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On Laplacian Energy

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Abstract

Let G be a connected graph of order n with Laplacian eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$. The Laplacian energy of the graph G is defined as

$$LE = LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| .$$

Upper bounds for LE are obtained, in terms of n and the number of edges m.

1 Introduction

Let $G = (V, E)$ be a simple connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, $|E(G)| = m$. Let d_i be the degree of the vertex v_i for $i = 1, 2, \ldots, n$. The minimum vertex degree is denoted by δ. Let $\mathbf{A}(G)$ be the $(0,1)$-adjacency matrix of G and $\mathbf{D}(G)$ be the diagonal matrix of vertex degrees. The Laplacian matrix of G is $\mathbf{L}(G) = \mathbf{D}(G) - \mathbf{A}(G)$. This matrix has nonnegative eigenvalues $n \geq \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. Denote by $Spec(G) = \{\mu_1, \mu_2, \ldots, \mu_n\}$ the spectrum of $\mathbf{L}(G)$, i.e., the Laplacian spectrum of G. When more than one graph is under consideration, then we write $\mu_i(G)$ instead of μ_i.

As well known [7],

$$\sum_{i=1}^{n} \mu_i = 2m \; . \tag{1}$$

The Laplacian energy of the graph G is defined as [5]

$$LE = LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| . \tag{2}$$

For its basic properties, including various upper and lower bounds, see [1, 8, 9, 11, 13, 15, 16].

As usual, K_n, P_n, and $K_{1,n-1}$, denote, respectively, the complete graph, the path, and the star on n vertices.

2 Bounds on Laplacian energy

In this section, we give two upper bounds on LE for graphs in terms of n and m. In order to obtain this result, we need to recall some previously known results.

Lemma 2.1. [3] *Let G be a graph of order n, different from K_n, and let δ be its smallest vertex degree. Then*

$$\mu_{n-1} \leq \delta \; . \tag{3}$$

In [6], Haemers et al. presented the following result for tree of order n:

$$\sum_{i=1}^{k} \mu_i \leq n + 2k - 2 \quad (1 \leq k \leq n) \, .$$

In [4], Fritscher et al. improved the above result for tree of order n in the following:

$$\sum_{i=1}^{k} \mu_i \leq n + 2k - 2 - \frac{2k-2}{n} \quad (1 \leq k \leq n)\,. \tag{4}$$

Moreover, equality is achieved only when $k = 1$ and $T \cong K_{1,\,n-1}$.

The following result was obtained by one of the present authors [12].

Lemma 2.2. [12] *Let G be a connected graph of order n with m edges. Then for $1 \leq k \leq n-2$,*

$$\sum_{i=1}^{k} \mu_i \leq \frac{1}{n-1}\left[2mk + \sqrt{mk(n-k-1)(n^2-n-2m)}\right] \,.$$

If $k = 1$, then equality holds if and only if either $G \cong K_{1,n-1}$ or $G \cong K_1$. If $2 \leq k \leq n-2$, then equality holds if and only if $G \cong K_n$.

Remark 2.3. Combining inequality (4) and Lemma 2.2 from the recent paper [2] by Du and one of the present authors, we get

$$\sum_{i=1}^{k} \mu_i \leq 2m - n + 2k - \frac{2k-2}{n}$$

which is another upper bound different from the one used in Lemma 2.2.

We are now ready to state an upper bound on LE.

Theorem 2.4. *Let G $(\ncong K_n)$ be a connected graph of order n with m edges. Then*

$$LE(G) < \frac{2m}{n} + \sqrt{m\,(n^2-n-2m) + \left(\frac{2m}{n}\right)^2} \,. \tag{5}$$

Proof: Since $G \ncong K_n$, therefore $n \geq 3$. We have to prove that the inequality (5) is strict.

By Lemma 2.1, $\mu_{n-1} \leq \delta$ because $G \ncong K_n$. This implies

$$\frac{2m}{n} \geq \delta \geq \mu_{n-1} \,.$$

Suppose that k $(\leq n-2)$ is an integer such that

$$\mu_k \geq \frac{2m}{n} \qquad \text{and} \qquad \mu_{k+1} < \frac{2m}{n} \,.$$

Then by the definition of Laplacian energy, Eq. (2),

$$\begin{aligned} LE(G) &= \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| = \sum_{i=1}^{k} \left(\mu_i - \frac{2m}{n} \right) + \sum_{i=k+1}^{n} \left(\frac{2m}{n} - \mu_i \right) \\ &= \sum_{i=1}^{k} \mu_i - \sum_{i=k+1}^{n} \mu_i + \frac{2m}{n}(n-2k) = 2\sum_{i=1}^{k} \mu_i - \frac{4mk}{n} \end{aligned} \tag{6}$$

because by Eq. (1),

$$\sum_{i=k+1}^{n-1} \mu_i = 2m - \sum_{i=1}^{k} \mu_i \; .$$

Then by Lemma 2.2,

$$\begin{aligned} LE(G) &\leq \frac{4mk + 2\sqrt{mk\,(n-k-1)(n^2-n-2m)}}{n-1} - \frac{4mk}{n} \\ &= \frac{4mk + 2n\sqrt{mk\,(n-k-1)(n^2-n-2m)}}{n(n-1)} \; . \end{aligned} \tag{7}$$

Consider now the function

$$f(x) = 4mx + 2n\sqrt{mx(n-x-1)(n^2-n-2m)} \quad , \quad 1 \leq x \leq n-2 \, .$$

