Maximal Balaban Index of Graphs *

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Abstract

The Balaban index (also called J index) of a connected graph $G$ is denoted as

$$J(G) = \frac{|E(G)|}{\mu + 1} \sum_{uv \in E(G)} \frac{1}{\sqrt{\sigma_G(u)\sigma_G(v)}}$$

where $\sigma_G(u) = \sum_{w \in V(G)} d_G(u, w)$ and $\mu$ is the cycloomatic number. It has been used in various QSAR and QSPR studies. Let $U^3_n$ be an unicyclic graph obtained from a triangle $C_3$ by attaching $n - 3$ pendent edges at one vertex of $C_3$. Let $B_n$ ($n \geq 4$) be a bicyclic graph obtained from $U^3_n$ by adding an edge between one pendent vertex and a vertex of degree 2 of $U^3_n$. In this paper, we show that $U^3_n$ and $B_n$ have the largest Balaban index among all $n$-vertex unicyclic graphs and $n$-vertex bicyclic graphs, respectively.

1 Introduction

For a simple graph $G$, denote the edge set and vertex set of $G$ by $E(G)$ and $V(G)$, respectively. Let $m = |E(G)|$ and $n = |V(G)|$, i.e., the edge number and vertex number of $G$. $N_G(v)$ denotes the set of neighbors of vertex $v$ in $G$. If $H$ is subgraph of $G$, its edge set and vertex set are denoted by $E_G(H)$ and $V_G(H)$, respectively. If $Y$ is a vertex subset of $V(G)$, the vertex-induced subgraph of $G$ induced by $Y$ is denoted by $G[Y]$. The distance between vertices $u$ and $v$ in $G$ is denoted by $d_G(u, v)$, and the sum of the distance between vertex $u$ and each vertex of $G$ is denoted by $\sigma_G(u)$. That is $\sigma_G(u) = \sum_{w \in V(G)} d_G(u, w)$.

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In 1982, A.T. Balaban in [1] introduced a new topological index for a connected graph $G$, which is called the Balaban index nowadays or $J$ index for short. It is defined as

$$J(G) = \frac{|E(G)|}{\mu + 1} \sum_{uv \in E(G)} \frac{1}{\sqrt{\sigma_G(u)\sigma_G(v)}}$$

where $\mu$ is the cyclomatic number.

The Balaban index has been widely used in various QSAR and QSPR studies. The Balaban index and Wiener index [2] are two kinds of important topological indices based on the distance. Balaban et al. [3] compared the sequence of the isomers of alkane with $k$ carbon atoms, where $6 \leq k \leq 9$. The result shows that the sequence of the isomers of alkane based on the Balaban index is parallel with that based on the Wiener index. Moreover, the former has smaller degeneracy than latter, which means that using the Balaban index to characterize molecular structure is better than the Wiener index. Until now, there are many results on the maximal and minimal Wiener index [4–8]. However, there are few similar results on the Balaban index [9–11].

We denoted by $C_n$ a $n$-vertex cycle. Let $U_n^3$ be an unicyclic graph obtained from $C_3$ by attaching $n - 3$ pendant edges at one vertex of $C_3$. Let $B_n$ be a bicyclic graph obtained from $U_n^3$ by adding an edge between one pendant vertex and a vertex of degree 2 of $U_n^3$. Two graphs $U_n^3$ and $B_n$ are shown in Fig 1. In this paper, we show that $U_n^3$ and $B_n$ have the largest Balaban index among all $n$-vertex unicyclic graphs and $n$-vertex bicyclic graphs, respectively.

![Graphs $U_n^3$ and $B_n$](image)

Fig. 1. Graphs $U_n^3$ and $B_n$

## 2 Preliminaries

Let $G$ be a graph, and $G - xy$ denotes the graph that arises from $G$ by deleting the edge $xy \in E(G)$. Similarly, $G + xy$ is a graph that arises from $G$ by adding an edge $xy \notin E(G)$, where $x, y \in V(G)$. We will use $\mathcal{U}_n$ and $\mathcal{B}_n$ to denote the sets of all unicyclic and bicyclic graphs with $n$ vertices, respectively.

For each $i$, $i = 1, 2, \cdots, g$, let $T_i$ be a rooted tree attached at the root vertex $w_i$ of the cycle $C_g$. Of course, some of $T_i$s may be just the root $w_i$. Then we may denote
the \( n \)-vertex unicyclic graph by \( U_n(T_{w_1}, T_{w_2}, \ldots, T_{w_g}) \). The center of a star means its vertex with maximum degree. If each \( T_{w_i}, i = 1, 2, \ldots, g \), is a star \( S_{w_i} \) whose root is \( w_i \), the center of \( S_{w_i} \), the \( n \)-vertex unicyclic graph \( U_n(T_{w_1}, T_{w_2}, \ldots, T_{w_g}) \) is denoted by \( U_n(S_{w_1}, S_{w_2}, \ldots, S_{w_g}) \). The set of all this kind of unicyclic graphs is denoted by \( U^*_n \).

3 Main results

The main result in this section is as follows.

**Theorem 1.** The graph \( U^3_n \) has the largest Balaban index among all \( n \)-vertex unicyclic graphs.

Before given the proof of Theorem 1, we need some more preparations.

Let \( G_1 \) and \( G_2 \) be two graphs, and \( v_i \in V(G_i), i = 1, 2 \). Suppose \( n_1 = |V(G_1)| \) and \( n_2 = |V(G_2)| \), where \( n_1 \geq 1, n_2 \geq 1 \). The graph \( H_1 \) is obtained from \( G_1 \) and \( G_2 \) by joining two vertices \( v_1 \) and \( v_2 \) by an edge \( v_1 v_2 \). The graph \( H_2 \) is obtained from \( H_1 \) by identifying two vertices \( v_1 \) and \( v_2 \) and changing the edge \( v_1 v_2 \) into a pendent edge attached at vertex \( v_1 \), that is,

\[
H_2 = H_1 - \{v_2 x | x \in N_{H_1}(v_2) \setminus \{v_1\}\} + \{v_1 x | x \in N_{H_1}(v_2) \setminus \{v_1\}\}
\]

Two graphs \( H_1 \) and \( H_2 \) are shown in Fig 2.

![Fig. 2. Graphs \( H_1 \) and \( H_2 \)](image)

**Lemma 1.** \( J(H_1) \leq J(H_2) \).

**Proof.** By the definition of the Balaban index, we have

\[
\frac{\mu + 1}{m} J(H_1) = \sum_{uv \in E_{H_1}(G_1)} \frac{1}{\sqrt{\sigma_{H_1}(u)\sigma_{H_1}(v)}} + \sum_{ab \in E_{H_1}(G_2)} \frac{1}{\sqrt{\sigma_{H_1}(a)\sigma_{H_1}(b)}} + \frac{1}{\sqrt{\sigma_{H_1}(v_1)\sigma_{H_1}(v_2)}}
\]
\[
\frac{\mu + 1}{m} J(H_2) = \sum_{uv \in E_{H_2}(G_1)} \frac{1}{\sqrt{\sigma_{H_2}(u)\sigma_{H_2}(v)}} + \sum_{ab \in E_{H_2}(G_2)} \frac{1}{\sqrt{\sigma_{H_2}(a)\sigma_{H_2}(b)}} + \frac{1}{\sqrt{\sigma_{H_2}(v_1)\sigma_{H_2}(v_2)}} \tag{2}
\]

For any vertex \( u \in V_{H_2}(G_1) \) (i.e., \( u \in V_{H_2}(G_1) \)), it’s obvious that \( \sigma_{H_1}(u) \geq \sigma_{H_2}(u) \). Thus,
\[
\sum_{uv \in E_{H_1}(G_1)} \frac{1}{\sqrt{\sigma_{H_1}(u)\sigma_{H_1}(v)}} \leq \sum_{uv \in E_{H_2}(G_1)} \frac{1}{\sqrt{\sigma_{H_2}(u)\sigma_{H_2}(v)}} \tag{3}
\]

Claim 1.1.
\[
\sum_{ab \in E_{H_1}(G_2)} \frac{1}{\sqrt{\sigma_{H_1}(a)\sigma_{H_1}(b)}} \leq \sum_{ab \in E_{H_2}(G_2)} \frac{1}{\sqrt{\sigma_{H_2}(a)\sigma_{H_2}(b)}}
\]

In fact, it is easy to see that
\[
V_{H_2}(G_1) = V_{H_1}(G_1), \quad V_{H_2}(G_2) = V_{H_1}(G_2) - \{v_2\} + \{v_1\}
\]

Let \( N_2 \) be the set of all neighbors of \( v_2 \) in the subgraph \( G_2 \) of \( H_1 \), i.e., \( N_2 = \{y|y \in E_{H_1}(G_2)\} \). Obviously, \( N_2 = \{y|v_1y \in E_{H_2}(G_2)\} \). For any vertex \( v \in V_{H_1}(G_2) - \{v_2\} \) (i.e., \( v \in V_{H_2}(G_2) - \{v_1\} \)), we have
\[
\sigma_{H_1}(v) = \sum_{x \in V_{H_1}(G_2) - \{v_2\}} d_{H_1}(v, x) + d_{H_1}(v, v_2) + \sum_{y \in V_{H_1}(G_1)} d_{H_1}(v, y) \tag{4}
\]
\[
\sigma_{H_2}(v) = \sum_{x \in V_{H_2}(G_2) - \{v_1\}} d_{H_2}(v, x) + d_{H_2}(v, v_2) + \sum_{y \in V_{H_2}(G_1)} d_{H_2}(v, y) \tag{5}
\]

By the structures of \( H_1 \) and \( H_2 \), one can find that
\[
\sum_{x \in V_{H_1}(G_2) - \{v_2\}} d_{H_1}(v, x) = \sum_{x \in V_{H_2}(G_2) - \{v_1\}} d_{H_2}(v, x)
\]

and
\[
d_{H_2}(v, v_2) = d_{H_1}(v, v_2) + 1
\]

