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Every Extremal Fullerene Graph with No Less than 60 Vertices is 2-Resonant*

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Abstract

A *fullerene graph* is a 3-regular 3-connected plane graph with only pentagonal and hexagonal faces. For a fullerene graph F, a set $\mathcal{H}$ of disjoint hexagons is called a *resonant pattern* (or *sextet pattern*) if it has a perfect matching such that every hexagon in $\mathcal{H}$ is M-alternating. F is said to be *k-resonant* if any i $(0 \leq i \leq k)$ disjoint hexagons of F form a resonant pattern. Every fullerene graph is shown to be 1-resonant and exactly nine fullerene graphs are k-resonant $(k \geq 3)$. But not all fullerene graphs are 2-resonant. For a fullerene graph F_n with n vertices, the size of a maximum resonant pattern of F_n is the *Clar number*, denoted by $c(F_n)$. It is known that $c(F_n) \leq \frac{n-12}{6}$. The fullerene graphs F_n with $c(F_n) = \frac{n-12}{6}$ are *extremal*. In this paper, we show that every extremal fullerene graph with no less than 60 vertices is 2-resonant. However, non-2-resonant extremal fullerene graphs with less than 60 vertices exist.

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1 Introduction

Fullerene graphs are 3-connected 3-regular plane graph having only pentagonal faces and hexagonal faces. In chemistry, a fullerene consists entirely of carbon atoms, each carbon atom is connected to three carbon atoms by chemical bonds. The archetype of fullerenes, icosahedral C_{60}, known as the buckministerfullerene, was firstly discovered by Kroto et al. [10] in 1985 and confirmed by later experiments [9, 16]. The stability of fullerene graphs is always the focus of investigations. Up to now a number of invariants that could be used as a predictor of fullerene stability has been proposed, such as Kekulé count, Clar number, etc., which stimulated dozens of works to investigate the number of perfect matching [3, 8, 25] and the Clar number [19, 22, 24] of fullerene graphs.

Let F be a fullerene graph. A *perfect matching* (or Kekulé structure) of F is a set of pairwise disjoint edges M such that every vertex of F is incident with an edge in M. A cycle of F is *M-alternating* if its edges appear alternately in and off M. A set $\mathcal{H}$ of mutually disjoint hexagons is called a *resonant pattern* (or *sextet pattern*) if F has a perfect matching M such that every hexagon in $\mathcal{H}$ is M-alternating. For a fullerene graph F_n on n vertices, we denote by $c(F_n)$ the size of a maximum resonant pattern and call this number the *Clar number* of F_n. It has shown that $c(F_n) \leq \frac{n-12}{6}$ [22]. If equality holds, we call F_n *extremal.*

A fullerene graph F is *k-resonant* if any i $(0 \leq i \leq k)$ disjoint hexagons of F form a resonant pattern. The concept of resonance originates from Clar's aromatic sextet theory [2]: within benzenoid hydrocarbon isomers, one with larger Clar number is more stable. The k-resonance was firstly proposed by Zheng [27] for benzenoid systems. Then the k-resonance of many other molecular graphs is investigated extensively [1, 5, 11, 13, 14, 15, 20, 21, 23, 26]. But 2-resonance for benzenoid systems, open-ended nanotubes and fullerenes remains open.

Ye et al. [18] showed that every fullerene graph is 1-resonant and there are exactly nine fullerene graphs which are k-resonant $(k \geq 3)$. But not all fullerene graphs are 2-resonant. It is known that all leapfrog fullerene graphs are 2-resonant (see [18]). The class of leapfrog fullerene graphs satisfies the IPR (a fullerene with no adjacent pentagons is said to satisfy the *isolated pentagon rule* and is called an IPR fullerene). This leads Ye et al. [18] to ask whether every IPR fullerene graph is 2-resonant. Kaiser et al. [6] answered the question in the affirmative.

Theorem 1.1. *([6] Theorem 1) Every IPR fullerene graph is 2-resonant.*

Note the IPR fullerene graphs do not include the subgraph L or R (see Fig. 1). So later Yang and Zhang [17] generalized the result of Theorem 1.1 to the fullerene graphs which do not contain the subgraph L or R.

Fig. 1. The subgraphs L and R.

Theorem 1.2. *([17] Theorem 1.1)The fullerene graphs which do not contain the subgraph L or R are 2-resonant except for* $F_{42}, F_{44}^1, F_{44}^2, F_{46}^1, F_{46}^2, F_{46}^3, F_{46}^4, F_{48}^1, F_{48}^2, F_{48}^3, F_{48}^4$.

In this paper, by means of Theorem 1.2 we prove the following result.

Theorem 1.3. *Every extremal fullerene graph F_n with $n \geq 60$ is 2-resonant.*

The above conclusion does not always hold for the extremal fullerene graphs with vertices less than 60. For example, the three fullerene graphs $F_{42}, F_{48}^3, F_{48}^4$ as shown in Fig. 2 are extremal but not 2-resonant since the two grey hexagons do not form a resonant pattern.

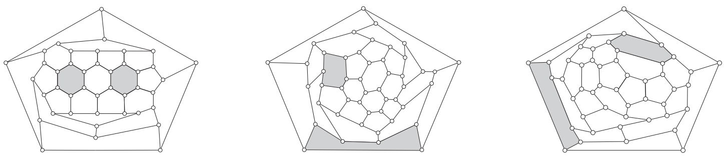

Fig. 2. Non-2-resonant extremal fullerene graphs F_{42}, F_{48}^3 and F_{48}^4.

2 Preliminaries

Let G be a connected plane graph with vertex-set $V(G)$ and edge-set $E(G)$. For $H \subseteq V(G) \cup E(G)$, we let $G - H$ be the subgraph of G obtained from G by removing the elements in H. If $X \subset V(G)$, then we write $\nabla(X)$ for the set of edges having one endpoint in X and the other in $\overline{X}$, where $\overline{X} = V(G) - X$. For a subgraph H of G, we also simply write $\nabla(H)$ for $\nabla(V(H))$, and $\overline{H}$ for $\overline{V(H)}$.