Then we have

$$f'(x) = 4m + \frac{(n-2x-1)n\sqrt{m(n^2-n-2m)}}{\sqrt{nx-x^2-x}} \, .$$

Thus $f(x)$ is an increasing function on

$$1 \leq x \leq \frac{n-1}{2} + \frac{(n-1)\sqrt{m}}{\sqrt{n^2(n^2-n-2m)+4m}}$$

and decreasing function on

$$\frac{n-1}{2} + \frac{(n-1)\sqrt{m}}{\sqrt{n^2(n^2-n-2m)+4m}} \leq x \leq n-2 \; .$$

Consequently, $f(x)$ has maximum value at

$$x = \frac{n-1}{2} + \frac{(n-1)\sqrt{m}}{\sqrt{n^2(n^2-n-2m)+4m}} \; .$$

Hence

$$f(x) \leq 2m(n-1) + (n-1)\sqrt{m\left[n^2(n^2-n-2m)+4m\right]} \, .$$

Bearing this in mind, from (7) we arrive at (5).

Suppose now that equality holds in (5). Then all the above inequalities must be equalities. Thus,

$$k = \frac{n-1}{2} + \frac{(n-1)\sqrt{m}}{\sqrt{n^2(n^2-n-2m)+4m}} \geq 2$$

as $n \geq 3$ and k is an integer. Since $G \not\cong K_n$, Lemma 2.2 would imply $k = 1$, a contradiction as $k \geq 2$. This completes the proof. ■

Remark 2.5. In [14], it was shown that under the conditions of Theorem 2.4,

$$LE(G) < 4m - \frac{4m}{n}$$

Comparing this bound with (5), we find that the new upper bound is better than the previous one if and only if

$$m > (n^2-n)\left(18 - \frac{48}{n} + \frac{32}{n^2}\right)^{-1} .$$

For obtaining our second bound on LE, we need additional earlier known lemma.

Lemma 2.6. [10] *Let* $\mathbf{B}$ *be a* $p \times p$ *symmetric matrix and let* $\mathbf{B}_k$ *be its leading* $k \times k$ *submatrix. Then, for* $i = 1, 2, \ldots, k$,

$$\lambda_{p-i+1}(\mathbf{B}) \leq \lambda_{k-i+1}(\mathbf{B}_k) \leq \lambda_{k-i+1}(\mathbf{B}) \tag{8}$$

where $\lambda_i(\mathbf{B})$ *is the* i*-th greatest eigenvalue of* $\mathbf{B}$.

We now give another upper bound on Laplacian energy $LE(G)$ of graph G in terms on n, m and number of pendent vertices p.

Theorem 2.7. *Let* G *be a connected graph of order* n *with* m *edges and number of pendent vertices* p $\left(p \geq \frac{n+1}{2}\right)$. *Then*

$$LE(G) \leq \frac{4m(n-p) + 2n\sqrt{m(n-p)\,(p-1)(n^2-n-2m)}}{n(n-1)} \tag{9}$$

with equality if and only if $G \cong K_{1,n-1}$.

Proof: Since G contains p pendent vertices (say $v_1, v_2, \ldots, v_p$), then by (8),

$$\mu_{n-p+1}(G) \leq \mu_1(\mathbf{B}_p)$$

where $\mathbf{B}_p$ is the $p \times p$ submatrix of $\mathbf{L}(G)$ consisting of the entries (1, 1), (1, 2), (1, 3), …, (1, p), (2, 1), (2, 2), (2, 3), …, (2, p), …, (p, 1), (p, 2), (p, 3), …, (p, p). Thus,

$$\mu_{n-p+1}(G) \leq \mu_1(\mathbf{B}_p) = \mu_1(\mathbf{I}_p) = 1 < \frac{2m}{n} \text{ as } G \text{ is connected and hence } m \geq n-1,$$

where $\mathbf{I}_p$ is the $p \times p$ unit matrix. From this we conclude that there exists an integer k $(k \leq n-p)$, such that

$$\mu_k \geq \frac{2m}{n} \qquad \text{and} \qquad \mu_{k+1} < \frac{2m}{n} \, .$$

From (7), we get

$$LE(G) \leq \frac{4mk + 2n\sqrt{mk\,(n-k-1)(n^2-n-2m)}}{n(n-1)} \, .$$

Since

$$f(x) = 4mx + 2n\sqrt{mx(n-x-1)(n^2-n-2m)}$$

is an increasing function on

$$1 \leq x \leq \frac{n-1}{2} + \frac{(n-1)\sqrt{m}}{\sqrt{n^2(n^2-n-2m)+4m}}$$

from the above, we get

$$LE(G) \leq \frac{4m(n-p) + 2n\sqrt{m(n-p)\,(p-1)(n^2-n-2m)}}{n(n-1)}$$

as $k \leq n-p \leq \frac{n-1}{2}$. By this, the first part of the proof is done.

Now suppose that equality holds in (9). Then all inequalities in the above argument must be equalities. Since $p \geq \frac{n+1}{2}$ $(n \geq 3)$, we have $G \not\cong K_n$. Lemma 2.2 would imply $k = 1$. Thus we have $k = n - p = 1$, by Lemma 2.2. Therefore $p = n-1$ and hence $G \cong K_{1,n-1}$.

Conversely, one can see easily that the equality holds in (9) for the star $K_{1,n-1}$.

■

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