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On AGX Conjectures Regarding Average Eccentricity

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Abstract

The eccentricity of a vertex is the maximum distance from it to another vertex and the average eccentricity $ecc(G)$ of a graph G is the mean value of eccentricities of all vertices of G. In this paper we resolve five conjectures, obtained by the system AutoGraphiX, about the average eccentricity and other graph parameters (independence number, chromatic number and the Randić index), and refute two AutoGraphiX conjectures about the average eccentricity and the spectral radius.

1 Introduction

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. Let $deg(v)$ denote the degree of the vertex v. For vertices $u, v \in V(G)$, the distance $d_G(u, v)$ is defined as the length of a shortest path from u to v in G. The eccentricity of a vertex in G is the maximum distance from it to another vertex,

$$\varepsilon_G(v) = \max_{u \in V(G)} d_G(u, v).$$

The average eccentricity of a graph G of order n is the mean value of eccentricities of all vertices of G,

$$ecc(G) = \frac{1}{n} \sum_{v \in V(G)} \varepsilon_G(v).$$

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Dankelmann, Goddard and Swart [6] presented some upper bounds and formulas for the average eccentricity regarding the diameter and the minimum vertex degree. Tang and Zhou [21] presented some lower and upper bounds for the average eccentricity in terms of the numbers of vertices and edges. Furthermore, they gave the lower and upper bounds for the average eccentricity of trees with fixed diameter, fixed number of pendent vertices and fixed matching number, respectively, and determined the n-vertex trees with the first four smallest and the first $n/2$-th largest average eccentricities for $n \geq 6$.

In theoretical chemistry molecular structure descriptors (also called topological indices) are used for modeling physico–chemical, pharmacologic, toxicologic, biological and other properties of chemical compounds. There exist several types of such indices, especially those based on vertex and edge distances [15]. Arguably the best known of these indices is the Wiener index, which is defined as the sum of distances between all (unordered) pairs of vertices of the molecular graph [8]

$$W(G) = \sum_{u,v \in V(G)} d_G(u,v).$$

Besides of the use in chemistry, it was independently studied due to its relevance in social science, architecture, and graph theory.

Sharma, Goswami and Madan [20] introduced a distance–based molecular structure descriptor, called the eccentric connectivity index, which is defined as

$$\xi^c(G) = \sum_{v \in V(G)} deg(v) \cdot \varepsilon_G(v).$$

The eccentric connectivity index is deeply connected to the average eccentricity, but for each vertex v in G, $\xi^c(G)$ takes one local property (vertex degree) and one global property (vertex eccentricity) into account. For a k-regular graph G, we have $\xi^c(G) = k \cdot n \cdot ecc(G)$. For the recent results and a survey see [13, 18, 22] and [12].

AutoGraphiX (AGX) computer system was developed by GERAD group from Montréal [1, 2, 3, 5]. AGX is an interactive software designed to help finding conjectures in graph theory. It uses the variable neighborhood search metaheuristic (Hansen and Mladenović [9, 10]) and data analysis methods to find extremal graphs with respect to one or more invariants. Recently there are vast researches regarding AGX conjectures and a series of papers on various graph invariants, e.g., [2, 5, 11]. In this paper we continue this work and resolve conjectures from [14] and [1].

The paper is organized as follows. In Section 3 we resolve two conjectures regarding the maximum values of $ecc(G)/\alpha(G)$ and $ecc(G)-\alpha(G)$, where $\alpha(G)$ is the independence number of G. In Section 4 we resolve the conjecture regarding the maximum value of the product of the average eccentricity and the chromatic number. Using hypercubes, in Section 5 we refute two conjectures regarding the minimum values of the sum and the product of the spectral radius and the average eccentricity. In Section 6 two conjectures regarding the minimum values of the sum and the product of the Randić index and the average eccentricity are resolved.

2 Preliminaries

Let P_n, S_n, K_n be the path, the star, the complete graph on n vertices, respectively.

