Upper Bounds for the Energy of Graphs

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Abstract

The energy of a graph $G$, denoted by $E(G)$, is defined as the sum of the absolute values of all eigenvalues of $G$. In this paper we present some new upper bounds for $E(G)$ in terms of number of vertices, number of edges, clique number, minimum degree, and the first Zagreb index.

1 Introduction

Let $G = (V, E)$ be a simple connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, $|E(G)| = m$. Let $d_i$ be the degree of the vertex $v_i$ for $i = 1, 2, \ldots, n$. The maximum and minimum vertex degrees are denoted by $\Delta$ and $\delta$, respectively. Let $N_i$ be the neighbor set of the vertex $v_i \in V$. Denote by $\omega$ the clique number of the graph $G$. If the vertices $v_i$ and $v_j$ are adjacent, we denote this by $v_iv_j \in E(G)$. The adjacency matrix $A(G)$ of $G$ is defined by its entries as $a_{ij} = 1$ if $v_iv_j \in E(G)$ and 0 otherwise. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n$ denote the eigenvalues of $A(G)$. $\lambda_1$ is called the spectral radius of the graph $G$. When more than one graphs are under consideration, then we write $\lambda_i(G)$ instead of $\lambda_i$.

Some well known results on graph eigenvalues are the following:

$$\sum_{i=1}^{n} \lambda_i = 0, \quad \sum_{i=1}^{n} \lambda_i^2 = 2m.$$
For more details see the monograph [1].

The energy of the graph $G$ is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$ 

For more details on the theory of graph energy see the monograph [3].

The paper is organized as follows. In Section 2, we give a list of some previously known results. In Section 3, we obtain two new upper bounds on energy $E(G)$ of the graph $G$.

### 2 Preliminaries

We list here some previously known results that will be needed in the subsequent sections.

**Lemma 2.1.** [6] Let $B$ be a $p \times p$ symmetric matrix and let $B_k$ be its leading $k \times k$ submatrix; that is, $B_k$ is matrix obtained from $B$ by deleting its last $p-k$ rows and columns. Then for $i = 1, 2, \ldots, k$

$$\rho_{p-i+1}(B) \leq \rho_{k-i+1}(B_k) \leq \rho_{k-i+1}(B)$$

where $\rho_i(B)$ is the $i$-th largest eigenvalue of $B$.

**Lemma 2.2.** [5] Let $F = \left\{ X = (x_1, x_2, \ldots, x_n) : x_i \geq 0, \sum_{i=1}^{n} x_i = 1 \right\}$. Then

$$1 - \frac{1}{\omega(G)} = \max_{X \in F} \langle X, AX \rangle.$$ 

**Lemma 2.3.** (Schur’s lemma [5]) For each rectangular array $\{c_{jk} : 1 \leq j \leq m, 1 \leq k \leq n\}$ and each pair of sequences $\{x_j : 1 \leq j \leq m\}$ and $\{y_k : 1 \leq k \leq n\}$,

$$\left| \sum_{j=1}^{m} \sum_{k=1}^{n} c_{jk} x_j y_k \right| \leq \sqrt{RC} \left( \sum_{j=1}^{m} |x_j|^2 \right)^{1/2} \left( \sum_{k=1}^{n} |y_k|^2 \right)^{1/2}$$

where $R$ and $C$ are the row sum and column sum maxima defined by

$$R = \max_j \sum_{k=1}^{n} |c_{jk}| \quad \text{and} \quad C = \max_k \sum_{j=1}^{m} |c_{jk}|.$$
3 Upper bound on energy of graphs

In this section we give two upper bounds on energy. First we mention three earlier known such bounds. McClelland’s famous bound is \[ E \leq \sqrt{2mn} \]. It was improved by Koolen and Moulton [2]:

\[
E(G) \leq \frac{2m}{n} + \sqrt{(n - 1) \left[ 2m - \left( \frac{2m}{n} \right)^2 \right]}.
\] (1)

Zhou [7] obtained an upper bound

\[
E(G) \leq \sqrt{\frac{M_1}{n} + \sqrt{(n - 1) \left( 2m - \frac{M_1}{n} \right)}}
\] (2)

where \( M_1 \) stands for the first Zagreb index.

We are now ready to give an upper bound on energy in terms of \( n, m, \delta, \) and \( \omega \).

**Theorem 3.1.** Let \( G \) be a graph of order \( n \), with \( m \) edges, with minimum degree \( \delta \), and clique number \( \omega \). Then

\[
E(G) \leq \sqrt{2m(n - \delta)} + 4\sqrt{m^3(1 - 1/\omega)}.
\] (3)

**Proof:** First we have to prove that

\[
\sum_{v_j, v_k \in E(G)} |\lambda_j \lambda_k| \leq (n - 1 - \delta)m.
\] (4)

Putting \( c_{jk} = a_{jk}(G^c) \), \( x_j = |\lambda_j| \) and \( y_k = |\lambda_k| \), in Lemma 2.3, we get

\[
\sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk}(G^c)|\lambda_j||\lambda_k| \leq \Delta(G^c) \sqrt{2m} \sqrt{2m} \quad \text{as} \quad \sum_{j=1}^{n} \lambda_j^2 = 2m
\]

that is,

\[
2 \sum_{v_j, v_k \in E(G)} |\lambda_j||\lambda_k| \leq (n - 1 - \delta)2m
\]

which gives the result in (4).