Since
\[
\sum_{y \in V_{H_1}(G_1)} d_{H_1}(v, y) - \sum_{y \in V_{H_2}(G_1)} d_{H_2}(v, y) \geq 1,
\]
by (4) and (5), we have
\[
\sigma_{H_1}(v) \geq \sigma_{H_2}(v) \tag{6}
\]

Thus,
\[
\sum_{ab \in E_{H_1}(G_2) - \{v_2y|y \in N_2\}} \frac{1}{\sqrt{\sigma_{H_1}(a)\sigma_{H_1}(b)}} \leq \sum_{ab \in E_{H_2}(G_2) - \{v_1y|y \in N_2\}} \frac{1}{\sqrt{\sigma_{H_2}(a)\sigma_{H_2}(b)}} \tag{7}
\]
By direct calculation, we have
\[
\sigma_{H_2}(v_1) = \sigma_{H_1}(v_1) - n_2 + 1
\]
\[
\sigma_{H_1}(v_2) = \sigma_{H_1}(v_1) - n_2 + n_1
\]
\[
\sigma_{H_2}(v_2) = \sigma_{H_2}(v_1) + n - 2
\]
\[
= \sigma_{H_1}(v_1) + n_1 - 1
\]
Since \( n_1 \geq 1 \), we have
\[
\sigma_{H_1}(v_2) \geq \sigma_{H_2}(v_1)
\] (8)
By (6) and (8), we have
\[
\sum_{a \in \{v_2, y \in N_2 \}} \frac{1}{\sqrt{\sigma_{H_1}(a)\sigma_{H_1}(b)}} \leq \sum_{a \in \{v_2, y \in N_2 \}} \frac{1}{\sqrt{\sigma_{H_2}(a)\sigma_{H_2}(b)}}
\] (9)
So Claim 1.1 holds by (7) and (9).
Moreover, since
\[
\sigma_{H_2}(v_1)\sigma_{H_2}(v_2) - \sigma_{H_1}(v_1)\sigma_{H_1}(v_2)
\]
\[
= (\sigma_{H_1}(v_1) - n_2 + 1)(\sigma_{H_1}(v_1) + n_1 - 1) - \sigma_{H_1}(v_1)(\sigma_{H_1}(v_1) + n_1 - n_2)
\]
\[
= (n_2 - 1)(1 - n_1) \leq 0,
\]
then
\[
\frac{1}{\sqrt{\sigma_{H_1}(v_1)\sigma_{H_1}(v_2)}} \leq \frac{1}{\sqrt{\sigma_{H_2}(v_1)\sigma_{H_2}(v_2)}}
\] (10)
So the result follows immediately from (1), (2), (3), (10) and Claim 1.1. □

Thus, by Lemma 1, we focus on seeking the unicyclic graph with the largest Balaban index in \( U_{n,g}^* \).

Let \( U_0 = U_n(S_{w_1}, S_{w_2}, \ldots, S_{w_g}) \) be an unicyclic graph in \( U_{n,g}^* \), where \( g > 3 \), and \( w_1 \) and \( w_2 \) be the centers of two adjacent stars \( S_{w_1}, S_{w_2} \) to which there are some pendent vertices, say \( \{u_1, u_2, \ldots, u_{n_1}\} \) and \( \{v_1, v_2, \ldots, v_{n_2}\} \) attached, respectively. Let \( U_1 \) be the graph obtained from \( U_0 \) by identifying two vertices \( w_1 \) and \( w_2 \) and changing the edge \( w_1w_2 \) into a pendent edge attached at \( w_1 \), that is,
\[
U_1 = U_0 - \{w_2x | x \in N_{U_0}(w_2) \setminus \{w_1\}\} + \{w_1x | x \in N_{U_0}(w_2) \setminus \{w_1\}\}
\]
Two graphs $U_0$ and $U_1$ are shown in Fig 3.

![Graphs $U_0$ and $U_1$](image)

**Fig. 3.** Graphs $U_0$ and $U_1$, where $x = \left\lfloor \frac{g}{2} \right\rfloor$

**Lemma 2.** $J(U_0) < J(U_1)$.

**Proof.** Let $Y_1(U_0)$ and $Y_2(U_0)$ be two subsets of the vertex set $V(U_0)$ such that

$$
Y_1(U_0) = V(S_{w_1}) \cup \{w_2\} \cup V(S_{w_{\lfloor \frac{g}{2} \rfloor +2}}) \cup V(S_{w_{\lfloor \frac{g}{2} \rfloor +3}}) \cdots \cup V(S_{w_g})
$$

$$
Y_2(U_0) = V(U_0) - Y_1(U_0)
$$

Let $E_1(U_0)$ and $E_2(U_0)$ be two subsets of edge set $E(U_0)$ such that

$$
E_1(U_0) = E(U_0[Y_1(U_0)]) - \{w_1w_2\}
$$

$$
E_2(U_0) = E(U_0) - E_1(U_0) - \{w_1w_2\}
$$

In a similar way, we define the corresponding items for $U_1$ as follows.

$$
Y_1(U_1) = Y_1(U_0), \quad Y_2(U_1) = Y_2(U_0)
$$

$$
E_1(U_1) = E(U_1[Y_1(U_1)]) - \{w_1w_2\}
$$

$$
E_2(U_1) = E(U_1) - E_1(U_1) - \{w_1w_2\}
$$

It is easy to see that

$$
E(U_0) = E_1(U_0) \cup E_2(U_0) \cup \{w_1w_2\}
$$

$$
E(U_1) = E_1(U_1) \cup E_2(U_1) \cup \{w_1w_2\}
$$

By the definition of the Balaban index, we have
\[
\frac{2}{m} J(U_0) = \sum_{uv \in E_1(U_0)} \frac{1}{\sqrt{\sigma_{U_0}(u)\sigma_{U_0}(v)}} + \sum_{ab \in E_2(U_0)} \frac{1}{\sqrt{\sigma_{U_0}(a)\sigma_{U_0}(b)}} + \frac{1}{\sqrt{\sigma_{U_0}(w_1)\sigma_{U_0}(w_2)}}
\]

\[
\frac{2}{m} J(U_1) = \sum_{uv \in E_1(U_1)} \frac{1}{\sqrt{\sigma_{U_1}(u)\sigma_{U_1}(v)}} + \sum_{ab \in E_2(U_1)} \frac{1}{\sqrt{\sigma_{U_1}(a)\sigma_{U_1}(b)}} + \frac{1}{\sqrt{\sigma_{U_1}(w_1)\sigma_{U_1}(w_2)}}
\]

When \( x \in Y_1(U_0) \) (i.e., \( x \in Y_1(U_1) \)) and \( x \neq w_2 \), since the length of the cycle in \( U_1 \) is less exactly one than that in \( U_0 \), thus,
\[
\sigma_{U_0}(x) > \sigma_{U_1}(x)
\]

So it is easy to see that
\[
\sum_{uv \in E_1(U_0)} \frac{1}{\sqrt{\sigma_{U_0}(u)\sigma_{U_0}(v)}} < \sum_{uv \in E_1(U_1)} \frac{1}{\sqrt{\sigma_{U_1}(u)\sigma_{U_1}(v)}}
\]

Claim 2.1.
\[
\sum_{ab \in E_2(U_0)} \frac{1}{\sqrt{\sigma_{U_0}(a)\sigma_{U_0}(b)}} < \sum_{ab \in E_2(U_1)} \frac{1}{\sqrt{\sigma_{U_1}(a)\sigma_{U_1}(b)}}
\]

By the structures of \( U_0 \) and \( U_1 \), one can find that
\[
E_2(U_0) = \bigcup_{1 \leq i \leq n_2} \{ w_2v_i \} \cup \{ w_2w_3 \} \cup \{ w_{j+1}w_{j+2} \} \cup \overline{E_2(U_0)}
\]
\[
E_2(U_1) = \bigcup_{1 \leq i \leq n_2} \{ w_1v_i \} \cup \{ w_1w_3 \} \cup \{ w_{j+1}w_{j+2} \} \cup \overline{E_2(U_1)}
\]

where
\[
\overline{E_2(U_0)} = E_2(U_0) - \bigcup_{1 \leq i \leq n_2} \{ w_2v_i \} \cup \{ w_2w_3 \} \cup \{ w_{j+1}w_{j+2} \}
\]
\[
\overline{E_2(U_1)} = E_2(U_1) - \bigcup_{1 \leq i \leq n_2} \{ w_1v_i \} \cup \{ w_1w_3 \} \cup \{ w_{j+1}w_{j+2} \}
\]

By direct computation, we have
\[
\sigma_{U_1}(w_1) = \sigma_{U_0}(w_1) - |Y_2(U_0)|
\]
\[
\sigma_{U_0}(u_i) = \sigma_{U_0}(w_1) + n - 2
\]
\[
\sigma_{U_1}(u_i) = \sigma_{U_0}(u_i) - |Y_2(U_0)|
\]
\[
= \sigma_{U_0}(w_1) + |Y_1(U_0)| - 2
\]
\[ \sigma_{U_i}(w_1) = \sigma_{U_0}(w_1) - |Y_2(U_0)| \]
\[ \sigma_{U_0}(w_2) = \sigma_{U_0}(w_1) + |Y_1(U_0)| - |Y_2(U_0)| - 2 \]
\[ \sigma_{U_0}(v_j) = 2(n_2 - 1) + \sigma_{U_0}(w_2) - 2n_2 + n \]
\[ = \sigma_{U_0}(w_1) + 2|Y_1(U_0)| - 4 \]

where \( 1 \leq i \leq n_1 \) and \( 1 \leq j \leq n_2 \).