The broom $B(n,\Delta)$ is the tree obtained from a star S_Δ and a path $P_{n-\Delta+1}$ by identifying the central vertex of S_Δ with one of the end vertices of $P_{n-\Delta+1}$. The lollipop graph $LP(n,t)$ is the graph obtained from a complete graph K_t and a path P_{n-t+1} by identifying one vertex of K_t with one of the end vertices of P_{n-t+1}. Clearly, $ecc(B(n,k)) = ecc(LP(n,k))$ for $2 \le k \le n-1$.

For integers n, d with $2 \le d \le n-1$, let

$$f(n,d) = \frac{1}{n} \cdot \left(\left\lfloor \frac{(d+1)(3d+1)}{4} \right\rfloor + (n-d-1) \cdot d \right).$$

If G is a connected graph on n vertices with diameter d, then

$$ecc(G) \;\le\; \frac{1}{n} \cdot [(d+1) \cdot ecc(P_{d+1}) + (n-d-1) \cdot d] = f(n,d)$$

with equality if and only if each vertex outside the diametrical path has eccentricity d.

Note that $f(n,d) = ecc(B(n,n-d+1)) = ecc(LP(n,n-d+1))$. It is easily seen that

$$f(n,d) = ecc(LP(n,n-d+1)) < ecc(LP(n,n-d)) = f(n,d+1),$$

i.e., $f(n,d)$ is increasing for d. It follows that the path is the unique connected graph with maximum average eccentricity.

3 Conjectures regarding the independence number

An independent set of a graph is a vertex subset in which no two vertices are adjacent. The independence number of a graph G, denoted by $\alpha(G)$, is the maximum cardinality of an independent set of G.

Theorem 3.1 (Conjecture A.479-U) *For every $n \geq 4$ the maximum value of $ecc(G)/\alpha(G)$ is achieved for some graph composed of two cliques linked by a path.*

Proof: Let G be a connected graph on n vertices with independence number $\alpha = \alpha(G)$.

If $\alpha(G) \geq \lceil \frac{n}{2} \rceil$, and note that $\alpha(P_n) = \lceil \frac{n}{2} \rceil$, then we have

$$ecc(G)/\alpha(G) \leq ecc(G) \Big/ \left\lceil \frac{n}{2} \right\rceil \leq ecc(P_n) \Big/ \left\lceil \frac{n}{2} \right\rceil = ecc(P_n)/\alpha(P_n).$$

Suppose that $\alpha(G) < \lceil \frac{n}{2} \rceil$. Let d be the diameter of G. Since $d \leq 2\alpha - 1$, we have

$$ecc(G) \leq f(n,d) \leq f(n, 2\alpha - 1) = ecc(LP(n, n - 2\alpha + 2)).$$

Note that the average eccentricity of each n-vertex graph composed of two cliques K_a and K_b with $a + b = n - 2\alpha + 4$ linked by a path is equal to $ecc(LP(n, n - 2\alpha + 2))$.

Then the result follows easily. ■

By the above proof, we have the maximum value of $ecc(G)/\alpha(G)$ is $f(n, 2\alpha - 1)/\alpha$ with $\alpha \leq \frac{n}{2}$. Furthermore, we have

$$\begin{aligned} ecc(G)/\alpha(G) &\leq f(n, 2\alpha - 1)/\alpha \\ &= \frac{1}{n}\left(2n + 1 - \left(\alpha + \frac{n}{\alpha}\right)\right) \\ &\leq \frac{1}{n}\left(2n + 1 - 2\sqrt{n}\right) \\ &= \left(1 - \frac{1}{\sqrt{n}}\right)^2 + 1. \end{aligned}$$

The equality can be achieved by the lollipop graph $LP(n, n - 2\sqrt{n} + 2)$ if $\sqrt{n}$ is an integer.

Theorem 3.2 (Conjecture A.477-U) *For every $n \geq 4$ it holds*

$$ecc(G) - \alpha(G) \leq \begin{cases} \frac{n}{4} - \frac{1}{2} & \text{if } n \text{ is even} \\ \frac{n}{4} - \frac{1}{2} - \frac{3}{4n} & \text{if } n \text{ is odd} \end{cases}$$

with equality if and only if $G \cong P_n$ for even n, and $G \cong LP(n, 3)$ for odd n.