An *edge-cut* of a connected graph G is a set of edges $C \subset E(G)$ such that $G - C$ is disconnected. An edge-cut C of G is *cyclic* if at least two components of $G - C$ contain a cycle. A graph G is *cyclically k-edge-connected* if deleting less than k edges from G can not separate it into two components such that each of them contains at least one cycle. The *cyclic edge-connectivity* of G, denoted by $c\lambda(G)$, is the greatest integer k such that G is cyclically k-edge-connected. For a fullerene graph F, Došlić [4] and Qi and Zhang [12] proved that $c\lambda(F) = 5$; Kardoš and Škrekovski [7] obtained the same result by three operations on cyclic edge-cuts. The edges pointing outward of each pentagonal (or hexagonal) face form a cyclic 5 (or 6) edge cut. We call these cyclic 5 and 6 edge-cuts *trivial*. A cyclic edge-cut C of a fullerene graph F is *non-degenerated* if both components of $F - C$ contain precisely six pentagons. Otherwise, C is *degenerated*. Obviously, the trivial cyclic edge-cuts are degenerated. There is a family of fullerene graphs, which have many non-degenerated cyclic edge-cuts—the nanotubes. *Nanotubes* are cylindrical in shape with two caps, each containing six pentagons and maybe some hexagons. The cylindrical part of the nanotube can be obtained from a planar hexagonal grid by identifying objects lying on two parallel lines. The way that the grid is wrapped is represented by a pair of indices (p_1, p_2). The numbers p_1 and p_2 denote the coefficients of the linear combination of the unit vectors a_1 and a_2 such that the vector $p_1a_1 + p_2a_2$ joins pairs of identified points, see Fig. 3.

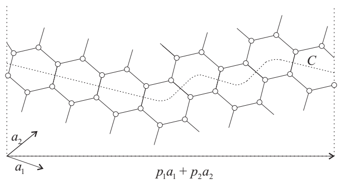

Fig. 3. An example of a nanotube of type (6,2).

Kardoš and Škrekovski [7] characterized the cyclic 5- and 6-edge-cuts in fullerene graphs.

Theorem 2.1. *([7] Theorem 2) A fullerene graph has non-trivial cyclic 5-edge-cuts if and only if it is isomorphic to the graph G_m for some integer $m \geq 1$, where G_m is the fullerene graph comprised of two caps formed of six pentagons joined by m layers of hexagons (see Fig. 4).*

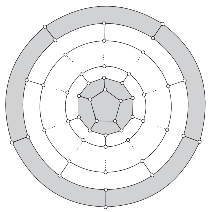

Fig. 4. The graphs G_m are the only fullerene graphs with non-trivial cyclic 5-edge-cuts.

Theorem 2.2. *([7] Theorems 3,4) (i)There are precisely seven non-isomorphic graphs that can be obtained as components of degenerated cyclic 6-edge-cuts with less than six pentagons (see Fig. 5).*

(ii) A fullerene graph has a non-degenerated cyclic 6-edge-cut if and only if it is a nanotube of type (p_1, p_2) with $p_1 + p_2 = 6$, or it is G_m with $m \geqslant 2$, where G_m is shown in Fig. 4.

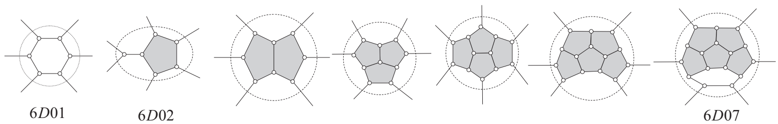

Fig. 5. Degenerated cyclic 6-edge-cuts.

Corollary 2.3. *([7] Corollary 5) There are exactly eight caps for the nanotubes of type (p_1, p_2) such that $p_1 + p_2 = 6$ (see Fig. 6).*

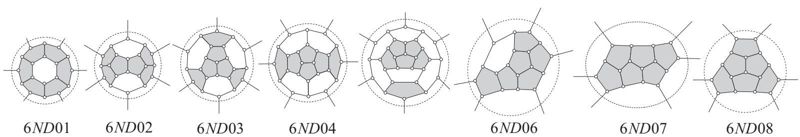

Fig. 6. The (minimal) caps of (p_1, p_2)-nanotubes with $p_1 + p_2 = 6$.

A *fragment* B of a fullerene graph F is a subgraph of F consisting of a cycle together with its interior. A *pentagonal fragment* is a fragment with only pentagonal inner faces. A face f of F is a *neighboring face* of B if f is not a face of B and f has at least one edge in common with B. A fragment B of F is *maximal* if all its neighboring faces are hexagonal.

For a fragment B, all 2-degree vertices of B lie on its boundary. We call a path P on the boundary of B connecting two 2-degree vertices *degree-saturated* if P contains no 2-degree vertices of B as intermediate vertices. Maximal pentagonal fragments play an important role in characterizing extremal fullerene graphs.

Theorem 2.4. *([19] Theorem 3.11) Let F_n $(n \geq 60)$ be an extremal fullerene graph and $B_1, B_2, \ldots, B_k$ all maximal pentagonal fragments of F_n. Then $B_i \in \{P, B_2, B_3, B_4, B_5, B_6\}$ for all $1 \leq i \leq k$, where $P, B_2, B_3, B_4, B_5, B_6$ are shown in Fig. 7.*

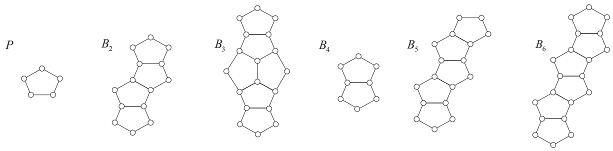

Fig. 7. Maximal pentagonal fragments $P, B_2, B_3, B_4, B_5, B_6$.