Next we have to prove that

\[
\sum_{v_j, v_k \in E(G)} |\lambda_j \lambda_k| \leq \sqrt{4m^3(1 - 1/\omega)}.
\] (5)

Putting \( X = \frac{1}{2m}(\lambda_1^2, \ldots, \lambda_n^2) \) in Lemma 2.2, we have

\[
1 - 1/\omega \geq \frac{1}{4m^2} \sum_{v_j, v_k \in E(G)} \lambda_j^2 \lambda_k^2.
\]
By the Cauchy–Schwarz inequality and using the above result, we get

\[
\frac{1}{m} \left( \sum_{v_jv_k \in E(G)} |\lambda_j\lambda_k| \right)^2 \leq \sum_{v_jv_k \in E(G)} (\lambda_j\lambda_k)^2 \leq 4m^2(1 - 1/\omega)
\]

that is,

\[
\left( \sum_{v_jv_k \in E(G)} |\lambda_j\lambda_k| \right)^2 \leq 4m^3(1 - 1/\omega)
\]

which gives the result in (5).

Now,

\[
\left( \sum_{i=1}^{n} |\lambda_i| \right)^2 = \sum_{i=1}^{n} \lambda_i^2 + 2 \sum_{1 \leq j < k \leq n} |\lambda_j\lambda_k|,
\]

that is,

\[
E^2 = 2m + 2 \sum_{v_jv_k \in E(G)} |\lambda_j\lambda_k| + 2 \sum_{v_jv_k \notin E(G)} |\lambda_j\lambda_k|
\]

\[
\leq 2m + 4\sqrt{m^3(1 - 1/\omega)} + 2m(n - 1 - \delta) \quad \text{by (4) and (5)}.
\]

This completes the proof. \(\square\)

**Theorem 3.2.** Let \(G\) be a connected graph of order \(n\) \((n \geq 6)\), with \(m\) edges, and with minimum degree \(\delta\). Then

\[
E(G) \leq \frac{2(m - \delta)}{n - 1} + \sqrt{(n - 1) \left[ 2m - 4(m - \delta)^2 \right]}.
\]

**Proof:** First we have to prove that

\[
\lambda_1(G) \geq \frac{2(m - \delta)}{n - 1}.
\]

By Lemma 2.1,

\[
\lambda_1(G) \geq \lambda_1(G')
\]

where \(\lambda_1(G') = \lambda_1(A_{n-1})\) and \(A_{n-1}\) is the leading \((n - 1) \times (n - 1)\) submatrix of \(A(G)\), obtained from \(A(G)\) by deleting its last row and column corresponding minimum degree vertex of degree \(\delta\). Thus we have

\[
\lambda_1(G) \geq \lambda_1(G') \geq \frac{2m'}{n'} = \frac{2(m - \delta)}{n - 1}
\]

(7)
where $m'$ and $n'$ are the number of edges and the number of vertices in $G'$, respectively.

Claim 1. For $n \geq 6$,
\[
\frac{2(m - \delta)}{n - 1} \geq \sqrt{\frac{2m}{n}}. \tag{8}
\]

Proof of Claim 1: We have to prove that
\[
\frac{(m - \delta)^2}{m} \geq \frac{(n - 1)^2}{2n}
\]
that is,
\[
m - 2\delta + \frac{\delta^2}{m} \geq \frac{n}{2} - 1 + \frac{1}{2n}
\]
that is,
\[
2m - 4\delta \geq n - 2 + \frac{1}{n} - \frac{2\delta^2}{m}. \tag{9}
\]

For $\delta = 1$, one can easily see that the result in (9) holds as $m \geq n - 1$ and $n \geq 6$. Otherwise, $\delta \geq 2$. We now assume that $d_1 \geq d_2 \geq \cdots \geq d_n$. Thus we have
\[
2m - 4\delta \geq \sum_{i=1}^{n-4} d_i \geq 2(n - 4) \geq n - 2 + \frac{1}{n} - \frac{2\delta^2}{m} \text{ as } n \geq 6.
\]

By the Cauchy–Schwarz inequality,
\[
E(G) \leq \lambda_1 + \sqrt{(n - 1)(2m - \lambda_1^2)}.
\]

One can see easily that
\[
f(x) = x + \sqrt{(n - 1)(2m - x^2)}
\]
is a decreasing function on $\left(\sqrt{\frac{2m}{n}}, \sqrt{2m}\right)$. By Claim 1,
\[
\sqrt{\frac{2m}{n}} \leq \frac{2(m - \delta)}{n - 1} \leq \lambda_1(G).
\]

From the above, we get
\[
E(G) \leq \frac{2(m - \delta)}{n - 1} + \sqrt{(n - 1) \left[2m - \frac{4(m - \delta)^2}{(n - 1)^2}\right]}.
\]

\qed

Corollary 3.3. Let $G$ be a connected graph of order $n$ ($n \geq 6$), with $m$ edges, and with at least a pendent vertex. Then
\[
E(G) \leq \frac{2(m - 1)}{n - 1} + \sqrt{(n - 1) \left[2m - \frac{4(m - 1)^2}{(n - 1)^2}\right]} \tag{10}
\]
Proof: Since $G$ has a pendent vertex, according to Theorem 3.2, we get

$$\lambda_1(G) \geq \frac{2(m - 1)}{n - 1}$$

and hence the result (10).

\[\square\]

**Remark 3.4.** Our result (6) is better than (1) for

$$\frac{2(m - \delta)}{n - 1} \geq \frac{2m}{n},$$

that is,

$$m \geq n\delta.$$

**Remark 3.5.** Our result (6) is better than (2) for

$$\frac{2(m - \delta)}{n - 1} \geq \sqrt{\frac{M_1(G)}{n}}$$

that is, if the condition $4n(m - \delta)^2 \geq M_1(G)(n - 1)^2$ is obeyed. In particular, (6) is better than (2) for the complete graph $K_{n-1}$ to which a pendent edge is attached.

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**References**