Proof: Let G be a connected graph on $n \geq 4$ vertices with maximum value $ecc(G) - \alpha(G)$. Similar to the proof of Theorem 3.1, we have G is a graph composed of two cliques linked by a path. Assume that $G \cong LP(n, n - 2t + 2)$. Clearly, $\alpha(G) = t$, and

$$ecc(G) = ecc(LP(n, n - 2t + 2)) = ecc(B(n, n - 2t + 2)).$$

Let $H_{n,t} = B(n, n-2t+2)$. Let P be a diametrical path of $H_{n,t}$, and v be the vertex with maximum degree $n-2t+2$ in $H_{n,t}$.

Suppose that v has at least two pendent neighbors, say x, y, outside P in $H_{n,t}$. Let H be the tree obtained from $H_{n,t}$ by deleting the edges xv, yv and attaching x and y, respectively, to an end vertex of P. It is easily seen that $\varepsilon_H(z) = \varepsilon_{H_{n,t}}(z) + 1$ for $z \in V(H_{n,t}) \setminus \{x, y\}$, $\varepsilon_H(z) = \varepsilon_{H_{n,t}}(z) + 2$ for $z = x, y$, and thus $ecc(H) > ecc(H_{n,t}) + 1$. Since both H and $H_{n,t+1}$ have the same diameter $2t+1$, we have $ecc(H) \leq f(n, 2t+1) = ecc(H_{n,t+1})$. It follows that $ecc(H_{n,t+1}) > ecc(H_{n,t}) + 1$, i.e.,

$$ecc(LP(n, n-2t)) = ecc(B(n, n-2t)) > ecc(B(n, n-2t+2)) + 1 = ecc(G) + 1.$$

Note that $\alpha(LP(n, n-2t)) = t+1$, and thus

$$ecc(LP(n, n-2t)) - \alpha(LP(n, n-2t)) > (ecc(G) + 1) - (t+1) = ecc(G) - \alpha(G),$$

which is a contradiction.

Thus v has at most one pendent neighbor outside P in $H_{n,t}$, i.e., $H_{n,t} \cong P_n$ or $B(n, 3)$, implying that $G \cong P_n$ or $LP(n, 3)$.

It is easily seen that

$$ecc(LP(n,3)) = ecc(B(n,3)) < ecc(P_n) < ecc(B(n,3)) + 1 = ecc(LP(n,3)) + 1,$$

and $\alpha(LP(n,3)) = \lceil \frac{n-1}{2} \rceil$, $\alpha(P_n) = \lceil \frac{n}{2} \rceil$. Now it follows that $G \cong P_n$ for even n and $G \cong LP(n, 3)$ for odd n. ■

4 Conjecture regarding the chromatic number

A graph G is said to be k-colorable if we may partition the vertices of G into k independent sets. The chromatic number of a graph G, denoted by $\chi(G)$, is the minimum k for which G is k-colorable.

Theorem 4.1 (Brooks) *For any connected graph G with maximum degree Δ, $\chi(G) \leq \Delta$ unless G is a clique or an odd cycle, in which case $\chi(G) = \Delta + 1$.*

Theorem 4.2 (Conjecture A.492-U) *For every $n \geq 4$ the maximum value of $ecc(G) \cdot \chi(G)$ is achieved for some lollipop graph.*

Proof: Let G be a connected graph on n vertices with chromatic number $\chi = \chi(G)$, maximum degree Δ, and diameter d.

If $\chi = 2$, then G is a bipartite graph, and thus $ecc(G) \leq ecc(P_n) = ecc(LP(n, 2))$.

If $\chi = 3$, then $d \leq n - 2$, and thus $ecc(G) \leq f(n, d) \leq f(n, n - 2) = ecc(LP(n, 3))$.

If $\chi = n$, then $G \cong K_n \cong LP(n, n)$.