Since \( w_2, v_j \) and \( u_i \) are the pendent vertices of \( w_1 \) on \( U_1 \), then \( \sigma_{U_i}(w_2) = \sigma_{U_i}(v_j) = \sigma_{U_i}(u_i) \). By \( \sigma_{U_0}(w_2) > \sigma_{U_1}(w_1) \) and \( \sigma_{U_0}(v_j) > \sigma_{U_1}(w_2) = \sigma_{U_1}(v_j) = \sigma_{U_1}(u_i) \), we have

\[ \sum_{1 \leq j \leq n_2} \frac{1}{\sqrt{\sigma_{U_1}(w_2)\sigma_{U_0}(v_j)}} < \sum_{1 \leq j \leq n_2} \frac{1}{\sqrt{\sigma_{U_1}(w_1)\sigma_{U_1}(v_j)}} \]  \hspace{1cm} (17)

We choose a vertex \( x \in Y_2(U_0) - \{v_1, v_2, \cdots, v_{n_2}\} \) (i.e., \( x \in Y_2(U_1) - \{v_1, v_2, \cdots, v_{n_2}\} \)).

Since \( d_{U_1}(x,w_2) = d_{U_0}(x,w_2) + 1 \) and

\[ \sum_{y \in V(U_0) - \{w_2\}} d_{U_0}(x,y) - \sum_{y \in V(U_1) - \{w_2\}} d_{U_1}(x,y) > 1, \]

then

\[ \sigma_{U_1}(x) = d_{U_1}(x,w_2) + \sum_{y \in V(U_1) - \{w_2\}} d_{U_1}(x,y) \]
\[ < d_{U_0}(x,w_2) + 1 + \sum_{y \in V(U_0) - \{w_2\}} d_{U_0}(x,y) - 1 \]
\[ = \sigma_{U_0}(x) \]  \hspace{1cm} (18)

Therefore,

\[ \sum_{ab \in E_2(U_0)} \frac{1}{\sqrt{\sigma_{U_0}(a)\sigma_{U_0}(b)}} < \sum_{ab \in E_2(U_1)} \frac{1}{\sqrt{\sigma_{U_1}(a)\sigma_{U_1}(b)}} \]  \hspace{1cm} (19)

By (13) and (18), we have

\[ \frac{1}{\sqrt{\sigma_{U_0}(w_{\lceil \frac{j}{2} \rceil + 1})\sigma_{U_0}(w_{\lceil \frac{j}{2} \rceil + 2})}} < \frac{1}{\sqrt{\sigma_{U_1}(w_{\lceil \frac{j}{2} \rceil + 1})\sigma_{U_1}(w_{\lceil \frac{j}{2} \rceil + 2})}} \]  \hspace{1cm} (20)

From (18) and \( \sigma_{U_0}(w_2) > \sigma_{U_1}(w_1) \), it is easy to see that

\[ \frac{1}{\sqrt{\sigma_{U_0}(w_2)\sigma_{U_0}(w_3)}} < \frac{1}{\sqrt{\sigma_{U_1}(w_1)\sigma_{U_1}(w_3)}} \]  \hspace{1cm} (21)

Thus, Claim 2.1 follows by (15)-(17) and (19)-(21).

When \( 1 \leq i \leq n_1 \), we know that \( \sigma_{U_1}(w_2) = \sigma_{U_1}(u_i) \). Since

\[ \sigma_{U_0}(w_1)\sigma_{U_0}(w_2) - \sigma_{U_1}(w_1)\sigma_{U_1}(w_2) = |Y_2(U_0)|||Y_1(U_0)| - 2| > 0, \]

we have
then
\[ \frac{1}{\sqrt{\sigma_{U_0}(w_1)\sigma_{U_0}(w_2)}} < \frac{1}{\sqrt{\sigma_{U_1}(w_1)\sigma_{U_1}(w_2)}} \] (22)

Thus, the result holds from (11), (12), (14), (22) and Claim 2.1. □

Hence, any unicyclic graph in $U_{n,g}$ can be transformed into the unicyclic graph $U_n(S_{w_1}, S_{w_2}, S_{w_3})$ by using Lemma 3 repeatedly and the corresponding Balaban index increases gradually along with the length of cycle decreasing.

Let $U'_0 = U_n(S_{w_1}, S_{w_2}, S_{w_3})$. Let $U'_1$ be the graph obtained from $U'_0$ by changing all the pendent edges attached at the vertex $w_1$ into the pendent edges attached at the vertex $w_3$, that is

\[ U'_1 = U'_0 - \{w_1x | x \in N_{U'_0}(w_1) \setminus \{w_2, w_3\}\} + \{w_3x | x \in N_{U'_0}(w_1) \setminus \{w_2, w_3\}\} \]

In $U'_0$, vertex $w_1$ has $n_1$ pendent vertices and $u_i$ is a pendent vertex of $w_i$, where $i = 1, 2, 3$ and $n_1 \geq 1$, $n_3 \geq 1$. The graphs $U'_0$ and $U'_1$ are shown in Fig 4.

**Lemma 3.** $J(U'_0) < J(U'_1)$.

**Proof.** From the definition of the Balaban index and the structures of the graphs $U'_0$ and $U'_1$, we know that

\[ \frac{2}{m} J(U'_0) = \frac{n_1}{\sqrt{\sigma_{U'_0}(w_1)\sigma_{U'_0}(u_1)}} + \frac{n_2}{\sqrt{\sigma_{U'_0}(w_2)\sigma_{U'_0}(u_2)}} + \frac{n_3}{\sqrt{\sigma_{U'_0}(w_3)\sigma_{U'_0}(u_3)}} + \frac{1}{\sqrt{\sigma_{U'_0}(w_1)\sigma_{U'_0}(w_2)}} + \frac{1}{\sqrt{\sigma_{U'_0}(w_1)\sigma_{U'_0}(w_3)}} \] (23)
\[
\frac{2}{m} J(U'_1) = \frac{n_2}{\sqrt{\sigma_{U'_1}(w_2)\sigma_{U'_0}(w_2)}} + \frac{n_1 + n_3}{\sqrt{\sigma_{U'_1}(w_3)\sigma_{U'_1}(w_3)}} \\
+ \frac{1}{\sqrt{\sigma_{U'_1}(w_1)\sigma_{U'_0}(w_2)}} + \frac{1}{\sqrt{\sigma_{U'_1}(w_3)\sigma_{U'_1}(w_2)}} \\
+ \frac{1}{\sqrt{\sigma_{U'_1}(w_1)\sigma_{U'_0}(w_3)}}
\]

By direct calculation, we have

\[
\sigma_{U'_0}(w_1) = 2n - n_1 - 4, \quad \sigma_{U'_0}(u_1) = 3n - n_1 - 6 \\
\sigma_{U'_0}(w_2) = 2n - n_2 - 4, \quad \sigma_{U'_0}(w_2) = 3n - n_2 - 6 \\
\sigma_{U'_0}(w_3) = 2n - n_3 - 4, \quad \sigma_{U'_0}(w_3) = 3n - n_3 - 6 \\
\sigma_{U'_1}(w_1) = 2n - 4, \quad \sigma_{U'_1}(w_2) = \sigma_{U'_1}(w_2) = 2n - n_2 - 4 \\
\sigma_{U'_1}(u_2) = \sigma_{U'_0}(u_2) = 3n - n_2 - 6, \quad \sigma_{U'_1}(w_3) = n + n_2 - 1 \\
\sigma_{U'_1}(u_3) = \sigma_{U'_1}(u_1) = 2n + n_2 - 3
\]

Thus, one can see that

\[
\sigma_{U'_0}(w_1)\sigma_{U'_0}(u_1) > \sigma_{U'_1}(w_2)\sigma_{U'_1}(u_3), \quad \sigma_{U'_0}(w_3)\sigma_{U'_0}(w_3) > \sigma_{U'_1}(w_3)\sigma_{U'_1}(u_3) \\
\sigma_{U'_0}(w_1)\sigma_{U'_0}(w_2) < \sigma_{U'_1}(w_1)\sigma_{U'_1}(w_2), \quad \sigma_{U'_0}(w_3)\sigma_{U'_0}(w_2) > \sigma_{U'_1}(w_3)\sigma_{U'_1}(w_2) \\
\sigma_{U'_0}(w_1)\sigma_{U'_0}(w_3) > \sigma_{U'_1}(w_1)\sigma_{U'_1}(w_3), \quad \sigma_{U'_0}(w_2)\sigma_{U'_0}(w_2) = \sigma_{U'_1}(w_2)\sigma_{U'_1}(w_2)
\]