3 Proof of Theorem 1.3

From now on, let F_n be an extremal fullerene graph with $n \geq 60$ and $B_1, B_2, \ldots, B_k$ all maximal pentagonal fragments of F_n. If there exists a vertex v belonging to three pentagons in F_n, then the maximal pentagonal fragment containing v must be B_3 by Theorem 2.4. So next we distinguish $B_3 \subset F_n$ or $B_3 \not\subset F_n$ to prove Theorem 1.3.

In what follows we assume $B_3 \subset F_n$. Denote by $P_1, \ldots, P_6$ and $f_1, \ldots, f_8$ the pentagonal inner faces and neighboring faces of B_3 as shown in Fig. 8(a). Then all of $f_1, \ldots, f_8$ are hexagons as B_3 is a maximal pentagonal fragment of F_n. Let $G = B_3 \cup f_2 \cup f_3 \cup f_6 \cup f_7$. Then we have the following result.

Proposition 3.1. *F_n is an (6,0)-nanotube with the two caps as the components of the non-degenerated cyclic 6-edge-cut $6ND04$ (see Fig. 8(b) the (6,0)-nanotube F_n).*

Proof. Since G $(=B_3 \cup f_2 \cup f_3 \cup f_6 \cup f_7)$ contains six pentagons and f_1, f_4, f_5, f_8 are hexagons, at least one pentagon is included in $\overline{G}$. In other words, $\nabla(G)$ forms a cyclic 6-edge-cut. Moreover, $\nabla(G)$ is a non-degenerate cyclic 6-edge-cut, otherwise, $n \leq 24 + 16 = 40$ by Theorem 2.2(i), contradicting that $n \geqslant 60$. So by Theorem 2.2(ii) F_n is either G_m with $m \geqslant 2$ (see Fig. 4 the graph G_m) or a nanotube of type (p_1, p_2) with $p_1 + p_2 = 6$. If the former case holds, then F_n has a maximal pentagonal fragment consisting of a central

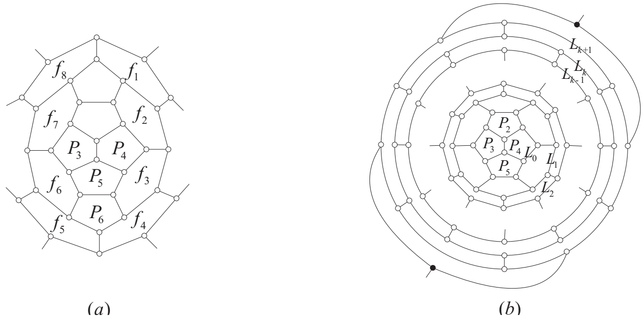

Fig. 8. (a) B_3 and its neighboring faces; (b) The (6,0)-nanotube F_n: the two black vertices are adjacent.

pentagon surrounded by five pentagons, contradicting Theorem 2.4. If the latter case holds, then the two caps of F_n can be choosen in Fig. 6 by Corollary 2.3. Now checking for all the caps shown in Fig. 6 and combining with Theorem 2.4 we can know F_n has the minimal cap G which is one component of the non-degenerated cyclic 6-edge-cut $6ND04$, and the Proposition holds. □

Divide G into the *inner* and *outer* layers, where the inner layer consists of P_2, P_3, P_4, P_5 and the outer layer is comprised of $P_1, P_6, f_2, f_3, f_6, f_7$. Next we give some labels for F_n. Let $H_0, H_1, \ldots, H_{k+1}$ be all the layers of F_n, where $H_0, H_{k+1}(H_1, H_k)$ are the inner (outer) layers of G and hexagonal layer H_i is adjacent to H_{i-1} and H_{i+1} for $2 \le i \le k-1$. For $1 \le i \le k+1$, we set $L_i = H_{i-1} \cap H_i$ (see Fig. 8(b)). Then L_1 (L_{k+1}) is a cycle of length 10 and L_i ($2 \le i \le k$) is a cycle of length 12. We call two vertices u, v on L_i *opposite* if the length of the two u, v paths on L_i is the same. Note that both H_1 and H_k have four hexagons and H_i ($2 \le i \le k-1$) has six hexagons, each of which is adjacent to two faces in every adjacent layer of H_i. For convenience, we label the hexagonal faces of F_n as follows: give the labels $h_1^1, h_1^2, h_1^3, h_1^4$ to the four hexagonal faces of H_1 along clockwise direction such that h_1^1, h_1^2 (h_1^3, h_1^4) are adjacent. Assume that the labels of the faces in $H_1, H_2, \ldots, H_i$ ($i \ge 1$) are given, then label the faces of H_{i+1} $h_{i+1}^1, h_{i+1}^2, \ldots, h_{i+1}^6$ along clockwise direction such that h_{i+1}^1, h_i^1, h_i^2 are pairwise adjacent. Finally give the labels $h_k^1, h_k^2, h_k^3, h_k^4$ to the four hexagonal faces of H_k arbitrarily. In the following, we always use the above symbols to indicate the (6,0)-nanotube F_n.

Regarding to the (6,0)-nanotube F_n, we have the following result.