Suppose that $4 \leq \chi \leq n - 1$. Then G is not a complete graph or an odd cycle. By Brook's theorem, we know that $\chi \leq \Delta$. Let v be a vertex with maximum degree Δ in G. Among v and its neighbors, there are at most three vertices lying on a diametrical path of G. It follows that $d + 1 \leq 3 + (n - \Delta - 1)$, i.e., $d \leq n - \Delta + 1$. Now we have

$$ecc(G) \leq f(n, d) \leq f(n, n - \Delta + 1) \leq f(n, n - \chi + 1) = ecc(LP(n, \chi)).$$

Combining the above arguments, we have $ecc(G) \leq ecc(LP(n, \chi))$ for fixed χ. Then the result follows easily. ■

By considering the maximum value of the function $ecc(LP(n, \chi)) \cdot \chi$ on χ, we have the maximum value of $ecc(G) \cdot \chi(G)$ is achieved by $LP(n, \chi)$ with χ being the integer closest to $\frac{1}{3}(2 - 2n + \sqrt{13n^2 - 2n + 4})$.

5 Conjectures regarding the spectral radius

Let $\lambda(G)$ be the spectral radius of G, i.e., the largest eigenvalue of the adjacency matrix of G.

Conjecture 5.1 (Conjecture A.458-L) *For every $n \geq 4$ it holds*

$$\lambda(G) + ecc(G) \geq \sqrt{n - 1} + 2 - \frac{1}{n}$$

with equality if and only if $G \cong S_n$.

Conjecture 5.2 (Conjecture A.460-L) *For every $n \geq 4$ it holds*

$$\lambda(G) \cdot ecc(G) \geq \sqrt{n - 1} \cdot \left(2 - \frac{1}{n}\right)$$

with equality if and only if $G \cong S_n$.

The k-dimensional hypercube Q_k is the graph whose vertices are the ordered k-tuples of 0's and 1's, two vertices being adjacent if and only if they differ in exactly one coordinate.

It is easily seen that Q_k is a k-regular graph (i.e., $\lambda(Q_k) = k$) with $n = 2^k$ vertices and $\varepsilon_{Q_k}(v) = k$ for each vertex $v \in V(Q_k)$.

Now we construct an n-vertex graph G_n which is a counterexample for Conjectures 5.1 and 5.2 for n being sufficiently large. If $n = 2^m$ for some integer m, then let $G_n = Q_m$. If $2^m < n < 2^{m+1}$, then let G_n be the n-vertex graph obtained from Q_m by attaching $n - 2^m$ pendent vertices each to one vertex of Q_m.

If $n = 2^m$ for some integer m, then

$$\lambda(G_n) + ecc(G_n) = \lambda(Q_m) + ecc(Q_m) = 2m < \sqrt{2^m - 1} + 2 - \frac{1}{2^m} = \sqrt{n-1} + 2 - \frac{1}{n}$$

for $m \geq 8$, and

$$\lambda(G_n) \cdot ecc(G_n) = \lambda(Q_m) \cdot ecc(Q_m) = m^2 < \sqrt{2^m - 1} \cdot \left(2 - \frac{1}{2^m}\right) = \sqrt{n-1} \cdot \left(2 - \frac{1}{n}\right)$$

for $m \geq 13$. Suppose that $2^m < n < 2^{m+1}$. Note that the $(m+1)$-dimensional hypercube Q_{m+1} is the Cartesian product of the m-dimensional hypercube Q_m and K_2. Clearly, G_n is a proper subgraph of Q_{m+1}, and thus $\lambda(G_n) < \lambda(Q_{m+1}) = m + 1$, and $\varepsilon_{G_n}(v) \leq m + 2$ for each vertex $v \in V(G_n)$. By direct verification we get

$$\lambda(G_n) + ecc(G_n) < (m+1) + (m+2) < \sqrt{2^m - 1} + 2 - \frac{1}{2^m} < \sqrt{n-1} + 2 - \frac{1}{n}$$

for $m \geq 9$, and

$$\lambda(G_n) \cdot ecc(G_n) < (m+1) \cdot (m+2) < \sqrt{2^m - 1} \cdot \left(2 - \frac{1}{2^m}\right) < \sqrt{n-1} \cdot \left(2 - \frac{1}{n}\right)$$

for $m \geq 14$.