So, we have

\[
\begin{align*}
\frac{n_2}{\sqrt{\sigma_{U'_0}(w_2)\sigma_{U'_0}(w_2)}} &= \frac{n_2}{\sqrt{\sigma_{U'_1}(w_2)\sigma_{U'_1}(w_2)}} \\
\frac{n_1}{\sqrt{\sigma_{U'_0}(w_1)\sigma_{U'_0}(w_1)}} &+ \frac{n_3}{\sqrt{\sigma_{U'_0}(w_3)\sigma_{U'_0}(w_3)}} < \frac{n_1 + n_3}{\sqrt{\sigma_{U'_1}(w_3)\sigma_{U'_1}(w_3)}} \\
\frac{1}{\sqrt{\sigma_{U'_0}(w_3)\sigma_{U'_0}(w_3)}} &< \frac{1}{\sqrt{\sigma_{U'_1}(w_3)\sigma_{U'_1}(w_2)}} \\
\frac{1}{\sqrt{\sigma_{U'_0}(w_1)\sigma_{U'_0}(w_3)}} &\leq \frac{1}{\sqrt{\sigma_{U'_1}(w_1)\sigma_{U'_1}(w_3)}} \\
\frac{1}{\sqrt{\sigma_{U'_0}(w_1)\sigma_{U'_0}(w_2)}} &> \frac{1}{\sqrt{\sigma_{U'_1}(w_1)\sigma_{U'_1}(w_2)}}
\end{align*}
\]
Claim 3.1.
\[
\frac{1}{\sqrt{\sigma_{U_0}(w_1)\sigma_{U_0}(w_2)}} + \frac{1}{\sqrt{\sigma_{U_1}(w_3)\sigma_{U_1}(w_2)}} < \frac{1}{\sqrt{\sigma_{U_1}(w_1)\sigma_{U_1}(w_2)}} + \frac{1}{\sqrt{\sigma_{U_1}(w_3)\sigma_{U_1}(w_2)}}
\]
i.e.,
\[
\frac{1}{\sqrt{2n - n_2 - 4}} + \frac{1}{\sqrt{2n - n_1 - 4}} + \frac{1}{\sqrt{2n - n_3 - 4}} - \frac{1}{\sqrt{2n - 4}} - \frac{1}{\sqrt{n + n_2 - 1}} < 0
\]
Let \( r = 2n - n_1 - 4 \) and \( t = 2n - n_3 - 4 \). Obviously, \( r, t > 0 \). Then
\[
\frac{1}{\sqrt{2n - n_1 - 4}} + \frac{1}{\sqrt{2n - n_3 - 4}} - \frac{1}{\sqrt{2n - 4}} - \frac{1}{\sqrt{n + n_2 - 1}} = \frac{1}{\sqrt{r}} + \frac{1}{\sqrt{t}} - \frac{1}{\sqrt{r + n_1}} - \frac{1}{\sqrt{t + n_1}}
\]
\[
= (-\sqrt{r(t - n_1)} - \sqrt{r(t + n_1)} + \sqrt{t(t - n_1)}(r + n_1) + \sqrt{t(t - n_1)}(r + n_1)) \quad (26)
\]
We just concern the sign of the numerator of above equation. Thus, from (26), we know
\[
(\sqrt{r(t - n_1)} + \sqrt{r(t + n_1)})^2 - (\sqrt{r(t - n_1)}(r + n_1) + \sqrt{t(t - n_1)}(r + n_1))^2
\]
\[
= -2r^{3/2}t^{3/2} + 2rt\sqrt{t - n_1}\sqrt{r + n_1} + (\sqrt{r} + \sqrt{t})^2n_1(r - t + n_1) \quad (27)
\]
Assume \( h = t - n_1 \). Obviously, \( h > 0 \). By \( n_3 = r - t + n_1 \) and (27), we have
\[
[2rt\sqrt{t - n_1}\sqrt{r + n_1} + (\sqrt{r} + \sqrt{t})^2n_1(r - t + n_1)]^2 - (2r^{3/2}t^{3/2})^2
\]
\[
= [2rt\sqrt{h(r + n_1)} + (\sqrt{r} + \sqrt{h})^2n_1n_3]^2 - (2r^{3/2}t^{3/2})^2
\]
\[
= [2rt\sqrt{h(r + n_1)} + (\sqrt{r} + \sqrt{h})^2n_1n_3 - 2r^{3/2}t^{3/2}]^2
\]
\[
[2rt\sqrt{h(r + n_1)} + (\sqrt{r} + \sqrt{h})^2n_1n_3 + 2r^{3/2}t^{3/2}] \quad (28)
\]
We choose the first factor from (28), it is easy to see that
\[
2rt\sqrt{h(r + n_1)} + (\sqrt{r} + \sqrt{h})^2n_1n_3 - 2r^{3/2}t^{3/2}
\]
\[
= 2\sqrt{rt}(-rt + \sqrt{rth(r + n_1)} + n_1n_3) + (r + t)n_1n_3
\]
\[
\geq 2\sqrt{rt}(-rt + \sqrt{rth(r + n_1)} + 2n_1n_3) \quad (29)
\]
So by (29), it suffices to prove that
\[
-rt + \sqrt{rth(r + n_1)} + 2n_1n_3 > 0 \quad (30)
\]
Let

\[ f_1(n) = 12n^2 - 6n(8 + n_1 + n_3) + 12n_1 + 12n_3 - n_1n_3 + 48 \]
\[ f_2(n) = 20n^2 - 10n(8 + n_1 + n_3) + 20n_1 + 20n_3 + 9n_1n_3 + 80 \]

Since \( n = n_1 + n_2 + n_3 + 3 \), then

\[ f_1(n_1 + n_2 + n_3 + 3) > 0, \quad f_2(n_1 + n_2 + n_3 + 3) > 0 \]

Thus, we have

\[
\begin{align*}
[2rt\sqrt{rth(r + n_1)}]^2 - [(rt)^2 + rth(r + n_1) - (2n_1n_3)^2]^2 \\
= n_1^2n_3^2f_1(n)f_2(n) \\
> 0
\end{align*}
\]

(31)

Note that

\( (rt)^2 + rth(r + n_1) - (2n_1n_3)^2 > 0 \)

From (31), we can find that

\[
\begin{align*}
2rt\sqrt{rth(r + n_1)} & > (rt)^2 + rth(r + n_1) - (2n_1n_3)^2 \\
\iff \\
(2n_1n_3)^2 & > (rt)^2 + rth(r + n_1) - 2rt\sqrt{rth(r + n_1)} \\
\iff \\
(2n_1n_3)^2 & > rt(\sqrt{rt} - \sqrt{h(r + n_1)})^2
\end{align*}
\]

(32)

Note that \( \sqrt{rt} - \sqrt{h(r + n_1)} > 0 \). By (32), we know that

\[
\sqrt{rt}(\sqrt{h(r + n_1)} - \sqrt{rt}) + 2n_1n_3 \\
= -rt + \sqrt{r}\sqrt{h}\sqrt{r + n_1} + 2n_1n_3 > 0
\]

Therefore, Claim 3.1 holds by (26)-(30).

Hence, from (23)-(25) and Claim 3.1, the result follows. □
Proof of Theorem 1. Let $G$ and $G'$ be two graphs and denote that graph $G$ is transformed into graph $G'$ by $G \rightarrow G'$. For any graph $U$ in $U_n$, we distinguish the following two cases.

Case 1. The length of the cycle in the unicyclic graph $U$ is 3. Then we take transformations as follow:

$$U \rightarrow U'_0 \rightarrow U_n^3$$

So by Lemma 1 and Lemma 3, we have $J(U) < J(U'_0) < J(U_n^3)$.

Case 2. The length of the cycle in the unicyclic graph $U$ is more than 3. Then we take the following transformations:

$$U \rightarrow U_0 \rightarrow U'_0 \rightarrow U_n^3$$

Thus, we have $J(U) < J(U_0) < J(U'_0) < J(U_n^3)$ by lemmas 1-3. This finishes the proof. □

With the similar proof as Theorem 1 but more tedious, we have the following result on bicyclic graphs and the proof is shown in the appendix.

**Theorem 2.** The graph $B_n$ has the largest Balaban index among all the $n$-vertex bicyclic graphs.

**References**


Appendix: The proof of Theorem 2.

Before given the proof of Theorem 2, we need some more preparations.

Suppose that \(v_1\) is a vertex of \(C_p\) and \(v_l\) is a vertex of \(C_q\). Joining \(v_1\) and \(v_l\) by a path \(v_1, v_2, \ldots, v_l\) of length \(l - 1\), where \(l \geq 1\) and \(l = 1\) means identifying \(v_1\) with \(v_l\), the resulting graph (see Fig 5.), denoted by \(B(p, l, q)\), is called an \(\infty\)-graph. If two cycles \(C_p\) and \(C_q\) share a common path of length \(l\), where \(l \leq \lfloor \frac{p}{2} \rfloor, l \leq \lfloor \frac{q}{2} \rfloor\), then the resulting graph (see Fig 5.), denoted by \(P(p, l, q)\), is called a \(\theta\)-graph. Obviously, the set of all bicyclic graphs \(B_n\) consists of two kinds of graphs. one kind, denoted by \(B_n^+\), are those graphs each of them is an \(\infty\)-graph with some trees attached; the other kind, denoted by \(B_n^{++}\), are those graphs each of them is a \(\theta\)-graph with some trees attached. Then we have \(B_n = B_n^+ \cup B_n^{++}\).

If \(G\) is a graph obtained by attaching their centers of some stars to some vertices of \(B(p, 1, q)\), then \(G \in B_n^+\) obviously. The set of all this kind of bicyclic graphs is denoted by \(B_n^+(p, q)\).

If \(G\) is obtained by attaching their centers of some stars to some vertices of \(P(p, l, q)\), then \(G \in B_n^{++}\). The set of all this kind of bicyclic graphs is denoted by \(B_n^{++}(p, l, q)\).

**Theorem 2.** The graph \(B_n\) has the largest Balaban index among all the \(n\)-vertex bicyclic graphs.

From Lemma 1, we focus on seeking the bicyclic graph with the largest Balaban index in \(B_n^+(p, q)\) or \(B_n^{++}(p, l, q)\).

Let \(B\) be a bicyclic graph in \(B_n^+(p, q)\), where \(p > 3\) and \(q \geq 3\), and \(w_1\) and \(w_2\) be the centers of two adjacent stars \(S_{w_1}, S_{w_2}\), where \(w_1\) is a vertex on the cycle \(C_p\) and \(w_2\) is the unique common vertex of two cycles \(C_p\) and \(C_q\). Let \(B'\) be the graph obtained from \(B\) by identifying vertices \(w_1\) and \(w_2\) and changing the edge \(w_1w_2\) into a pendent edge attached at \(w_1\), that is,

\[
B' = B - \{w_2x | x \in N_B(w_2) \setminus \{w_1\} \} + \{w_1x | x \in N_B(w_2) \setminus \{w_1\} \}
\]

Two graphs \(B\) and \(B'\) are shown in Fig 6.
With the same method as Lemma 2, we have

Lemma 4. $J(B) < J(B')$.

Let $B_0$ be a bicyclic graph in $B_n^{**}(p, l, q)$, where $p, q > 3$, with $\{w_1, w_2, \ldots, w_{l+1}, u_{l+2}, \ldots, u_p\}$ and $\{w_1, w_2, \ldots, w_{l+1}, v_{l+2}, \ldots, v_q\}$ as the vertex sets of two cycles $C_p$ and $C_q$, respectively, and the common vertices $w_1$ and $w_2$ of $C_p$ and $C_q$ be the centers of two adjacent stars $S_{w_1}, S_{w_2}$ to which $n_1$ and $n_2$ pendant vertices attached, respectively. Let $B'_0$ be the graph obtained from $B_0$ by identifying vertices $w_1$ and $w_2$ and changing the edge $w_1w_2$ into a pendent edge attached at $w_1$, that is,

$$B'_0 = B_0 - \{w_2x \mid x \in N_{B_0}(w_2) \setminus \{w_1\}\} + \{w_1x \mid x \in N_{B_0}(w_2) \setminus \{w_1\}\}$$

The graphs $B_0$ and $B'_0$ are shown in Fig 7.

