \frac{\sin(\pi t)}{\pi t} & \text{if } t \neq 0 \end{cases}$$

and which allows to recomposed maps under certain conditions.

In this setting, [2] states the following asymptotic property of sampling Shannon's theorem type where the convergence of the series is considered in the Cauchy's principal value.

Property $\mathcal{P}$. Let $f : \mathbb{R} \to \mathbb{R}$ be a map and $\tau \in \mathbb{R}^+$. We say that f satisfies the property $\mathcal{P}$ for τ if

$$f(t) = \lim_{n\to\infty} \left(\sum_{k\in\mathbb{Z}} f^{\frac{1}{n}} \left(\frac{k}{\tau} \right) \operatorname{sinc}(\tau t - k) \right)^n. \tag{1}$$

[2] proves that every constant signal holds property $\mathcal{P}$ for every given $\tau \in \mathbb{R}^+$ and conjectures that the Gaussian maps, i.e., maps of the form $e^{-\lambda t^2}$, $\lambda \in \mathbb{R}^+$ hold property $\mathcal{P}$ for every given $\tau \in \mathbb{R}^+$. To support the conjecture [2] proves that the Gaussian map e^{-t^2} holds expression (1) for the three first coefficients of the power series representation of e^{-t^2}.

Note that the veracity of the conjecture is also suggested by the Boas's estimation [3]. Finally, [1] and [5] present proof the conjecture using different approaches.

Following the spirit of [4] and [5] we shall extends the working of property $\mathcal{P}$ to pseudo–radioactive materials whose dynamics is not, strictly speaking, a Gaussian function. The statement of our main results is the following.

Theorem 1. *The function* $f(t) = e^{\operatorname{sinc}(x-t)}$ *satisfies the property* $\mathcal{P}$ *for all* $x \in \mathbb{R} \setminus \mathbb{Z}$, *i.e.,*

$$\lim_{n\to\infty} \left(\sum_{k\in\mathbb{Z}} e^{\frac{\operatorname{sinc}(x-k)}{n}} \operatorname{sinc}(t-k) \right)^n = e^{\operatorname{sinc}(x-t)}.$$

We remark that following the ideas of the proof of Theorem 1 is possible to proof that property $\mathcal{P}$ is also satisfies by the maps

$$f_1(t) = e^{\frac{\cos \pi t}{t} - \frac{\sin \pi t}{\pi t^2}}$$

and

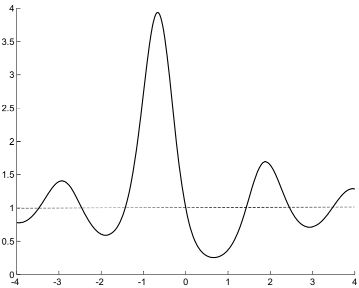

Figure 1: $f_1(t) = e^{\frac{\cos \pi t}{t} - \frac{\sin \pi t}{\pi t^2}}$.

$$f_2(t) = e^{\frac{\cos \pi t}{1+t^2} + \frac{\cosh \pi}{\sinh \pi} \frac{t \sin \pi t}{1+t^2}}.$$

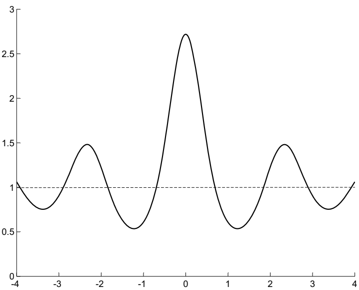

Figure 2: $f_2(t) = e^{\frac{\cos \pi t}{1+t^2} + \frac{\cosh \pi}{\sinh \pi} \frac{t \sin \pi t}{1+t^2}}$.

One consequence of the working of property $\mathcal{P}$ for trigonometrical maps implies that is possible to use the recomposition property for chemical reactions models with oscillators.

The structure of the paper is as follows. In section 2 we present some auxiliary results that we need for stating the proof of Theorem 1 in section 3.