These refute Conjectures 5.1 and 5.2 for n being sufficiently large.

6 Conjectures regarding the Randić index

The Randić index of G is defined as [19]

$$Ra(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{deg(u)deg(v)}}.$$

For a recent survey see [16].

Lemma 6.1 ([4]) *Let G be a connected graph on $n \geq 2$ vertices. Then*

$$Ra(G) \geq \sqrt{n-1}$$

with equality if and only if $G \cong S_n$.

For graphs G and H with disjoint vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$, let $G \cup H$ denote the vertex-disjoint union of G and H, and let $G \vee H$ denote the join of G and H which is the graph union $G \cup H$ together with all the edges joining $V(G)$ and $V(H)$.

Lemma 6.2 ([7]) *Let G be a connected graph on n vertices with minimum degree at least 2. Then*

$$Ra(G) \geq \sqrt{2(n-1)} + \frac{1}{n-1} - \frac{\sqrt{2}}{\sqrt{n-1}}$$

with equality if and only if G is the join of K_2 and the complement of K_{n-2}.

Theorem 6.1 (Conjecture A.462-L) *For every $n \geq 4$ it holds*

$$Ra(G) + ecc(G) \geq \sqrt{n-1} + 2 - \frac{1}{n}$$

with equality if and only if $G \cong S_n$.

Proof: We have verified the result on the set of all connected graphs with $n \leq 10$ vertices (with the help of Nauty [17] for the generation of non-isomorphic graphs). In the following let G be a connected graph on $n \geq 11$ vertices.

Suppose that there is no vertex of degree $n-1$ in G. Then $ecc(G) \geq 2$. And by Lemma 6.1, $Ra(G) \geq \sqrt{n-1}$. We have

$$Ra(G) + ecc(G) \geq \sqrt{n-1} + 2 > \sqrt{n-1} + 2 - \frac{1}{n}.$$

Suppose that there are $s \geq 1$ vertices of degree $n-1$ in G. Then $ecc(G) = 2 - \frac{s}{n}$. If $s = 1$, then the result follows from Lemma 6.1 directly. Now suppose that $s \geq 2$. Clearly, the minimum degree of G is at least $s \geq 2$. By Lemma 6.2,

$$Ra(G) \geq \sqrt{2(n-1)} + \frac{1}{n-1} - \frac{\sqrt{2}}{\sqrt{n-1}}.$$

Now we have

$$\begin{aligned} Ra(G) + ecc(G) &\geq \left(\sqrt{2(n-1)} + \frac{1}{n-1} - \frac{\sqrt{2}}{\sqrt{n-1}} \right) + 1 \\ &> \sqrt{n-1} + 2 - \frac{1}{n}. \end{aligned}$$

The last inequality above holds for $11 \leq n \leq 24$ by direct calculation, and holds for $n \geq 25$ since $\sqrt{2(n-1)} > \sqrt{n-1} + 2$. ■

For the vertex $v \in V(G)$, let $G - v$ denote the graph obtained from G by deleting the vertex v and its incident edges. For the edge $e \in E(G)$, let $G - e$ denote the graph obtained from G by deleting the edge e.

Lemma 6.3 ([11]) *Let G be a graph with minimum degree δ and maximum degree Δ. For a vertex v with minimum degree δ,*

$$Ra(G) \geq Ra(G - v) + \frac{1}{2}\sqrt{\frac{\delta}{\Delta}}.$$

For $e = uv \in E(G)$, the value $\frac{1}{\sqrt{deg(u)deg(v)}}$ is called the weight of e.

Lemma 6.4 ([4]) *Let e be an edge with maximal weight in G. Then $Ra(G) > Ra(G-e)$.*

Theorem 6.2 (Conjecture A.464-L) *For every $n \geq 4$ it holds*

$$Ra(G) \cdot ecc(G) \;\; \geq \;\; \begin{cases} \frac{n}{2} \cdot 1 & \text{if } n \leq 13 \\ \sqrt{n-1}\left(2 - \frac{1}{n}\right) & \text{if } n \geq 14 \end{cases}$$

with equality if and only if $G \cong K_n$ for $n \leq 13$, and $G \cong S_n$ for $n \geq 14$.

