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Upper Bounds for the Energy of Graphs

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Abstract

The energy of a graph G, denoted by $E(G)$, is defined as the sum of the absolute values of all eigenvalues of G. In this paper we present some new upper bounds for $E(G)$ in terms of number of vertices, number of edges, clique number, minimum degree, and the first Zagreb index.

1 Introduction

Let $G = (V, E)$ be a simple connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, $|E(G)| = m$. Let d_i be the degree of the vertex v_i for $i = 1, 2, \ldots, n$. The maximum and minimum vertex degrees are denoted by Δ and δ, respectively. Let N_i be the neighbor set of the vertex $v_i \in V$. Denote by ω the clique number of the graph G. If the vertices v_i and v_j are adjacent, we denote this by $v_iv_j \in E(G)$. The adjacency matrix $A(G)$ of G is defined by its entries as $a_{ij} = 1$ if $v_iv_j \in E(G)$ and 0 otherwise. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n$ denote the eigenvalues of $A(G)$. λ_1 is called the spectral radius of the graph G. When more than one graphs are under consideration, then we write $\lambda_i(G)$ instead of λ_i.

Some well known results on graph eigenvalues are the following:

$$\sum_{i=1}^{n} \lambda_i = 0 \qquad , \qquad \sum_{i=1}^{n} {\lambda_i}^2 = 2m \; .$$

For more details see the monograph [1].

The energy of the graph G is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i| \ .$$

For more details on the theory of graph energy see the monograph [3].

The paper is organized as follows. In Section 2, we give a list of some previously known results. In Section 3, we obtain two new upper bounds on energy $E(G)$ of the graph G.

2 Preliminaries

We list here some previously known results that will be needed in the subsequent sections.

Lemma 2.1. [6] *Let B be a $p \times p$ symmetric matrix and let B_k be its leading $k \times k$ submatrix; that is, B_k is matrix obtained from B by deleting its last $p-k$ rows and columns. Then for $i = 1, 2, \ldots, k$*

$$\rho_{p-i+1}(B) \leq \rho_{k-i+1}(B_k) \leq \rho_{k-i+1}(B)$$

where $\rho_i(B)$ is the i-th largest eigenvalue of B.

Lemma 2.2. [5] *Let $\mathbf{F} = \left\{ X = (x_1, x_2, \ldots, x_n) \ : \ x_i \geq 0 \ , \ \sum_{i=1}^{n} x_i = 1 \right\}$. Then*

$$1 - \frac{1}{\omega(G)} = \max_{X \in \mathbf{F}} \langle X, \, AX \rangle \ .$$

Lemma 2.3. *(Schur's lemma [5]) For each rectangular array $\{c_{jk} : 1 \leq j \leq m, 1 \leq k \leq n\}$ and each pair of sequences $\{x_j : 1 \leq j \leq m\}$ and $\{y_k : 1 \leq k \leq n\}$,*

$$\left| \sum_{j=1}^{m} \sum_{k=1}^{n} c_{jk} \ x_j y_k \right| \leq \sqrt{RC} \left(\sum_{j=1}^{m} |x_j|^2 \right)^{1/2} \left(\sum_{k=1}^{n} |y_k|^2 \right)^{1/2}$$

where R and C are the row sum and column sum maxima defined by

$$R = \max_{j} \sum_{k=1}^{n} |c_{jk}| \qquad \text{and} \qquad C = \max_{k} \sum_{j=1}^{m} |c_{jk}| \ .$$

3 Upper bound on energy of graphs

In this section we give two upper bounds on energy . First we mention three earlier known such bounds. McClelland's famous bound is [4] $E \leq \sqrt{2mn}$. It was improved by Koolen and Moulton [2]:

$$E(G) \leq \frac{2m}{n} + \sqrt{(n-1)\left[2m - \left(\frac{2m}{n}\right)^2\right]} . \tag{1}$$

Zhou [7] obtained an upper bound

$$E(G) \leq \sqrt{\frac{M_1}{n}} + \sqrt{(n-1)\left(2m - \frac{M_1}{n}\right)} \tag{2}$$

where M_1 stands for the first Zagreb index.

We are now ready to give an upper bound on energy in terms of n, m, δ, and ω .

Theorem 3.1. *Let G be a graph of order n, with m edges, with minimum degree δ, and clique number ω. Then*

$$E(G) \leq \sqrt{2m(n-\delta) + 4\sqrt{m^3(1-1/\omega)}} . \tag{3}$$

Proof: First we have to prove that

$$\sum_{v_j v_k \notin E(G)} |\lambda_j \, \lambda_k| \leq (n-1-\delta)m . \tag{4}$$

Putting $c_{jk} = a_{jk}(G^c)$, $x_j = |\lambda_j|$ and $y_k = |\lambda_k|$, in Lemma 2.3, we get

$$\sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk}(G^c)|\lambda_j||\lambda_k| \leq \Delta(G^c)\sqrt{2m}\sqrt{2m} \quad \text{as} \quad \sum_{j=1}^{n} \lambda_j^2 = 2m$$

that is,

$$2 \sum_{v_j v_k \notin E(G)} |\lambda_j||\lambda_k| \leq (n-1-\delta)2m$$

which gives the result in (4).