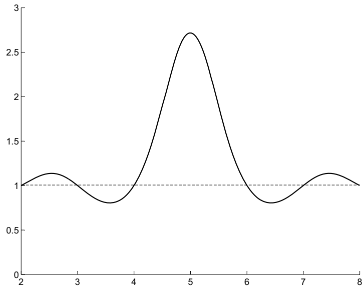

Figure 3: $f(t) = e^{\mathrm{sinc}\,(x-t)}$ for $x = 5$.

2 Auxiliary results

We start with a couple of result that can be respectively found in [6] and [8] for which we present an alternative proofs that we consider more direct.

Lemma 1. *Let* $\{f(k)\}_{k\in\mathbb{Z}}$ *be a sequence of positive real numbers. The series*

$$\sum_{k\in\mathbb{Z}} \log f(k)\mathrm{sinc}\,(t-k)$$

is absolutely convergent for every $t \in \mathbb{R}$ *if and only if*

$$\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \left|\frac{\log f(k)}{k}\right| < \infty. \tag{2}$$

Proof. Since

$$\begin{aligned}\sum_{k\in\mathbb{Z}} |\log f(k)\mathrm{sinc}\,(t-k)| &= \sum_{k\in\mathbb{Z}} \left|\frac{\log f(k)\sin(\pi(t-k))}{\pi(t-k)}\right| \\ &\leq \frac{1}{\pi}\sum_{k\in\mathbb{Z}} \left|\frac{\log f(k)}{t-k}\right|\end{aligned}$$

the condition (2) is sufficient. The necessity is obtained directly taking $t = \dfrac{1}{2}$. ■

Lemma 2. *For every* $x, t \in \mathbb{R}$*, there holds*

$$\sum_{k\in\mathbb{Z}} \mathrm{sinc}\,(x-k)\mathrm{sinc}\,(t-k) = \mathrm{sinc}\,(x-t).$$

Proof. Recall that Paley–Wiener space

$$PW_\pi = \{f \in L^2(\mathbb{R}); Supp\widehat{f} \subset [-\pi, \pi]\}$$

($\widehat{f}$ is the Fourier's transform of f) is a reproducing kernel Hilbert space with reproducing kernel $K(x,t) = \operatorname{sinc}(x-t)$.

Moreover, if $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for a Hilbert space, then the reproducing kernel can be expressed as

$$K(x,t) = \sum_{k=1}^{\infty} e_k(x)\overline{e_k(t)} \qquad x,t \in \mathbb{R}.$$

Applying this result with the fact that $\{\operatorname{sinc}(x-k)\}_{k=-\infty}^{k=\infty}$ is an orthonormal basis for PW_π, the proof is over. ■

Lemma 3. *Let $\{f(k)\}_{k\in\mathbb{Z}}$ be a sequence of postive real numbers. For every $k \in \mathbb{Z}$ and for every $n \in \mathbb{N}$ then*

$$\left| f(k)^{\frac{1}{n}} - 1 \right| \leq \frac{|\log f(k)|}{n} \max\left\{1, f(k)^{\frac{1}{n}}\right\}.$$

In particular, if $\{f(k)\}_{k\in\mathbb{Z}}$ is bounded above then there is $L \geq 1$ such that

$$\left| f(k)^{\frac{1}{n}} - 1 \right| \leq L\frac{|\log f(k)|}{n}.$$

Proof. Applying the Mean Value Theorem to the function $f(x) = f(k)^x$ in the interval $\left[0, \frac{1}{n}\right]$ follows that there exists $c \in \left(0, \frac{1}{n}\right)$ such that

$$\left| f(k)^{\frac{1}{n}} - 1 \right| = \frac{|\log f(k)|}{n} f(k)^c.$$

Given that if $0 < f(k) \leq 1$, then $f(k)^c \leq 1$. Finally, if $f(k) > 1$ then $f(k)^c < f(k)^{\frac{1}{n}} < f(k)$ ending the proof. ■

Lemma 4. $\sum_{k\in\mathbb{Z}} \operatorname{sinc}(z-k) = 1$ *for every $z \in \mathbb{C}$.*