Lemma 5. $J(B_0) < J(B'_0)$.

Proof. Let $Y_1(B_0)$ and $Y_2(B_0)$ be two subsets of the vertex set $V(B_0)$ such that

$$Y_2(B_0) = (V(S_{w_2}) - \{w_2\}) \cup V(S_{w_3}) \cup \cdots \cup V(S_{w_{l+1}}) \cup V(S_{u_{l+2}}) \cup V(S_{u_{l+3}}) \cup \cdots \cup V(S_{u_{l+2}u_{l+3}})$$

$$\cup V(S_{v_{l+2}}) \cup V(S_{v_{l+3}}) \cup \cdots \cup V(S_{v_{l+2}v_{l+3}})$$

$$= \left(\left\lceil \frac{p}{2} \right\rceil \right) \left(\left\lceil \frac{q}{2} \right\rceil \right)$$
\[ Y_1(B_0) = V(B_0) - Y_2(B_0) \]

Let \( E_1(B_0) \) and \( E_2(B_0) \) be two subsets of edge set \( E(B_0) \) such that
\[
E_1(B_0) = E(B_0[Y_1(B_0)]) - \{ w_1w_2 \}
E_2(B_0) = E(B_0) - E_1(B_0) - \{ w_1w_2 \}
\]

In a similar way, we define the corresponding items for \( B'_0 \) as follows.
\[
Y_1(B'_0) = Y_1(B_0), \quad Y_2(B'_0) = Y_2(B_0)
E_1(B'_0) = E(B'_0[Y_1(B'_0)]) - \{ w_1w_2 \}
E_2(B'_0) = E(B'_0) - E_1(B'_0) - \{ w_1w_2 \}
\]

It is easy to see that
\[
E(B_0) = E_1(B_0) \cup E_2(B_0) \cup \{ w_1w_2 \}
E(B'_0) = E_1(B'_0) \cup E_2(B'_0) \cup \{ w_1w_2 \}
\]

From the definition of the Balaban index, we know that
\[
\frac{3}{m} J(B_0) = \sum_{uv \in E_1(B_0)} \frac{1}{\sqrt{\sigma_{B_0}(u)\sigma_{B_0}(v)}} + \sum_{xy \in E_2(B_0)} \frac{1}{\sqrt{\sigma_{B_0}(x)\sigma_{B_0}(y)}} + \frac{1}{\sqrt{\sigma_{B_0}(w_1)\sigma_{B_0}(w_2)}} \quad (33)
\]
\[
\frac{3}{m} J(B'_0) = \sum_{uv \in E_1(B'_0)} \frac{1}{\sqrt{\sigma_{B'_0}(u)\sigma_{B'_0}(v)}} + \sum_{xy \in E_2(B'_0)} \frac{1}{\sqrt{\sigma_{B'_0}(x)\sigma_{B'_0}(y)}} + \frac{1}{\sqrt{\sigma_{B'_0}(w_1)\sigma_{B'_0}(w_2)}} \quad (34)
\]

By direct computation, we have
\[
\sigma_{B'_0}(w_1) = \sigma_{B_0}(w_1) - |Y_2(B_0)|
\]
\[
\sigma_{B_0}(w_2) = \sigma_{B_0}(w_1) + |Y_1(B_0)| - |Y_2(B_0)| - 2
\]

Let \( V \) and \( V' \) be two vertex sets such that
\[
V = \{ u_i | w_1u_i \in E_{B_0}(S_{w_1}), 1 \leq i \leq n_1 \}
V' = \{ u_j | w_1u_j \in E_{B'_0}(S_{w_1}), 1 \leq j \leq n_1 + n_2 + 1 \}
\]

When \( u_i \in V \), we can find that
\[
\sigma_{B_0}(u_i) = \sigma_{B_0}(w_1) + n - 2
\]
When \(1 \leq j \leq n_1\), \(u_j \in V\) (i.e., \(u_j \in V'\)). So we have
\[
\sigma_{B_0'}(u_j) = \sigma_{B_0'}(w_2) \\
= \sigma_{B_0}(u_j) - |Y_2(B_0)| \\
= \sigma_{B_0}(u_i) - |Y_2(B_0)| \\
= \sigma_{B_0}(w_1) + |Y_1(B_0)| - 2
\]
When \(n_1 + 1 \leq j \leq n_1 + n_2 + 1\), \(u_j \in V'\) and we obtain
\[
\sigma_{B_0'}(u_j) = \sigma_{B_0'}(w_2) = \sigma_{B_0}(w_1) + |Y_1(B_0)| - 2
\]
Let \(U\) be a vertex set such that
\[
U = \{u_i | w_2u_i \in E_{B_0}(S_{w_2}), 1 \leq i \leq n_2\}
\]
When \(v \in U\), one can find that
\[
\sigma_{B_0}(v) = \sigma_{B_0}(w_2) + n - 2 = \sigma_{B_0}(w_1) + 2|Y_1(B_0)| - 4
\]
From the structures of two graphs \(B_0\) and \(B_0'\), it is easy to see that
\[
E_2(B_0) = \bigcup_{u_i \in U, 1 \leq i \leq n_2} \{w_2u_i\} \cup \{w_2w_3\} \cup \overline{E_2}(B_0) \tag{35}
\]
\[
E_2(B_0') = \bigcup_{u_i \in V', 1 \leq i \leq n_2} \{w_1u_i\} \cup \{w_1w_3\} \cup \overline{E_2}(B_0') \tag{36}
\]
where
\[
\overline{E_2}(B_0) = E_2(B_0) - \bigcup_{u_i \in U, 1 \leq i \leq n_2} \{w_2u_i\} \cup \{w_2w_3\}
\]
\[
\overline{E_2}(B_0') = E_2(B_0') - \bigcup_{u_i \in V', 1 \leq i \leq n_2} \{w_1u_i\} \cup \{w_1w_3\}
\]
When \(u \in Y_2(B_0)\) (i.e., \(u \in Y_2(B_0')\)), we have
\[
\sigma_{B_0}(u) = d_{B_0}(u, w_2) + \sum_{x \in V(B_0) - \{w_2\}} d_{B_0}(u, x)
\]
\[
\sigma_{B_0'}(u) = d_{B_0'}(u, w_2) + \sum_{x \in V(B_0') - \{w_2\}} d_{B_0'}(u, x)
\]
Since \(d_{B_0'}(u, w_2) = d_{B_0}(u, w_2) + 1\) and
\[
\sum_{x \in V(B_0) - \{w_2\}} d_{B_0}(u, x) - \sum_{x \in V(B_0') - \{w_2\}} d_{B_0'}(u, x) > 1,
\]
then
\[
\sigma_{B_0}(u) > \sigma_{B_0'}(u) \tag{37}
\]
Thus, by (37), we can see that $\sigma_{B_0}(w_3) > \sigma_{B_0'}(w_3)$ and
\[
\sum_{xy \in E_2(B_0)} \frac{1}{\sqrt{\sigma_{B_0}(x)\sigma_{B_0}(y)}} < \sum_{xy \in E_2(B_0')} \frac{1}{\sqrt{\sigma_{B_0'}(x)\sigma_{B_0'}(y)}}
\]  
(38)

Since $\sigma_{B_0}(w_2) > \sigma_{B_0'}(w_1)$, $\sigma_{B_0}(w_3) > \sigma_{B_0'}(w_3)$ and $\sigma_{B_0}(u_i) > \sigma_{B_0'}(u_i)$, where $u_i \in U$ (i.e., $u_i \in V'$), $0 \leq i \leq n_2$, then
\[
\sum_{xy \in \{u_i|w_i \in U, 1 \leq i \leq n_2\}} \frac{1}{\sqrt{\sigma_{B_0}(x)\sigma_{B_0}(y)}} < \sum_{xy \in \{u_i|w_i \in V', 1 \leq i \leq n_2\}} \frac{1}{\sqrt{\sigma_{B_0'}(x)\sigma_{B_0'}(y)}}
\]  
(39)

By (35), (36), (38) and (39), we have
\[
\sum_{xy \in E_2(B_0)} \frac{1}{\sqrt{\sigma_{B_0}(x)\sigma_{B_0}(y)}} < \sum_{xy \in E_2(B_0')} \frac{1}{\sqrt{\sigma_{B_0'}(x)\sigma_{B_0'}(y)}}
\]  
(40)

Since $\sigma_{B_0}(w_1)\sigma_{B_0}(w_2) - \sigma_{B_0'}(w_1)\sigma_{B_0'}(w_2) = |Y_2(B_0)|(|Y_1(B_0)| - 2) > 0$, then
\[
\frac{1}{\sqrt{\sigma_{B_0}(w_1)\sigma_{B_0}(w_2)}} < \frac{1}{\sqrt{\sigma_{B_0'}(w_1)\sigma_{B_0'}(w_2)}}
\]  
(41)

By the structures of two graphs $B_0$ and $B_0'$, it is easy to see that
\[
E_1(B_0) = \{u_{|v|+1}u_{|v|+2}\} \cup \{v_{|v|+1}v_{|v|+2}\} \cup \overline{E_1}(B_0)
\]  
(42)
\[
E_1(B_0') = \{u_{|v|+1}u_{|v|+2}\} \cup \{v_{|v|+1}v_{|v|+2}\} \cup \overline{E_1}(B_0')
\]  
(43)

where
\[
\overline{E_1}(B_0) = E_1(B_0) - \{u_{|v|+1}u_{|v|+2}\} \cup \{v_{|v|+1}v_{|v|+2}\}
\]
\[
\overline{E_1}(B_0') = E_1(B_0') - \{u_{|v|+1}u_{|v|+2}\} \cup \{v_{|v|+1}v_{|v|+2}\}
\]