Lemma 3.2. *Let h_1, h_2 be two disjoint hexagons of F_n. Then for any $1 \le i \le k$, there is a matching M_i of $F_n - (h_1 \cup h_2)$ such that for any $1 \le j \le i$, each vertex of L_j is covered either by h_1 or h_2 or M_i and one of the following three cases holds.*

(1) two of $h_j^1, h_j^2, \ldots, h_j^6$ (the case $j \ne 1, k$) or two of $h_1^1, h_1^2, h_1^3, h_1^4$ or two of $h_k^1, h_k^2, h_k^3, h_k^4$ are h_1, h_2, and no vertex of L_j matches to a vertex of L_{j+1} in M_i;

(2) one of $h_j^1, h_j^2, \ldots, h_j^6$ (the case $j \ne 1, k$) or one of $h_1^1, h_1^2, h_1^3, h_1^4$ or one of $h_k^1, h_k^2, h_k^3, h_k^4$ is h_1 or h_2, and a unique vertex of L_j matches to a vertex of L_{j+1} in M_i;

(3) none of $h_j^1, h_j^2, \ldots, h_j^6$ (the case $j \ne 1, k$) or none of $h_1^1, h_1^2, h_1^3, h_1^4$ or none of $h_k^1, h_k^2, h_k^3, h_k^4$ is h_1 or h_2, and two opposite vertices of L_j match to two opposite vertices of L_{j+1}.

Proof. We use induction on i to prove the lemma.

Label the vertices of H_0 as shown in Fig. 9(a). Let v_{13}, v_{14} be the neighbors of v_1, v_6, respectively, not on H_0. For the first step, if two of $h_1^1, h_1^2, h_1^3, h_1^4$ are h_1 and h_2, then the two hexagons are h_1^1, h_1^3 or h_1^1, h_1^4 by symmetry. If h_1^1, h_1^3 are the two hexagons h_1, h_2, then we can set $M_1 = \{v_1v_{10}, v_{11}v_{12}, v_5v_6\}$, and M_1 satisfies the conditions (see Fig. 9(b)). If h_1^1, h_1^4 are the two hexagons h_1, h_2, then $M_1 = \{v_5v_6, v_7v_8, v_{11}v_{12}\}$ also satisfies the conditions. If one of $h_1^1, h_1^2, h_1^3, h_1^4$ equals h_1 or h_2, say $h_1^1 = h_1$, then $M_1 = \{v_1v_{13}, v_5v_6, v_7v_8, v_9v_{10}, v_{11}v_{12}\}$ is the desired matching and M_1 satisfies the conditions (see Fig. 9(c)). If none of $h_1^1, h_1^2, h_1^3, h_1^4$ is h_1 or h_2, then $M_1 = \{v_1v_{13}, v_2v_3, v_4v_5, v_6v_{14}, v_7v_8, v_9v_{10}, v_{11}v_{12}\}$ is the desired matching (see Fig. 9(d)).

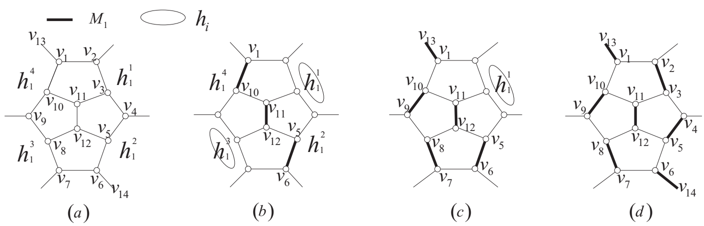

Fig. 9. Illustration for the first step in the proof of Lemma 3.2.

Suppose the conclusion is true for $i, 1 \le i \le k$ and M_i has been already constructed by inductive procedure. We consider the case $i + 1$. By the inductive hypothesis, three cases are considered.

Case 1. Two of $h_i^1, h_i^2, \ldots, h_i^6$ are h_1, h_2. Then H_{i+1} does not contain h_1 or h_2. By symmetry, the two hexagons are h_i^1, h_i^3 or h_i^1, h_i^4. If h_i^1, h_i^3 equal h_1, h_2, then six vertices on L_{i+1} are covered by h_1, h_2 and the other six are divided into a path $v_1v_2v_3v_4v_5$ and a vertex v_6 (see Fig. 10(a)). Denote by e_1 (e_2) the edge connecting v_3 (v_6) to its neighbor on L_{i+2}. Then $M_{i+1} = M_i \cup \{e_1, e_2, v_1v_2, v_4v_5\}$ satisfies the conditions (see Fig. 10(a)). The case for h_i^1, h_i^4 being h_1, h_2 is the same (see Fig. 10(b) the desired matching $M_{i+1} = M_i \cup \{e_1, e_2, v_1v_2, v_4v_5\}$).

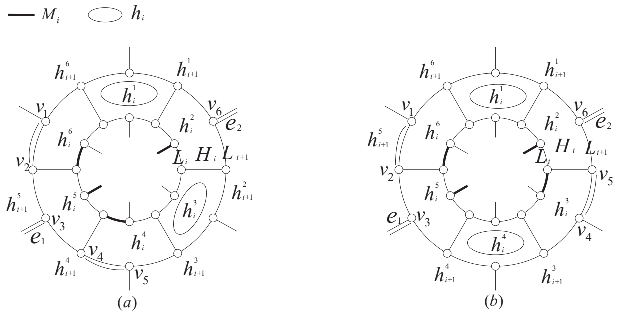

Fig. 10. Illustration for Case 1 in the proof of Lemma 3.2.

Case 2. One of $h_i^1, h_i^2, \ldots, h_i^6$ is h_1 or h_2, say $h_i^1 = h_1$. By the induction hypothesis, a unique vertex of L_i matches to a vertex of L_{i+1} in M_i, say $e = uv \in M_i$, where $u \in V(L_i), v \in V(L_{i+1})$. Then by symmetry $e \in E(h_i^2) \cap E(h_i^3)$ or $e \in E(h_i^3) \cap E(h_i^4)$. We only consider the former situation as the latter case can be analyzed similarly. If $h_{i+1}^2 \neq h_2$, then at most one of $h_{i+1}^3, h_{i+1}^4, h_{i+1}^5$ is h_2. If one of $h_{i+1}^3, h_{i+1}^4, h_{i+1}^5$, say h_{i+1}^4, equals h_2, then six vertices on L_{i+1} are covered by $h_1 \cup h_2$, and the remaining six vertices on L_{i+1} are divided into two paths: an edge v_1v_2 and a path $v_3v_4vv_5$ (see Fig. 11(a)). Denote by e_1 the edge connecting v_5 to its neighbor on L_{i+2}. Then $M_{i+1} = M_i \cup \{e_1, v_1v_2, v_3v_4\}$ satisfies the conditions. If none of $h_{i+1}^3, h_{i+1}^4, h_{i+1}^5$ equals h_2, then three vertices on L_{i+1} are covered by h_1 and the remaining nine vertices form a path $v_1v_2v_3v_4v_5v_6v_7vv_8$ (see Fig. 11(b)). Let e_1 (e_2) be the edge connecting v_8 (v_3) to its neighbor on L_{i+2}. Then $M_{i+1} = M_i \cup \{e_1, e_2, v_1v_2, v_4v_5, v_6v_7\}$ satisfies the conditions.