Proof. First of all we shall show that the result works for every $t \in \mathbb{R}$. Indeed, if $t \in \mathbb{Z}$, the result follows since

$$\sum_{k\in\mathbb{Z}} \operatorname{sinc}(t-k) = 1 + \sum_{\substack{k\in\mathbb{Z}\\ k\neq t}} \operatorname{sinc}(t-k) = 1 + 0 = 1.$$

Therefore, from now on we assume that $t \in \mathbb{R}\setminus\mathbb{Z}$. Taking simetric terms in the series we obtain

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \operatorname{sinc}(t-k) &= \frac{\sin(\pi t)}{\pi t} + \sum_{k\in\mathbb{N}} \left(\frac{\sin(\pi(t-k))}{\pi(t-k)} + \frac{\sin(\pi(t+k))}{\pi(t+k)} \right) \qquad (3) \\ &= \frac{\sin(\pi t)}{\pi t} + \frac{2t\sin(\pi t)}{\pi} \sum_{k\in\mathbb{N}} \frac{(-1)^{k+1}}{k^2 - t^2}. \end{aligned}$$

On the other hand, for a given $t \in \mathbb{R} \setminus \mathbb{Z}$ is known that

$$\frac{t\pi}{\sin(t\pi)} = 1 + 2t^2 \sum_{k\in\mathbb{N}} \frac{(-1)^k}{t^2 - k^2}$$

and therefore

$$\sum_{k\in\mathbb{N}} \frac{(-1)^{k+1}}{k^2 - t^2} = \frac{-1}{2t^2} + \frac{\pi}{2t\sin(\pi t)}. \tag{4}$$

Finally, replacing (4) in expression (3) the proof is completed for every real number t.

The proof of the result for complex numbers is a consequence of the use of the Analytic Prolongation Principle. For applying it, is enough to prove that the series $\sum_{k\in\mathbb{Z}} \operatorname{sinc}(z-k)$ is an analytic function. Indeed, by (3) the series can be written in the form

$$\sum_{k\in\mathbb{Z}} \operatorname{sinc}(z-k) = \frac{\sin(\pi z)}{\pi z} + \frac{2z\sin(\pi z)}{\pi} \sum_{k\in\mathbb{N}} \frac{(-1)^{k+1}}{k^2 - z^2}\ .$$

Obviously, the first term of the previous sum is an analytic map. For proving the analyticity of the second term of the sum we shall prove that the series $\sum_{k\in\mathbb{N}} \frac{(-1)^{k+1}}{k^2-z^2}$ uniformly converges on every compact set $L \subset \mathbb{C}\backslash\mathbb{N}$. In fact, let $s = \max\{|z| : z \in L\}$ and k_0 be such that $k_0 > 2s$, then for every $k \geq k_0$ is $|z| < \frac{k}{2}$ for every $z \in L$. Therefore,

$$\left|\frac{(-1)^{k+1}}{k^2 - z^2}\right| \leq \frac{4}{3k^2}$$

which guarantees the uniformly convergency of the series in L and the proof is over. ■

3 Proof of the main result

The objective of this section is to present the proof of our main result Theorem 1.

Proof of Theorem 1. Using the notation

$$h(t,n) = \sum_{k\in\mathbb{Z}} e^{\frac{\operatorname{sinc}(x-k)}{n}} \operatorname{sinc}(t-k),$$

we have to prove that

$$\lim_{n\to\infty} (h(t,n))^n = e^{\operatorname{sinc}(x-t)}.$$

Let $\{f(k)\}_{k\in\mathbb{Z}} = \left\{e^{\operatorname{sinc}(x-k)}\right\}_{k\in\mathbb{Z}}$ be for every $x \in \mathbb{R} \setminus \mathbb{Z}$. Since

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \left|\frac{\log f(k)}{k}\right| &= \sum_{k\in\mathbb{Z}} \left|\frac{\operatorname{sinc}(x-k)}{k}\right| = \sum_{k\in\mathbb{Z}} \left|\frac{\sin\pi(x-k)}{\pi k(x-k)}\right| \\ &\leq \frac{1}{\pi} \sum_{k\in\mathbb{Z}} \left|\frac{1}{k(x-k)}\right| < \infty \end{aligned} \tag{5}$$

by Lemma 1, we have that the series

$$\sum_{k\in\mathbb{Z}} \log f(k)\text{sinc}\,(t-k) = \sum_{k\in\mathbb{Z}} \text{sinc}\,(x-k)\text{sinc}\,(t-k)$$

is absolutely convergent.