When $v \in Y_1(B_0)$ (i.e., $v \in Y_1(B_0')$), since the length of each cycle of $\{C_p, C_q\}$ in $B_0'$ is less exactly one that that in $B_0$, thus,
\[
\sigma_{B_0}(v) > \sigma_{B_0'}(v)
\]  
(44)

Thus, we have
\[
\sum_{xy \in E_1(B_0)} \frac{1}{\sqrt{\sigma_{B_0}(x)\sigma_{B_0}(y)}} < \sum_{xy \in E_1(B_0')} \frac{1}{\sqrt{\sigma_{B_0'}(x)\sigma_{B_0'}(y)}}
\]

By (37) and (44), we can see that
\[
\frac{1}{\sqrt{\sigma_{B_0}(u_{|v|+1})\sigma_{B_0}(u_{|v|+2})}} < \frac{1}{\sqrt{\sigma_{B_0'}(u_{|v|+1})\sigma_{B_0'}(u_{|v|+2})}}
\]
two triangles such that each vertex
Le
B
By the definition of the Balaban index and the structures of the graphs
Proof.

\[ \sum_{uv \in E_1(B_0)} \frac{1}{\sqrt{\sigma_{B_0}(u)\sigma_{B_0}(v)}} < \sum_{uv \in E_1(B'_0)} \frac{1}{\sqrt{\sigma_{B'_0}(u)\sigma_{B'_0}(v)}} \]

(45)

Hence, the result holds from (33), (34), (40), (41) and (45). □

By using Lemma 4 or Lemma 5 repeatedly, the bicyclic graphs \( B \) or \( B_0 \) may be transformed into the graph \( B_1 \) (see Fig 8.), and the corresponding Balaban index increases gradually along with the length of the cycles decreasing.

![Graphs B1 and B'_1](image)

Fig. 8. Graphs \( B_1 \) and \( B'_1 \)

Let \( B_1 \) be a bicyclic graph in \( B^*_n(3,3) \) with \( \{v_1, v_2, v_3, v_4, v_5\} \) as the vertex set of its two triangles such that each vertex \( v_i \) has \( n_i \) pendent vertices and \( u_i \) is a pendent vertex of \( v_i \), where \( 1 \leq i \leq 5 \), and \( v_1 \) be the unique common vertex of the triangles. Let \( B'_1 \) be the graph obtained from \( B_1 \) by changing all the pendent edges attached at \( v_2 \) into the ones attached at \( v_1 \) such that there are \( n_1 + n_2 \) pendent edges attached at \( v_1 \), that is,

\[ B'_1 = B_1 - \{v_2x|x \in N_{B_1}(v_2) \setminus \{v_1, v_3\}\} + \{v_1x|x \in N_{B_1}(v_2) \setminus \{v_1, v_3\}\} \]

**Lemma 6.** \( J(B_1) < J(B'_1) \).

**Proof.** By the definition of the Balaban index and the structures of the graphs \( B_1 \) and \( B'_1 \), we know that

\[
\frac{3}{m} J(B_1) = \frac{n_1}{\sqrt{\sigma_{B_1}(v_1)\sigma_{B_1}(u_1)}} + \frac{n_2}{\sqrt{\sigma_{B_1}(v_2)\sigma_{B_1}(v_2)}} + \frac{n_3}{\sqrt{\sigma_{B_1}(v_3)\sigma_{B_1}(u_3)}} + \\
\frac{n_4}{\sqrt{\sigma_{B_1}(v_4)\sigma_{B_1}(u_4)}} + \frac{n_5}{\sqrt{\sigma_{B_1}(v_5)\sigma_{B_1}(u_5)}} + \frac{1}{\sqrt{\sigma_{B_1}(v_1)\sigma_{B_1}(v_1)}} + \\
\frac{1}{\sqrt{\sigma_{B_1}(v_1)\sigma_{B_1}(v_3)}} + \frac{1}{\sqrt{\sigma_{B_1}(v_2)\sigma_{B_1}(v_3)}} + \frac{1}{\sqrt{\sigma_{B_1}(v_2)\sigma_{B_1}(v_4)}} + \\
\frac{1}{\sqrt{\sigma_{B_1}(v_3)\sigma_{B_1}(v_3)}} + \frac{1}{\sqrt{\sigma_{B_1}(v_3)\sigma_{B_1}(v_5)}} + \frac{1}{\sqrt{\sigma_{B_1}(v_4)\sigma_{B_1}(v_5)}}
\]

(46)
\[
\frac{3}{m} J(B'_1) = \frac{n_1 + n_2}{\sqrt{\sigma_{B'_1}(v_1)\sigma_{B'_1}(u_1)}} + \frac{n_3}{\sqrt{\sigma_{B'_1}(v_3)\sigma_{B'_1}(u_3)}} + \frac{n_4}{\sqrt{\sigma_{B'_1}(v_4)\sigma_{B'_1}(u_4)}} + \\
\frac{1}{\sqrt{\sigma_{B'_1}(v_5)\sigma_{B'_1}(u_5)}} + \frac{1}{\sqrt{\sigma_{B'_1}(v_2)\sigma_{B'_1}(v_2)}} + \frac{1}{\sqrt{\sigma_{B'_1}(v_1)\sigma_{B'_1}(v_3)}} + \frac{1}{\sqrt{\sigma_{B'_1}(v_1)\sigma_{B'_1}(v_4)}} + \frac{1}{\sqrt{\sigma_{B'_1}(v_1)\sigma_{B'_1}(v_5)}}
\]

(47)

By direct calculation, we have

\[\sigma_{B'_1}(v_1) = 2n - n_1 - 6, \quad \sigma_{B'_1}(u_1) = 3n - n_1 - 8\]
\[\sigma_{B'_1}(v_2) = 2n - n_2 + n_4 + n_5 - 4, \quad \sigma_{B'_1}(u_2) = 3n - n_2 + n_4 + n_5 - 6\]
\[\sigma_{B'_1}(v_3) = 2n - n_3 + n_4 + n_5 - 4, \quad \sigma_{B'_1}(u_3) = 3n - n_3 + n_4 + n_5 - 6\]
\[\sigma_{B'_1}(v_4) = 2n - n_4 + n_2 + n_3 - 4, \quad \sigma_{B'_1}(u_4) = 3n - n_4 + n_2 + n_3 - 6\]
\[\sigma_{B'_1}(v_5) = 2n - n_5 + n_2 + n_3 - 4, \quad \sigma_{B'_1}(u_5) = 3n - n_5 + n_2 + n_3 - 6\]

and

\[\sigma_{B'_1}(v_1) = 2n - n_1 - n_2 - 6\]
\[\sigma_{B'_1}(u_1) = \sigma_{B'_1}(u_2) = 3n - n_1 - n_2 - 8\]
\[\sigma_{B'_1}(v_2) = 2n + n_4 + n_5 - 4\]
\[\sigma_{B'_1}(v_3) = \sigma_{B'_1}(v_3), \quad \sigma_{B'_1}(u_3) = \sigma_{B'_1}(u_3)\]
\[\sigma_{B'_1}(v_4) = 2n + n_3 - n_4 - 4, \quad \sigma_{B'_1}(v_5) = 2n - n_5 + n_3 - 4\]
\[\sigma_{B'_1}(u_4) = 3n + n_3 - n_4 - 6, \quad \sigma_{B'_1}(u_5) = 3n - n_5 + n_3 - 6\]

Thus, we easily find that

\[\sigma_{B'_1}(v_1)\sigma_{B'_1}(u_1) > \sigma_{B'_1}(v_1)\sigma_{B'_1}(u_1), \quad \sigma_{B'_1}(v_2)\sigma_{B'_1}(u_2) > \sigma_{B'_1}(v_1)\sigma_{B'_1}(u_1)\]
\[\sigma_{B'_1}(v_3)\sigma_{B'_1}(u_3) = \sigma_{B'_1}(v_3)\sigma_{B'_1}(u_3), \quad \sigma_{B'_1}(v_5)\sigma_{B'_1}(u_5) > \sigma_{B'_1}(v_1)\sigma_{B'_1}(u_5)\]
\[\sigma_{B'_1}(v_4)\sigma_{B'_1}(u_4) > \sigma_{B'_1}(v_4)\sigma_{B'_1}(u_4), \quad \sigma_{B'_1}(v_1)\sigma_{B'_1}(v_2) > \sigma_{B'_1}(v_1)\sigma_{B'_1}(v_2)\]
\[\sigma_{B'_1}(v_1)\sigma_{B'_1}(v_3) > \sigma_{B'_1}(v_1)\sigma_{B'_1}(v_3), \quad \sigma_{B'_1}(v_1)\sigma_{B'_1}(v_4) > \sigma_{B'_1}(v_1)\sigma_{B'_1}(v_4)\]
\[\sigma_{B'_1}(v_1)\sigma_{B'_1}(v_5) > \sigma_{B'_1}(v_1)\sigma_{B'_1}(v_5), \quad \sigma_{B'_1}(v_4)\sigma_{B'_1}(v_5) > \sigma_{B'_1}(v_4)\sigma_{B'_1}(v_5)\]
\[\sigma_{B'_1}(v_2)\sigma_{B'_1}(v_3) < \sigma_{B'_1}(v_2)\sigma_{B'_1}(v_3)\]

So we have

\[\frac{n_1}{\sqrt{\sigma_{B'_1}(v_1)\sigma_{B'_1}(u_1)}} + \frac{n_2}{\sqrt{\sigma_{B'_1}(v_2)\sigma_{B'_1}(u_2)}} < \frac{n_1 + n_2}{\sqrt{\sigma_{B'_1}(v_1)\sigma_{B'_1}(u_1)}}\]
Claim 6.1.

\[ \frac{1}{\sqrt{\sigma_{B_1}(v_1)\sigma_{B_1}(v_3)}} + \frac{1}{\sqrt{\sigma_{B_1}(v_2)\sigma_{B_1}(v_3)}} < \frac{1}{\sqrt{\sigma_{B'_1}(v_1)\sigma_{B'_1}(v_3)}} + \frac{1}{\sqrt{\sigma_{B'_1}(v_2)\sigma_{B'_1}(v_3)}} \]

i.e.

\[ \frac{1}{\sqrt{2n - n_3 + n_4 + n_5 - 4}} \left( \frac{1}{\sqrt{2n - n_1 - 6}} + \frac{1}{\sqrt{2n - n_2 + n_4 + n_5 - 4}} \right) < 0 \]

With the same method as Claim 3.1, Claim 6.1 follows. So the result holds by (46)-(48) and Claim 6.1. □

Hence, by Lemma 6, the bicyclic graph \( B_1 \) is transformed into the bicyclic graph \( B'' \) (see Fig 11.) and the corresponding Balaban index increases.