Proof: Let G be a connected graph on $n \geq 4$ vertices with minimum degree δ. Denote by s the number of vertices of degree $n-1$ in G. Note that $ecc(G) = 2 - \frac{s}{n}$ if $s \geq 1$.

We have verified the result on the set of all connected graphs with $n \leq 14$ vertices (with the help of Nauty [17] for the generation of non-isomorphic graphs). Suppose in the following that $n \geq 15$.

If $s = 1$, then the result follows from Lemma 6.1 directly.

In the following we will prove that

$$Ra(G) \cdot ecc(G) > \sqrt{n-1}\left(2 - \frac{1}{n}\right) \tag{1}$$

if $s \neq 1$.

If $s = 0$, then $ecc(G) \geq 2$, and together with Lemma 6.1, we have

$$Ra(G) \cdot ecc(G) \geq \sqrt{n-1} \cdot 2 > \sqrt{n-1}\left(2 - \frac{1}{n}\right).$$

If $s = n$, then $G \cong K_n$, and thus

$$Ra(G) \cdot ecc(G) = \frac{n}{2} \cdot 1 > \sqrt{n-1}\left(2 - \frac{1}{n}\right).$$

The inequality above holds since $\frac{n}{2} > 2\sqrt{n-1}$ for $n \geq 15$.

Suppose that $s = n-2$. Then G is the join of the complete graph K_{n-2} and the complement of K_2. Let u be a vertex of degree $n-2$ in G. Clearly, $G-u \cong K_{n-1}$. By Lemma 6.3,

$$Ra(G) \geq Ra(G-u) + \frac{1}{2}\sqrt{\frac{n-2}{n-1}} = \frac{n-1}{2} + \frac{1}{2}\sqrt{\frac{n-2}{n-1}},$$

and thus

$$\begin{aligned} Ra(G)\cdot ecc(G) &\geq \left(\frac{n-1}{2} + \frac{1}{2}\sqrt{\frac{n-2}{n-1}}\right)\left(2 - \frac{n-2}{n}\right) \\ &= \left(\frac{n-1}{2} + \frac{1}{2}\sqrt{\frac{n-2}{n-1}}\right)\left(1 + \frac{2}{n}\right) \\ &> \sqrt{n-1}\left(2 - \frac{1}{n}\right). \end{aligned}$$

The last inequality above holds for $n = 15, 16$ by direct calculation, and holds for $n \geq 17$ since $\frac{n-1}{2} \geq 2\sqrt{n-1}$.

Suppose that $2 \leq s \leq n-3$. Note that $\delta \geq s$.

Case 1. $\delta = s$.

We will prove the inequality (1) by induction on n. Denote by v a vertex of degree $\delta = s$ in G. By Lemma 6.1, $Ra(G-v) \geq \sqrt{n-2}$, and by Lemma 6.3,

$$Ra(G) \geq Ra(G-v) + \frac{1}{2}\sqrt{\frac{s}{n-1}}.$$

Denote by t the number of vertices of degree $n-2$ in G. Then $ecc(G-v) = 2 - \frac{s+t}{n-1}$. By the induction hypothesis, we have

$$Ra(G-v)\cdot\left(2 - \frac{s+t}{n-1}\right) = Ra(G-v)\cdot ecc(G-v) \geq \sqrt{n-2}\cdot\left(2 - \frac{1}{n-1}\right).$$