Now suppose $h_{i+1}^2 = h_2$. Then six vertices on L_{i+1} are covered by $h_1 \cup h_2$ and the remaining six form a path $v_1v_2v_3v_4v_5v_6$ (see Fig. 11(c)). Moreover, three vertices on L_i are covered by h_1 and the remaining nine form a path $u_1u_2u_3u_4u_5u_6u_7uu_8$ (see Fig. 11(c)). Since H_{i-1} does not contain h_1 or h_2, two opposite vertices of L_{i-1} match to two opposite vertices of L_i in M_i by the inductive hypothesis, that is, $\{u_1u_2, u_4u_5, u_6u_7\} \subset M_i$ (see Fig. 11(c)). Let $M_{i+1} = M_i - \{e, u_6u_7\} \cup \{uu_7, u_6v_6, e_1, v_1v_2, v_3v_4\}$. Then M_{i+1} again satisfies the conditions (see Fig. 11(d)), where e_1 is the edge connecting v_5 to its neighbor on L_{i+2}.

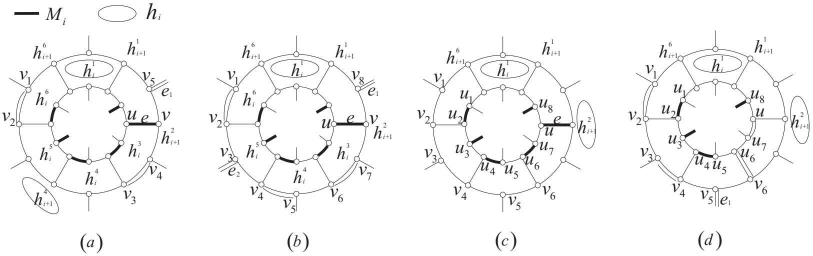

Fig. 11. Illustration for Case 2 in the proof of Lemma 3.2.

Case 3. None of $h_i^1, h_i^2, \ldots, h_i^6$ equals h_1 or h_2. By the hypothesis, two opposite vertices of L_i match to two opposite vertices of L_{i+1} in M_i. Let the two edges in M_i connecting L_i to L_{i+1} be u_1v_1, u_7v_7 such that $u_1, u_7 \in V(L_i)$, $v_1, v_7 \in V(L_{i+1})$. Denote by $u_1u_2\cdots u_{12}$ $(v_1v_2\cdots v_{12})$ the boundary labeling of L_i (L_{i+1}) along anticlockwise direction. Without lose of generality we may assume $v_1 \in V(h_{i+1}^6)$.

If two of $h_{i+1}^1, h_{i+1}^2, h_{i+1}^4, h_{i+1}^5$ are the hexagons h_1 and h_2, say $h_{i+1}^1 = h_1, h_{i+1}^4 = h_2$, then the six vertices $v_4, v_5, v_6, v_{10}, v_{11}, v_{12}$ are covered by $h_1 \cup h_2$ and $M_{i+1} = M_i \cup \{v_2v_3, v_8v_9\}$ satisfies the conditions (see Fig. 12(a)). If one of h_{i+1}^3, h_{i+1}^6 equals h_t and one of $h_{i+1}^1, h_{i+1}^2, h_{i+1}^4, h_{i+1}^5$ equals h_{3-t} for $t \in \{1,2\}$, say $h_{i+1}^6 = h_1, h_{i+1}^2 = h_2$, then H_{i-1} does not contain h_1 or h_2 and two opposite vertices of L_{i-1} match to two opposite vertices of L_i in M_i by the inductive hypothesis. Let s, t be the two opposite vertices on L_i. Then $s, t \in V(h_i^1 \cup h_i^4)$ or $s, t \in V(h_i^2 \cup h_i^5)$ or $s, t \in V(h_i^3 \cup h_i^6)$. If $s, t \in V(h_i^1 \cup h_i^4)$, say, $s = u_{12}, t = u_6$, then by the induction hypothesis, the vertices on L_i are covered by M_i, that is, $\{u_2u_3, u_4u_5, u_8u_9, u_{10}u_{11}\} \subset M_i$ (see Fig. 12(b)). Let $M_{i+1} = M_i - \{u_1v_1, u_7v_7, u_2u_3, u_4u_5, u_8u_9, u_{10}u_{11}\} \cup \{u_1u_2, u_3u_4, u_5v_5, u_7u_8, u_9u_{10}, u_{11}v_{11}, v_3v_4, v_6v_7\}$ (see Fig. 12(c)). Then M_{i+1} satisfies the conditions. If $s, t \in V(h_i^2 \cup h_i^5)$, say, $s = u_{10}, t = u_4$, then also by the induction hypothesis $\{u_2u_3, u_5u_6, u_8u_9, u_{11}u_{12}\} \subset M_i$ and $M_{i+1} = M_i - \{u_1v_1, u_7v_7, u_5u_6, u_{11}u_{12}\} \cup \{u_5v_5, u_{11}v_{11}, u_1u_{12}, u_6u_7, v_3v_4, v_6v_7\}$ satisfies the conditions. The case for $s, t \in V(h_i^3 \cup h_i^6)$ is the same. If h_{i+1}^3, h_{i+1}^6 equal h_1 and h_2, respectively, then H_{i-1} does not contain h_1 or h_2 and by the induction hypothesis, two opposite vertices of L_{i-1} match to two opposite vertices of L_i in M_i. Let s, t be the two opposite vertices on L_i. Then by symmetry $s, t \in V(h_i^1 \cup h_i^4)$ or $s, t \in V(h_i^2 \cup h_i^5)$. Also if $s, t \in V(h_i^1 \cup h_i^4)$, say, $s = u_{12}, t = u_6$, then $\{u_2u_3, u_4u_5, u_8u_9, u_{10}u_{11}\} \subset M_i$ (see Fig. 12(d)) and $M_{i+1} = M_i - \{u_1v_1, u_7v_7, u_2u_3, u_8u_9\} \cup \{u_1u_2, u_3v_3, u_7u_8, u_9v_9, v_4v_5, v_{10}v_{11}\}$