![Fig. 9. Graphs \( B_2 \) and \( B'_2 \)](image)

Let \( B_2 \) be a bicyclic graph in \( B^{n*}_n(p, 1, 3) \), where \( p > 3 \), with \( \{w_1, w_2, \ldots, w_p\} \) and \( \{w_2, w_3, v\} \) as the vertex sets of its cycles \( C_p \) and \( C_3 \), respectively, where \( w_2 \) and \( w_3 \) are the common vertices of the cycles, and \( w_1 \) and \( w_2 \) be the centers of two adjacent stars.

Let \( B'_2 \) be the graph obtained from \( B_2 \) by identifying \( w_1 \) and \( w_2 \) and changing the edge \( w_1w_2 \) into a pendent edge attached at \( w_1 \), that is,

\[ B'_2 = B_2 - \{w_2x | x \in N_{B_2}(w_2) \setminus \{w_1\}\} + \{w_1x | x \in N_{B_2}(w_2) \setminus \{w_1\}\} \]
The graphs $B_2$ and $B'_2$ are shown in Fig 9.

With the same method as Lemma 2, we have

**Lemma 7.** $J(B_2) < J(B'_2)$.

So using Lemma 7 repeatedly, the bicyclic graphs $B_2$ can be transformed into the graph $B_3$ (see Fig 10.), and the corresponding Balaban index increases gradually along with the length of the cycle decreasing.

Let $B_3$ be a bicyclic graph in $B^*_n(3,1,3)$ with $\{v_1, v_2, v_3, v_4\}$ as the vertex set of its two triangles such that each vertex $v_i$ has $n_i$ pendent vertices and $u_i$ is a pendent vertex of vertex $v_i$, where $1 \leq i \leq 4$, and $v_1$ and $v_3$ be the common vertices of the triangles. Let $B'_3$ be the graph obtained from $B_3$ by changing all the pendent edges attached at $v_2$ into the ones attached at $v_1$ such that there are $n_1 + n_2$ pendent edges attached at $v_1$, that is,

$$B'_3 = B_3 - \{v_2x|x \in N_{B_3}(v_2) \setminus \{v_1, v_3\}\} + \{v_1x|x \in N_{B_3}(v_2) \setminus \{v_1, v_3\}\}.$$

The graphs $B_3$ and $B'_3$ are shown in Fig 10.

**Lemma 8.** $J(B_3) < J(B'_3)$.

**Proof.** By the definition of the Balaban index and the structures of $B_3$ and $B'_3$, we know that

$$\frac{3}{m} J(B_3) = \frac{n_1}{\sqrt{\sigma_{B_3}(v_1)\sigma_{B_3}(u_1)}} + \frac{n_2}{\sqrt{\sigma_{B_3}(v_2)\sigma_{B_3}(u_2)}} + \frac{n_3}{\sqrt{\sigma_{B_3}(v_3)\sigma_{B_3}(u_3)}} + \frac{n_4}{\sqrt{\sigma_{B_3}(v_4)\sigma_{B_3}(u_4)}} + \frac{1}{\sqrt{\sigma_{B_3}(v_1)\sigma_{B_3}(v_2)}} + \frac{1}{\sqrt{\sigma_{B_3}(v_2)\sigma_{B_3}(v_3)}} + \frac{1}{\sqrt{\sigma_{B_3}(v_1)\sigma_{B_3}(v_3)}} + \frac{1}{\sqrt{\sigma_{B_3}(v_3)\sigma_{B_3}(v_4)}} \quad (49)$$
\[
\frac{3}{m} J(B'_3) = \frac{n_1 + n_2}{\sqrt{\sigma_{B'_3}(v_1) \sigma_{B'_3}(u_1)}} + \frac{n_3}{\sqrt{\sigma_{B'_3}(v_3) \sigma_{B'_3}(u_3)}}
\]
\[
+ \frac{n_4}{\sqrt{\sigma_{B'_3}(v_4) \sigma_{B'_3}(u_4)}} + \frac{1}{\sqrt{\sigma_{B'_3}(v_1) \sigma_{B'_3}(v_2)}} + \frac{1}{\sqrt{\sigma_{B'_3}(v_1) \sigma_{B'_3}(v_3)}} + \frac{1}{\sqrt{\sigma_{B'_3}(v_1) \sigma_{B'_3}(v_4)}}
\]

(50)

By direct calculation, we have

\[
\sigma_{B_3}(v_1) = 2n - n_1 - 5, \quad \sigma_{B_3}(u_1) = 3n - n_1 - 7
\]
\[
\sigma_{B_3}(v_3) = 2n - n_3 - 5, \quad \sigma_{B_3}(u_3) = 3n - n_3 - 7
\]
\[
\sigma_{B_3}(v_2) = 2n + n_4 - n_2 - 4, \quad \sigma_{B_3}(u_2) = 3n + n_4 - n_2 - 6
\]
\[
\sigma_{B_3}(v_4) = 2n + n_2 - n_4 - 4, \quad \sigma_{B_3}(u_4) = 3n + n_2 - n_4 - 6
\]

\[
\sigma_{B'_3}(v_1) = n + n_3 + n_4 - 1, \quad \sigma_{B'_3}(u_1) = 2n + n_3 + n_4 - 3
\]
\[
\sigma_{B'_3}(v_2) = 2n + n_4 - 4, \quad \sigma_{B'_3}(v_3) = 2n - n_3 - 5
\]
\[
\sigma_{B'_3}(v_3) = 3n - n_3 - 7, \quad \sigma_{B'_3}(v_4) = 2n - n_4 - 4
\]
\[
\sigma_{B'_3}(u_4) = 3n - n_4 - 6
\]

Thus, we easily find that

\[
\sigma_{B_3}(v_1) \sigma_{B_3}(u_1) > \sigma_{B'_3}(v_1) \sigma_{B'_3}(u_1)
\]
\[
\sigma_{B_3}(v_2) \sigma_{B_3}(u_2) > \sigma_{B'_3}(v_1) \sigma_{B'_3}(u_1)
\]
\[
\sigma_{B_3}(v_1) \sigma_{B_3}(v_2) > \sigma_{B'_3}(v_1) \sigma_{B'_3}(v_2)
\]
\[
\sigma_{B_3}(v_2) \sigma_{B_3}(v_3) < \sigma_{B'_3}(v_2) \sigma_{B'_3}(v_3)
\]

So by (49) and (50), we have

\[
\frac{n_1}{\sqrt{\sigma_{B_3}(v_1) \sigma_{B_3}(v_1)}} + \frac{n_2}{\sqrt{\sigma_{B_3}(v_2) \sigma_{B_3}(u_2)}} < \frac{n_1 + n_2}{\sqrt{\sigma_{B'_3}(v_1) \sigma_{B'_3}(u_1)}}
\]
\[
\frac{n_3}{\sqrt{\sigma_{B_3}(v_3) \sigma_{B_3}(u_3)}} = \frac{n_4}{\sqrt{\sigma_{B'_3}(v_4) \sigma_{B'_3}(u_4)}}
\]
\[
\frac{1}{\sqrt{\sigma_{B_3}(v_2)\sigma_{B_3}(v_3)}} > \frac{1}{\sqrt{\sigma_{B_3'}(v_2)\sigma_{B_3'}(v_3)}}
\] (51)

Claim 8.1.
\[
\frac{1}{\sqrt{\sigma_{B_3}(v_2)\sigma_{B_3}(v_3)}} + \frac{1}{\sqrt{\sigma_{B_3}(v_1)\sigma_{B_3}(v_3)}} < \frac{1}{\sqrt{\sigma_{B_3'}(v_2)\sigma_{B_3'}(v_3)}} + \frac{1}{\sqrt{\sigma_{B_3'}(v_1)\sigma_{B_3'}(v_3)}}
\]
i.e.
\[
\frac{1}{\sqrt{2n-n_3-5}} \left( \frac{1}{\sqrt{2n+n_4-n_2-4}} + \frac{1}{\sqrt{2n-n_1-5}} \right) < 0
\]

With the same method as Claim 3.1, Claim 8.1 follows. Hence, the result follows by (49)-(51) and Claim 8.1. □

From Lemma 8, the bicyclic graphs \(B_3\) can be transformed into the graph \(B'_0\) (see Fig 11.), and the corresponding Balaban index increases.

![Graphs B'_0, B_s, and B'](Fig 11)

Let \(B'_0\) be a bicyclic graph in \(B_n^{**}(3, 1, 3)\) with \(\{v_1, v_2, v_3, v_4\}\) as the vertex set of its two triangles such that vertex \(v_i\) has \(n_i\) pendent vertices and \(u_i\) is a pendent vertex of \(v_i\), where \(1 \leq i \leq 4\) and \(n_2 = n_4 = 0\) when \(i = 2, 4\), and \(v_1\) and \(v_3\) be the common vertices of the triangles. The bicyclic graph \(B_n\) is obtained from \(B'_0\) by changing all the pendent edges attached at \(v_3\) into the ones attached at \(v_1\) such that there are \(n_1 + n_3\) pendent edges attached at \(v_1\), that is,

\[
B_n = B'_0 - \{v_3 x| x \in N_{B'_0}(v_3) \setminus \{v_1, v_2, v_4\}\} + \{v_1 x| x \in N_{B'_0}(v_3) \setminus \{v_1, v_2, v_4\}\}
\]

Let \(B''\) be a bicyclic graph in \(B_n^{**}(3, 3)\) with \(\{v'_1, v'_2, v'_3, v'_4, v'_5\}\) as the vertex set of its two triangles and \(u'_1\) be a pendent vertex of vertex \(v'_1\), where \(v'_1\) is the unique common vertex of the triangles to which there are \(n-5\) pendent edges attached so that there are no ones attached at other vertices on the triangles. Three graphs \(B'_0, B_n\) and \(B''\) are shown in Fig 11.