It follows that

$$\begin{aligned} Ra(G)\cdot ecc(G) &\geq \left(Ra(G-v) + \frac{1}{2}\sqrt{\frac{s}{n-1}}\right)\cdot\left(2 - \frac{s}{n}\right) \\ &= Ra(G-v)\cdot\left(2 - \frac{s}{n}\right) + \frac{1}{2}\sqrt{\frac{s}{n-1}}\cdot\left(2 - \frac{s}{n}\right) \\ &= Ra(G-v)\cdot\left(2 - \frac{s+t}{n-1} + \frac{s+t}{n-1} - \frac{s}{n}\right) + \frac{1}{2}\sqrt{\frac{s}{n-1}}\cdot\left(2 - \frac{s}{n}\right) \\ &= Ra(G-v)\cdot\left(2 - \frac{s+t}{n-1}\right) + Ra(G-v)\cdot\left(\frac{s+t}{n-1} - \frac{s}{n}\right) \\ &\quad + \frac{1}{2}\sqrt{\frac{s}{n-1}}\cdot\left(2 - \frac{s}{n}\right) \\ &\geq \sqrt{n-2}\cdot\left(2 - \frac{1}{n-1}\right) + \sqrt{n-2}\cdot\left(\frac{s+t}{n-1} - \frac{s}{n}\right) \end{aligned}$$

$$
\begin{aligned}
& + \frac{1}{2}\sqrt{\frac{s}{n-1}} \cdot \left(2 - \frac{s}{n}\right) \\
= \ & \sqrt{n-2} \cdot \left(2 - \frac{1}{n-1} + \frac{s+t}{n-1} - \frac{s}{n}\right) + \frac{1}{2}\sqrt{\frac{s}{n-1}} \cdot \left(2 - \frac{s}{n}\right) \\
> \ & \sqrt{n-2} \cdot \left(2 - \frac{1}{n-1} + \frac{1+0}{n-1} - \frac{1}{n}\right) + \frac{1}{2}\sqrt{\frac{s}{n-1}} \cdot \left(2 - \frac{s}{n}\right) \\
= \ & \sqrt{n-2} \cdot \left(2 - \frac{1}{n}\right) + \frac{1}{2}\sqrt{\frac{s}{n-1}} \cdot \left(2 - \frac{s}{n}\right).
\end{aligned}
$$

Let $f(x) = \frac{1}{2}\sqrt{\frac{x}{n-1}} \cdot \left(2 - \frac{x}{n}\right)$ for $2 \leq x \leq n$. Note that $f(x)$ is increasing for x if $2 \leq x \leq \frac{2n}{3}$, and decreasing for x if $\frac{2n}{3} \leq x \leq n$. This implies that the minimum value of $f(x)$ is achieved when $x = 2$ and therefore we need to prove that

$$
\begin{aligned}
Ra(G) \cdot ecc(G) \ & > \ \sqrt{n-2} \cdot \left(2 - \frac{1}{n}\right) + \frac{1}{2}\sqrt{\frac{2}{n-1}} \cdot \left(2 - \frac{2}{n}\right) \\
& > \ \sqrt{n-1} \cdot \left(2 - \frac{1}{n}\right),
\end{aligned}
$$

which is equivalent to prove that

$$
\sqrt{2}\left((n-1) - \sqrt{(n-1)(n-2)}\right) < \frac{2n-2}{2n-1}.
$$

The last inequality above holds since it holds for $n = 4$, and the left side is decreasing for $n \geq 4$ and the right side is increasing for $n \geq 4$.

Case 2. $\delta \geq s + 1$.

Let A be the set of vertices of degree $n-1$, and $B = V(G) \setminus A$. Then every vertex in B has at least one neighbor in B. Let e be an edge with maximal weight in G. Clearly, the two end vertices of e are both in B. By Lemma 6.4, $Ra(G) > Ra(G-e)$. Together with the fact that $ecc(G) = ecc(G-e) = 2 - \frac{s}{n}$, we know that

$$
Ra(G) \cdot ecc(G) > Ra(G-e) \cdot ecc(G-e).
$$

By deleting an edge with maximal weight repeatedly, we can obtain a connected graph on n vertices, say H, such that its minimum degree is s, and

$$
Ra(G) \cdot ecc(G) > Ra(H) \cdot ecc(H).
$$

Now applying the arguments in Case 1 to H, we have

$$
Ra(G) \cdot ecc(G) > Ra(H) \cdot ecc(H) > \sqrt{n-1}\left(2 - \frac{1}{n}\right).
$$

Combining the above arguments, the inequality (1) follows, and thus the result follows easily. ■

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