again satisfies the conditions (see Fig. 12(e)). Similarly for $s, t \in V(h_i^2 \cup h_i^5)$.

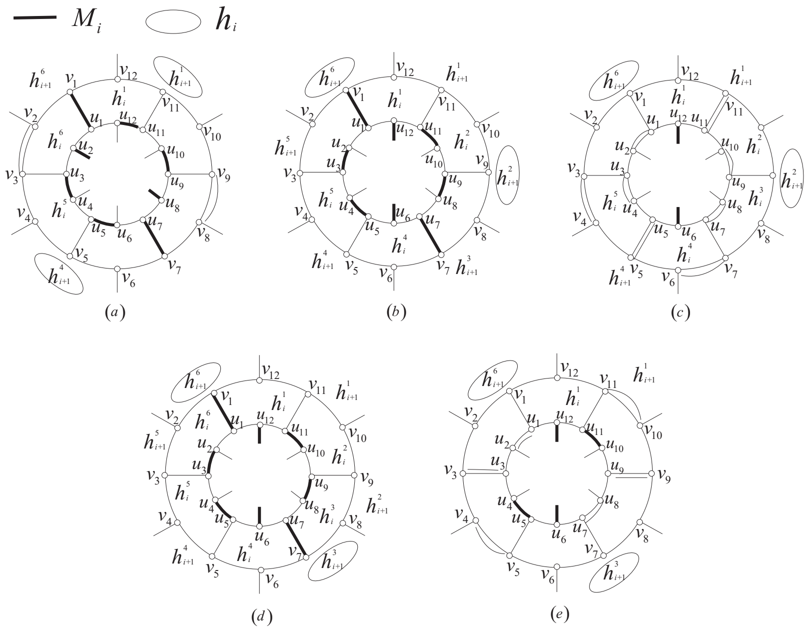

Fig. 12. Two of $h_{i+1}^1, \ldots, h_{i+1}^6$ being the hexagons h_1, h_2.

If one of $h_{i+1}^1, h_{i+1}^2, h_{i+1}^4, h_{i+1}^5$ equals h_1 or h_2, without lose of generality suppose $h_{i+1}^1 = h_1$. Then $M_{i+1} = M_i \cup \{e, v_2v_3, v_4v_5, v_8v_9\}$ satisfies the conditions, where e is the edge connecting v_6 to its neighbor on L_{i+2} (see Fig. 13(a)). If one of h_{i+1}^3, h_{i+1}^6 equals h_1 or h_2, say $h_{i+1}^6 = h_1$, then H_{i-1} does not contain h_2 or one of $h_{i-1}^2, h_{i-1}^3, h_{i-1}^5, h_{i-1}^6$ equals h_2. If the former case holds, then by the inductive hypothesis two opposite vertices of L_{i-1} match to two opposite vertices of L_i in M_i. Let s, t be the two opposite vertices on L_i. Then $s, t \in V(h_i^1 \cup h_i^4)$ or $s, t \in V(h_i^2 \cup h_i^5)$ by symmetry. If $s, t \in V(h_i^1 \cup h_i^4)$, say, $s = u_{12}, t = u_6$, then by the induction hypothesis, the vertices on L_i are covered by M_i, in other words, $\{u_2u_3, u_4u_5, u_8u_9, u_{10}u_{11}\} \subset M_i$ (see Fig. 13(b). Let $M_{i+1} = M_i - \{u_1v_1, u_7v_7, u_2u_3, u_8u_9\} \cup \{e, u_1u_2, u_3v_3, u_7u_8, u_9v_9, v_5v_6, v_7v_8, v_{10}v_{11}\}$. Then M_{i+1} is the desired matching (see Fig. 13(c)), where e is the edge connecting v_4 to its neighbor on L_{i+2}. The case for $s, t \in V(h_i^2 \cup h_i^5)$ is the same. If the latter case holds, then by symmetry $h_{i-1}^2 = h_2$ or $h_{i-1}^3 = h_2$. If $h_{i-1}^2 = h_2$, then a unique vertex of L_{i-1} matches to a vertex (say u) of L_i. Since by the induction hypothesis the vertices on L_i must be covered by h_1 or h_2 or M_i, $u \in V(h_i^4)$ or $u \in V(h_i^5)$ or $u \in V(h_i^6)$. As the similarity we only consider the case $u \in V(h_i^4)$. Then we have the M_i edges u_2u_3, u_4u_5, u_8u_9 on L_i as shown in Fig. 13(d). Exchange some edges and we obtain $M_{i+1} = M_i - \{u_1v_1, u_7v_7, u_2u_3, u_8u_9\} \cup \{e_1, u_1u_2, u_3v_3, u_7u_8, u_9v_9, v_5v_6, v_7v_8, v_{10}v_{11}\}$ satisfies the conditions (see Fig. 13(e)), where e_1 is the edge connecting v_4 to its neighbor on L_{i+2}. If $h_{i-1}^3 = h_2$, then analogously a unique vertex of L_{i-1} matches to a vertex (say u) of L_i and $u \in V(h_i^4)$ or $u \in V(h_i^5)$ or $u \in V(h_i^6)$. Using the same analysis as the case $h_{i-1}^2 = h_2$ we can find the desired matching M_{i+1} in each cases.