Moreover, as $\{f(k)\}_{k\in\mathbb{Z}}$ is bounded, by Lemma 3 there exists $L \geq 1$ such that

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \left|\left(e^{\frac{\text{sinc}\,(x-k)}{n}} - 1\right)\text{sinc}\,(t-k)\right| &\leq L\sum_{k\in\mathbb{Z}} \left|\frac{\text{sinc}\,(x-k)}{n}\right| |\text{sinc}\,(t-k)| \\ &\leq L\sum_{k\in\mathbb{Z}} |\text{sinc}\,(x-k)\,\text{sinc}\,(t-k)| < \infty.\end{aligned}$$

Therefore, applying Lemma 4, the Dominated Convergence Theorem and that

$$\lim_{n\to\infty}\left(e^{\frac{\text{sinc}\,(x-k)}{n}} - 1\right)\text{sinc}\,(t-k) = 0,$$

we have that

$$\begin{aligned}\lim_{n\to\infty}(h(t,n)-1) &= \lim_{n\to\infty}\left(\sum_{k\in\mathbb{Z}} e^{\frac{\text{sinc}\,(x-k)}{n}}\text{sinc}\,(t-k) - \sum_{k\in\mathbb{Z}}\text{sinc}\,(t-k)\right) \\ &= \lim_{n\to\infty}\sum_{k\in\mathbb{Z}}\left(e^{\frac{\text{sinc}\,(x-k)}{n}} - 1\right)\text{sinc}\,(t-k) \\ &= \sum_{k\in\mathbb{Z}}\lim_{n\to\infty}\left(e^{\frac{\text{sinc}\,(x-k)}{n}} - 1\right)\text{sinc}\,(t-k) = 0,\end{aligned}$$

hence it follows that

$$\lim_{n\to\infty} h(t,n) = 1\ . \tag{6}$$

Using the Lemma 3 and the condition (5) we have that

$$\sum_{k\in\mathbb{Z}}\left|n\left(e^{\frac{\text{sinc}\,(x-k)}{n}} - 1\right)\text{sinc}\,(t-k)\right| \leq L\sum_{k\in\mathbb{Z}}|\text{sinc}\,(x-k)\text{sinc}\,(t-k)| < \infty.$$

Therefore, applying the Dominated Convergence Theorem and that

$$\lim_{n\to\infty} n\left(e^{\frac{\text{sinc}\,(x-k)}{n}} - 1\right)\text{sinc}\,(t-k) = \text{sinc}\,(x-k)\,\text{sinc}\,(t-k),$$

we obtain

$$\begin{aligned}\lim_{n\to\infty} n(h(t,n)-1) &= \lim_{n\to\infty}\left(\sum_{k\in\mathbb{Z}} n\left(e^{\frac{\text{sinc}\,(x-k)}{n}} - 1\right)\text{sinc}\,(t-k)\right) \\ &= \sum_{k\in\mathbb{Z}}\lim_{n\to\infty} n\left(e^{\frac{\text{sinc}\,(x-k)}{n}} - 1\right)\text{sinc}\,(t-k) \\ &= \sum_{k\in\mathbb{Z}}\text{sinc}\,(x-k)\text{sinc}\,(t-k).\end{aligned}$$

Thus, by Lemma 2, we have

$$\lim_{n\to\infty} n(h(t,n)-1) = \operatorname{sinc}(x-t). \tag{7}$$

Finally, from (7) and (6), we conclude that

$$\lim_{n\to\infty} (h(t,n))^n = e^{\lim_{n\to\infty} n(h(t,n)-1)} = e^{\operatorname{sinc}(x-t)}$$

finishing the proof of the theorem. ■

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