Lemma 9. \(J(B'_0) < J(B_n)\).
Proof. From the structures of the graphs \(B'_0\) and \(B_n\), we obtain
\[
\sigma_{B'_0}(v_2) = \sigma_{B'_0}(v_4) = \sigma_{B_n}(v_2) = \sigma_{B_n}(v_4)
\]
Thus, by the definition of the Balaban index, we know that
\[
\frac{3}{m} J(B_0'') = \frac{n_1}{\sqrt{\sigma_{B_0''}(v_1)\sigma_{B_0''}(u_1)}} + \frac{n_3}{\sqrt{\sigma_{B_0''}(v_3)\sigma_{B_0''}(u_3)}}
+ \frac{1}{\sqrt{\sigma_{B_0''}(v_1)\sigma_{B_0''}(v_3)}} + \frac{2}{\sqrt{\sigma_{B_0''}(v_1)\sigma_{B_0''}(v_2)}}
+ \frac{2}{\sqrt{\sigma_{B_0''}(v_2)\sigma_{B_0''}(v_3)}} \tag{52}
\]
\[
\frac{3}{m} J(B_n) = \frac{n_1 + n_3}{\sqrt{\sigma_{B_n}(v_1)\sigma_{B_n}(u_1)}} + \frac{1}{\sqrt{\sigma_{B_n}(v_1)\sigma_{B_n}(v_3)}} + \frac{2}{\sqrt{\sigma_{B_n}(v_2)\sigma_{B_n}(v_3)}} \tag{53}
\]
By direct computation, we have
\[
\sigma_{B_0''}(v_1) = 2n - n_1 - 5, \quad \sigma_{B_0''}(u_1) = 3n - n_1 - 7
\]
\[
\sigma_{B_0''}(v_3) = 2n - n_3 - 5, \quad \sigma_{B_0''}(u_3) = 3n - n_3 - 7
\]
\[
\sigma_{B_0''}(v_2) = \sigma_{B_0''}(v_4) = 2n - 4, \quad \sigma_{B_n}(v_1) = n - 1
\]
\[
\sigma_{B_n}(u_1) = \sigma_{B_n}(u_3) = 2n - 3, \quad \sigma_{B_n}(v_2) = \sigma_{B_n}(v_4) = 2n - 4
\]
\[
\sigma_{B_n}(v_3) = 2n - 5
\]
So we easily find that
\[
\sigma_{B_0''}(v_1)\sigma_{B_0''}(u_1) > \sigma_{B_n}(v_1)\sigma_{B_n}(u_1), \quad \sigma_{B_0''}(v_3)\sigma_{B_0''}(u_3) > \sigma_{B_n}(v_1)\sigma_{B_n}(u_1)
\]
\[
\sigma_{B_0''}(v_1)\sigma_{B_0''}(v_3) > \sigma_{B_n}(v_1)\sigma_{B_n}(v_3), \quad \sigma_{B_0''}(v_1)\sigma_{B_0''}(v_2) > \sigma_{B_n}(v_1)\sigma_{B_n}(v_2)
\]
\[
\sigma_{B_0''}(v_2)\sigma_{B_0''}(v_3) < \sigma_{B_n}(v_2)\sigma_{B_n}(v_3)
\]
Thus,
\[
\frac{n_1}{\sqrt{\sigma_{B_0''}(v_1)\sigma_{B_0''}(u_1)}} + \frac{n_3}{\sqrt{\sigma_{B_0''}(v_3)\sigma_{B_0''}(u_3)}} < \frac{n_1 + n_3}{\sqrt{\sigma_{B_n}(v_1)\sigma_{B_n}(u_1)}} \tag{54}
\]
and
\[
\frac{1}{\sqrt{\sigma_{B_0''}(v_2)\sigma_{B_0''}(v_3)}} < \frac{1}{\sqrt{\sigma_{B_n}(v_1)\sigma_{B_n}(v_3)}} \tag{55}
\]
\[
\frac{2}{\sqrt{\sigma_{B_0''}(v_1)\sigma_{B_0''}(v_2)}} < \frac{2}{\sqrt{\sigma_{B_n}(v_1)\sigma_{B_n}(v_2)}}
\]
\[
\frac{2}{\sqrt{\sigma_{B_0''}(v_2)\sigma_{B_0''}(v_3)}} > \frac{2}{\sqrt{\sigma_{B_n}(v_2)\sigma_{B_n}(v_3)}}
\]
Claim 9.1.

\[
\frac{2}{\sqrt{\sigma_{B''}(v_1)\sigma_{B''}(v_2)}} + \frac{2}{\sqrt{\sigma_{B''}(v_2)\sigma_{B''}(v_3)}} < \frac{2}{\sqrt{\sigma_{B_n}(v_1)\sigma_{B_n}(v_2)}} + \frac{2}{\sqrt{\sigma_{B_n}(v_2)\sigma_{B_n}(v_3)}}
\]

i.e.,

\[
\frac{2}{\sqrt{2n - 4}} \left( \frac{1}{\sqrt{2n - n_1 - 5}} + \frac{1}{\sqrt{2n - n_3 - 5}} - \frac{1}{\sqrt{n - 1}} - \frac{1}{\sqrt{2n - 5}} \right) < 0
\]

With the same method as Claim 3.1, Claim 9.1 follows. So the result follows by (52)-(55) and Claim 9.1. □

From above lemmas, we focus on seeking the bicyclic graph with the largest Balaban index in the bicyclic graphs $B''$ and $B_n$.

**Lemma 10.** $J(B'') < J(B_n)$.

**Proof.** By direct calculation, we have

\[
\begin{align*}
\sigma_{B_n}(v_1) &= n - 1, \quad \sigma_{B_n}(u_1) = 2n - 3, \quad \sigma_{B_n}(v_2) = 2n - 4, \quad \sigma_{B_n}(v_3) = 2n - 5 \\
\sigma_{B''}(v'_1) &= n - 1, \quad \sigma_{B''}(u'_1) = 2n - 3, \quad \sigma_{B''}(v'_2) = 2n - 4
\end{align*}
\]

From the structures of two graphs $B_n$ and $B''$, we obtain

\[
\begin{align*}
\sigma_{B_n}(v_2) &= \sigma_{B_n}(v_4) = \sigma_{B''}(v'_2) = \sigma_{B''}(v'_3) = \sigma_{B''}(v'_4) = \sigma_{B''}(v'_5) \\
\sigma_{B_n}(v_1) &= \sigma_{B''}(v'_1), \quad \sigma_{B_n}(u_1) = \sigma_{B''}(u'_1)
\end{align*}
\]

Thus,

\[
J(B_n) - J(B'') = \frac{1}{\sqrt{(n - 1)(2n - 5)}} + \frac{2}{\sqrt{(2n - 4)(2n - 5)}} + \frac{1}{\sqrt{(n - 1)(2n - 3)}}
\]

\[
- \left( \frac{2}{\sqrt{(n - 1)(2n - 4)}} + \frac{2}{\sqrt{(2n - 4)(2n - 4)}} \right)
\]

\[
= \frac{2}{\sqrt{2n - 4}} \left( \frac{1}{\sqrt{2n - 5}} - \frac{1}{\sqrt{2n - 4}} \right)
\]

\[
+ \frac{1}{\sqrt{n - 1}} \left( \frac{1}{\sqrt{2n - 5}} - \frac{2}{\sqrt{2n - 4}} + \frac{1}{\sqrt{2n - 3}} \right) > 0
\]

This finishes the proof. □

**Proof of Theorem 2.** Let $G$ and $G'$ be two graphs and denote by $G \rightarrow G'$ graph $G$ is transformed into graph $G'$. When $G \in \mathcal{B}_n$, we distinguish the following two cases.

**Case 1.** $G \in \mathcal{B}_n^+$. Then we take the following transformations:

\[
G \rightarrow B \rightarrow B_1 \rightarrow B''
\]
So by Lemma 1, Lemma 4 and Lemma 6, we can see that

\[ J(G) < J(B) < J(B_1) < J(B'') \]

Hence, the result follows from Lemma 10.

**Case 2.** \( G \in B_n^{++} \). We discuss two subcases as follows.

**Subcase 2.1.** The graph \( G \) has no triangle \( C_3 \) as its subgraph. Then we take one of the following two types of transformations \((I)\) and \((II)\):

\[
(\text{(I)}) \quad G \rightarrow B_0 \rightarrow B \rightarrow B_1 \rightarrow B''
\]

\[
(\text{(II)}) \quad G \rightarrow B_0 \rightarrow B_2 \rightarrow B_3 \rightarrow B''_0 \rightarrow B_n
\]

When we take transformations \((I)\), by Lemma 1 and lemmas 4-6, we obtain

\[ J(G) < J(B_0) < J(B) < J(B_1) < J(B'') \]

So the result holds by Lemma 10.

When we take transformations \((II)\), by Lemma 1, Lemma 5 and lemmas 7-9, we have

\[ J(G) < J(B_0) < J(B_2) < J(B_3) < J(B''_0) < J(B_n) \]

Thus, the result follows.

**Subcase 2.2.** The graph \( G \) has at least one triangle \( C_3 \) as its subgraph. Then we take the following transformations :

\[ G \rightarrow B_2 \rightarrow B_3 \rightarrow B''_0 \rightarrow B_n \]

Therefore, by Lemma 1 and lemmas 7-9, we have

\[ J(G) < J(B_2) < J(B_3) < J(B''_0) < J(B_n) \]

Hence, this finishes the proof. □