If none of $h_{i+1}^1, h_{i+1}^2 \cdots h_{i+1}^6$ equals h_1 or h_2, then $M_{i+1} = M_i \cup \{e_1, e_2, v_2v_3, v_4v_5, v_8v_9, v_{10}v_{11}\}$ satisfies the conditions, where e_1 (e_2) is the edge connecting v_6 (v_{12}) to its neighbor on L_{i+2}. □

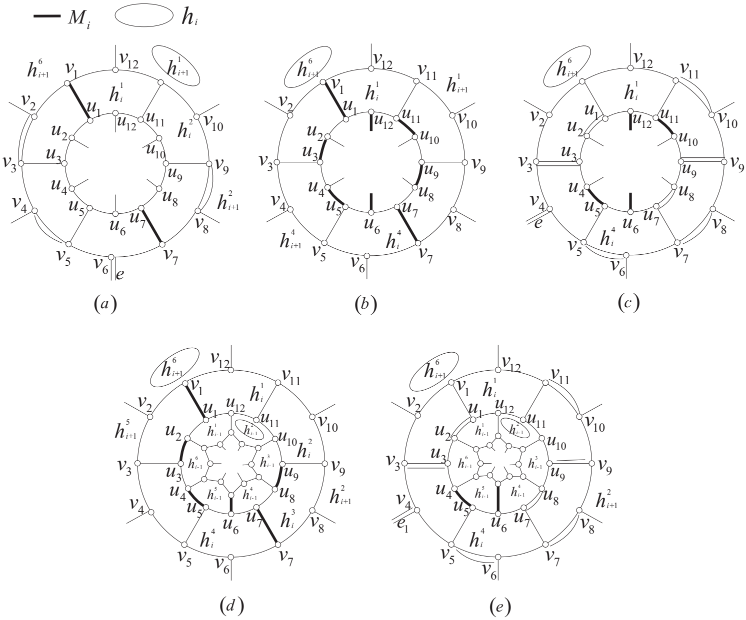

Fig. 13. One of $h_{i+1}^1, \ldots, h_{i+1}^6$ being the hexagon h_1 or h_2.

From Lemma 3.2 we can easily gain our main result.

Lemma 3.3. *For the (6,0)-nanotube F_n, it is 2-resonant.*

Proof. To prove the lemma, it suffices to show $F_n - (h_1 \cup h_2)$ has a perfect matching for any two disjoint hexagons h_1, h_2. By Lemma 3.2 M_k has been constructed. Next we find a perfect matching M_{k+1} of $F_n - (h_1 \cup h_2)$ from M_k as follows. For convenience, we give the labels of H_{k+1} as shown in Fig. 14(a).

If two of $h_k^1, h_k^2, h_k^3, h_k^4$ are h_1 and h_2, then by symmetry we may assume $h_k^1 = h_1, h_k^3 = h_2$ or $h_k^1 = h_1, h_k^4 = h_2$. By Lemma 3.2 no vertex of L_k matches to a vertex of L_{k+1} in M_k. So if $h_k^1 = h_1, h_k^3 = h_2$, then $M_{k+1} = M_k \cup \{v_1v_{10}, v_5v_6, v_{11}v_{12}\}$ is a perfect matching of $F_n - (h_1 \cup h_2)$ (see Fig. 14(b)). If $h_k^1 = h_1, h_k^4 = h_2$, then $M_{k+1} = M_k \cup \{v_5v_6, v_7v_8, v_{11}v_{12}\}$ is a perfect matching of $F_n - (h_1 \cup h_2)$ (see Fig. 14(c)).

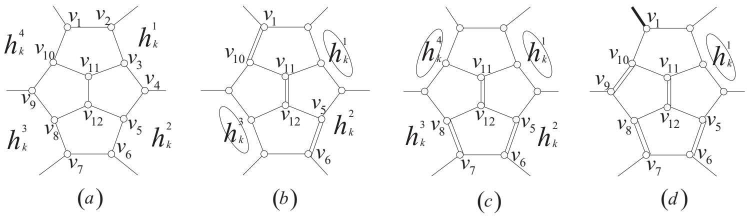

Fig. 14. (a) The labeling of H_{k+1}; (b)-(d) At least one of $h_k^1, \ldots, h_k^4$ being the hexagon h_1, h_2.