Next we have to prove that

$$\sum_{v_j v_k \in E(G)} |\lambda_j \, \lambda_k| \leq \sqrt{4m^3(1-1/\omega)} . \tag{5}$$

Putting $X = \frac{1}{2m}(\lambda_1^2, \ldots, \lambda_n^2)$ in Lemma 2.2, we have

$$1 - 1/\omega \geq \frac{1}{4m^2} \sum_{v_j v_k \in E(G)} \lambda_j^2 \lambda_k^2 .$$

By the Cauchy–Schwarz inequality and using the above result, we get

$$\frac{1}{m}\left(\sum_{v_jv_k\in E(G)}|\lambda_j\lambda_k|\right)^2 \leq \sum_{v_jv_k\in E(G)}(\lambda_j\lambda_k)^2 \leq 4m^2(1-1/\omega)$$

that is,

$$\left(\sum_{v_jv_k\in E(G)}|\lambda_j\lambda_k|\right)^2 \leq 4m^3(1-1/\omega)$$

which gives the result in (5).

Now,

$$\left(\sum_{i=1}^{n}|\lambda_i|\right)^2 = \sum_{i=1}^{n}\lambda_i^2 + 2\sum_{1\leq j<k\leq n}|\lambda_j\lambda_k|\,,$$

that is,

$$\begin{aligned} E^2 &= 2m + 2\sum_{v_jv_k\in E(G)}|\lambda_j\,\lambda_k| + 2\sum_{v_jv_k\notin E(G)}|\lambda_j\,\lambda_k| \\ &\leq 2m + 4\sqrt{m^3(1-1/\omega)} + 2m(n-1-\delta) \quad \text{by (4) and (5)}\,. \end{aligned}$$

This completes the proof. □

Theorem 3.2. *Let G be a connected graph of order n ($n \geq 6$), with m edges, and with minimum degree δ. Then*

$$E(G) \leq \frac{2(m-\delta)}{n-1} + \sqrt{(n-1)\left[2m - \frac{4(m-\delta)^2}{(n-1)^2}\right]}\,. \tag{6}$$

Proof: First we have to prove that

$$\lambda_1(G) \geq \frac{2(m-\delta)}{n-1}\,.$$

By Lemma 2.1,

$$\lambda_1(G) \geq \lambda_1(G')$$

where $\lambda_1(G') = \lambda_1(A_{n-1})$ and A_{n-1} is the leading $(n-1)\times(n-1)$ submatrix of $A(G)$, obtained from $A(G)$ by deleting its last row and column corresponding minimum degree vertex of degree δ. Thus we have

$$\lambda_1(G) \geq \lambda_1(G') \geq \frac{2m'}{n'} = \frac{2(m-\delta)}{n-1} \tag{7}$$

where m' and n' are the number of edges and the number of vertices in G', respectively.

Claim 1. For $n \geq 6$,

$$\frac{2(m-\delta)}{n-1} \geq \sqrt{\frac{2m}{n}} . \tag{8}$$

Proof of Claim 1: We have to prove that

$$\frac{(m-\delta)^2}{m} \geq \frac{(n-1)^2}{2n}$$

that is,

$$m - 2\delta + \frac{\delta^2}{m} \geq \frac{n}{2} - 1 + \frac{1}{2n}$$

that is,

$$2m - 4\delta \geq n - 2 + \frac{1}{n} - \frac{2\delta^2}{m} . \tag{9}$$

For $\delta = 1$, one can easily see that the result in (9) holds as $m \geq n-1$ and $n \geq 6$. Otherwise. $\delta \geq 2$. We now assume that $d_1 \geq d_2 \geq \cdots \geq d_n$. Thus we have

$$2m - 4\delta \geq \sum_{i=1}^{n-4} d_i \geq 2(n-4) \geq n - 2 + \frac{1}{n} - \frac{2\delta^2}{m} \quad \text{as } n \geq 6.$$

By the Cauchy–Schwarz inequality,

$$E(G) \leq \lambda_1 + \sqrt{(n-1)(2m - \lambda_1^2)} .$$

One can see easily that

$$f(x) = x + \sqrt{(n-1)(2m - x^2)}$$

is a decreasing function on $\left(\sqrt{\frac{2m}{n}}, \sqrt{2m}\right)$. By **Claim 1**,

$$\sqrt{\frac{2m}{n}} \leq \frac{2(m-\delta)}{n-1} \leq \lambda_1(G) .$$

From the above, we get

$$E(G) \leq \frac{2(m-\delta)}{n-1} + \sqrt{(n-1)\left[2m - \frac{4(m-\delta)^2}{(n-1)^2}\right]} .$$

□

Corollary 3.3. *Let G be a connected graph of order n ($n \geq 6$), with m edges, and with at least a pendent vertex. Then*

$$E(G) \leq \frac{2(m-1)}{n-1} + \sqrt{(n-1)\left[2m - \frac{4(m-1)^2}{(n-1)^2}\right]} . \tag{10}$$

Proof: Since G has a pendent vertex, according to Theorem 3.2, we get

$$\lambda_1(G) \geq \frac{2(m-1)}{n-1}$$

and hence the result (10). □

Remark 3.4. *Our result (6) is better than (1) for*

$$\frac{2(m-\delta)}{n-1} \geq \frac{2m}{n},$$

that is,

$$m \geq n\delta .$$

Remark 3.5. *Our result (6) is better than (2) for*

$$\frac{2(m-\delta)}{n-1} \geq \sqrt{\frac{M_1(G)}{n}}$$

that is, if the condition $4n(m-\delta)^2 \geq M_1(G)(n-1)^2$ *is obeyed. In particular, (6) is better than (2) for the complete graph* K_{n-1} *to which a pendent edge is attached.*

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