If one of $h_k^1, h_k^2, h_k^3, h_k^4$ is the hexagon h_1 or h_2, say $h_k^1 = h_1$, then by Lemma 3.2 a unique vertex of L_k matches to a vertex of L_{k+1} in M_k. Let the matched vertex on L_{k+1} be u. Then $u = v_1$ or $u = v_9$ or $u = v_7$ or $u = v_6$. If $u = v_1$, then $M_{k+1} = M_k \cup \{v_5v_6, v_7v_8, v_9v_{10}, v_{11}v_{12}\}$ is a perfect matching of $F_n - (h_1 \cup h_2)$ (see Fig. 14(d)). If $u = v_9$, then $M_{k+1} = M_k \cup \{v_5v_6, v_7v_8, v_1v_{10}, v_{11}v_{12}\}$ is a perfect matching of $F_n - (h_1 \cup h_2)$. If $u = v_7$, then $M_{k+1} = M_k \cup \{v_5v_6, v_8v_9, v_1v_{10}, v_{11}v_{12}\}$ is a perfect matching of $F_n - (h_1 \cup h_2)$. Lastly suppose $u = v_6$. Let $v_{13}, v_{14}, \ldots, v_{24}$ be the boundary of L_k and $h_{k-1}^1, h_{k-1}^2, \ldots, h_{k-1}^6$ the six faces of H_{k-1} as shown in Fig. 15(a). They by Lemma 3.2 none of $h_{k-1}^1, h_{k-1}^2, h_{k-1}^3$ equals h_2 and $v_{18}v_{25} \in M_k$, where $v_{18}v_{25}$ is an edge connecting v_{18} to its neighbor on L_{k-2} (see Fig. 15(a)). Now at most one of $h_{k-1}^4, h_{k-1}^5, h_{k-1}^6$ equals h_2. If none does, then two opposite vertices of L_{k-1} match to two opposite vertices of L_k by Lemma 3.2 and the two opposite vertices on L_k are v_{18}, v_{24}. Thus $\{v_{13}v_{14}, v_{22}v_{23}, v_{20}v_{21}\} \subset M_k$ (see Fig. 15(b)). Let $M_{k+1} = M_k - \{v_{20}v_{21}, v_6v_{19}\} \cup \{v_{19}v_{20}, v_7v_{21}, v_5v_6, v_8v_9, v_{11}v_{12}, v_1v_{10}\}$ (see Fig. 15(c)). Then M_{k+1} is a perfect matching of $F_n - (h_1 \cup h_2)$. If h_{k-1}^5 or h_{k-1}^6 equals h_2, say h_{k-1}^5, then analogously $\{v_{13}v_{14}, v_{20}v_{21}\} \subset M_k$ and $M_{k+1} = M_k - \{v_{20}v_{21}, v_6v_{19}\} \cup \{v_{19}v_{20}, v_7v_{21}, v_5v_6, v_8v_9, v_{11}v_{12}, v_1v_{10}\}$ is a perfect matching of $F_n - (h_1 \cup h_2)$. If h_{k-1}^4 equals

h_2, then two opposite vertices of L_{k-2} match to two opposite vertices of L_{k-1} by Lemma 3.2, and the two opposite vertices on L_{k-1} are assured, say s, t (see Fig. 15(d)). Now the M_k edges on L_k and L_{k-1} are known (see Fig. 15(d) the M_k edges $v_{26}v_{27}, v_{28}v_{29}, v_{30}v_{31}$ on L_{k-1} and $v_{13}v_{14}, v_{23}v_{24}$ on L_k). Let $M_{k+1} = M_k - \{v_6v_{19}, v_{18}v_{25}, v_{26}v_{27}, v_{28}v_{29}, v_{13}v_{14}\} \cup \{v_{18}v_{19}, v_{25}v_{26}, v_{27}v_{28}, v_{29}v_{14}, v_1v_{13}, v_9v_{10}, v_{11}v_{12}, v_7v_8, v_5v_6\}$ (see Fig. 15(e)). Then M_{k+1} forms a perfect matching of $F_n - (h_1 \cup h_2)$.

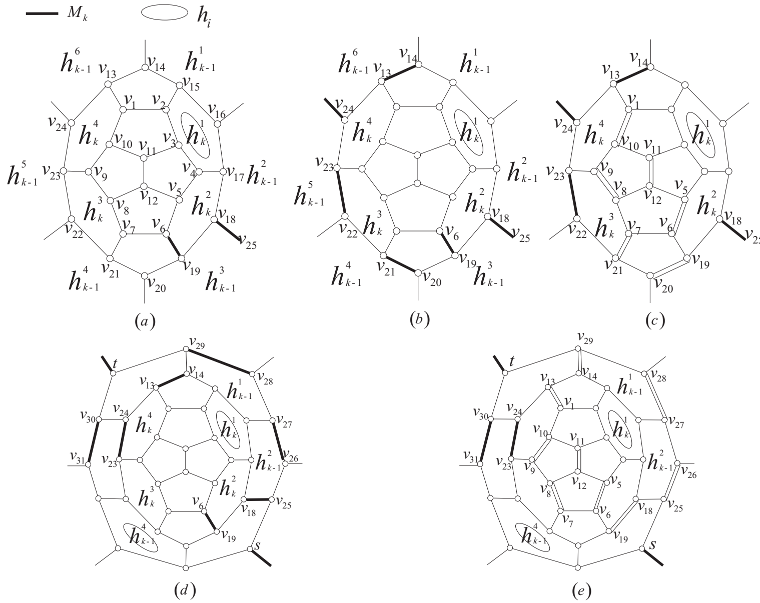

Fig. 15. The case $u = v_6$.

If none of $h_k^1, h_k^2, h_k^3, h_k^4$ equals h_1 or h_2, then by Lemma 3.2 two opposite vertices of L_k match to two opposite vertices of L_{k+1}. By symmetry, the two opposite vertices on L_{k+1} are v_2, v_7 or v_4, v_9. If v_2, v_7 do, then $M_{k+1} = M_k \cup \{v_1v_{10}, v_{11}v_{12}, v_8v_9, v_5v_6, v_3v_4\}$ is a perfect matching of $F_n - (h_1 \cup h_2)$. If v_4, v_9 do, then $M_{k+1} = M_k \cup \{v_1v_{10}, v_{11}v_{12}, v_7v_8, v_5v_6, v_2v_3\}$ is a perfect matching of $F_n - (h_1 \cup h_2)$. Until now the proof of Lemma 3.3 is completed. □

Proof of Theorem 1.3: From Proposition 3.1 and Lemma 3.3 we can know if $B_3 \subset F_n$, then F_n is 2-resonant. So next we may assume $B_3 \not\subseteq F_n$. For every B_i $(1 \leq i \leq k)$, let f be a face of B_i with t 2-degree vertices on its boundary. Then t 2-degree vertices separate f into t degree-saturated paths and the length of these degree-saturated paths is no more than three by Theorem 2.4, which means F_n cannot possess L as subgraph (see Fig. 1 the subgraphs L, R). Thus F_n is also 2-resonant by Theorem 1.2. Now Theorem 1.3 is proved. □

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