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Interpolation Method and Topological Indices: 2-Parametric Families of Graphs

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Abstract

Interpolation method can be used to obtain closed formulas for topological indices of families of graphs. In this paper we discuss the method on 2-parametric families of graphs. The method is illustrated on the family of carbon nanocones $CNC_k[n]$ and on four distance based topological indices: the Wiener index, the edge Wiener index, the reverse Wiener index, and the Szeged index. Closed formulas are presented and several exceptions are pointed out for small values of k. For larger values of k, parity cases must be usually consider to obtain desired polynomials.

1 Introduction

In mathematical chemistry, one way to obtain closed formulas for topological indices of families of graphs is the so-called interpolation method [7, 8]. This method and the way how it can be used to obtain desired formulas was recently discussed in [2]. The method was illustrated on fullerenes C_{12k+4} and four topological indices: the Wiener index, the edge Wiener index, the reverse Wiener index, and the eccentric connectivity index. In this paper we extend the method to the case when a family of graphs is defined with two (or

more) parameters, so that formulas we are searching for are polynomials in two (or more) variables. The method is then applied on carbon nanocones $CNC_k[n]$ for the Wiener index, the edge Wiener index, the reverse Wiener index, and the Szeged index. (We do not consider the eccentric connectivity index of carbon nanocones $CNC_k[n]$ because this has already been done in [1].)

Each of the four distance-based topological indices we are considering is of recent interest. Not much has to be said about the famous Wiener index, let us only mention that it is still extensively investigated, cf. [5, 10, 12, 21, 28]. The edge Wiener index was recently introduced in [17] but already received a lot of attention, cf. [9, 27]. In particular, the cut method [19] for this topological index was developed in [29]. The reverse Wiener index was introduced by Balaban et al. [25] and studied for instance in [6, 11, 13]. Finally, the Szeged index that was introduced in 1995 [18], received a recent revival of interest, see [3, 4, 16, 22, 23, 24].

In the rest of this section necessary definition are given. In the subsequent section the 2-parametric interpolation method is discussed while in Section 3 the method is applied to the carbon-nanocone graphs $CNC_k[n]$. In the final section we give related formulas for the haxagonal-parallelogram graphs $P(n,k)$ and point out that in principle the method can be applied for families of graphs defined with more than two parameters.

Let G be a connected graph. Then the *distance* between vertices u and v is denoted by $d(u,v)$. The *diameter* of G is the maximum distance between its vertices and is denoted by $\mathrm{diam}(G)$. The distance between edges $g = u_1v_1$ and $f = u_2v_2$ is defined as

$$d_e(g,f) = \min\{d(u_1,u_2), d(u_1,v_2), d(v_1,u_2), d(v_1,v_2)\} + 1\,.$$

Equivalently, this is the distance between the vertices g and f in the line graph of G [17]. The degree of a vertex u will be denoted by $\deg(u)$.

Let G be a connected graph on n vertices. Then the *Wiener index* $W(G)$, the *edge Wiener index* $W_e(G)$, the *reverse Wiener index* $RW(G)$, and the *Szeged index* $Sz(G)$, are respectively defined as follows:

$$\begin{aligned} W(G) &= \sum_{\{u,v\}\in\binom{V(G)}{2}} d(u,v)\,, \\ W_e(G) &= \sum_{\{f,g\}\in\binom{E(G)}{2}} d_e(f,g)\,, \\ RW(G) &= \frac{1}{2}n(n-1)\mathrm{diam}(G) - W(G)\,, \end{aligned}$$

$$Sz(G) \;=\; \sum_{uv \in E(G)} n_u n_v \,.$$

Here n_u (resp. n_v) is the number of vertices that are closer to vertex u (resp. v) than to vertex v (resp. u).

2 The method on 2-parametric families of graphs

Let $\{G_{n,k}\}_{n,k\geq 1}$ be a series of graphs and let I be a topological index. Suppose that there exist $n_0, k_0 \in \mathbb{N}$ such that for any $n \geq n_0$ and $k \geq k_0$, $I(G_{n,k}) = p(n,k)$, where p is a polynomial in two variables. We will say that p *interpolates* I on $\{G_{n,k}\}_{n,k\geq 1}$. If this is the case, then $I(G_{n,k})$ is completely determined by the polynomial $p(n,k)$ and the initial values $I(G_{n,k})$, where $1 \leq n < n_0$, and $1 \leq k < k_0$. To determine the degree of p, we can use:

Lemma 2.1 *Suppose that $p(n,k)$ interpolates I on $\{G_{n,k}\}_{n,k\geq 1}$. If there exists a positive constant α such that $I(G_{n,k}) < \alpha n^s k^t$, then* $\deg(p) \leq s+t$.

Proof. Suppose on the contrary that $\deg(p) > s+t$. Then $\frac{I(G_{n,k})}{p(n,k)} < \frac{\alpha n^s k^t}{p(n,k)} \xrightarrow[n,k\to\infty]{} 0$. On the other hand p interpolates I on $\{G_{n,k}\}_{n,k\geq 1}$, that is, there exist $n_0, k_0 \in \mathbb{N}$ such that for any $n \geq n_0$, and $k \geq k_0$, $I(G_{n,k}) = p(n,k)$. Hence $\frac{I(G_{n,k})}{p(n,k)} \xrightarrow[n,k\to\infty]{} 1$, a contradiction. □

Similarly as discussed in [2] for 1-parametric families and consequently for polynomials in one variable, in order to apply the interpolation method, we need to (i) find out the smallest n_0 and k_0 such that the interpolation works for all $n \geq n_0$ and $k \geq k_0$ (ii) prove that $I(G_{n,k})$ is a fixed polynomial function, and (iii) determine a constant upper bound on the degree of p, more precisely, the largest power in n and the largest power in k. We emphasize that as far as we know, in the literature condition (ii) was eventually never formally proved. This means that the obtained formulas are only conditionally true. But if they were checked for large n and k, a practical use of them is safe.

Suppose now that in $p(n,k)$ the largest powers in n and in k are s and t, respectively. Then $p(n,k)$ can be written as

$$p(n,k) = q_s(k)n^s + q_{s-1}(k)n^{s-1} + \cdots + q_1(k)n + q_0(k)\,,$$

where $q_i(k)$ is a polynomial of degree at most t and at least one of the $q_i(k)$'s is of degree t. Suppose now that conditions (i)-(iii) are fulfilled and let $k_0 < k_1 < \cdots < k_t$. Then for any k_i,

$$p(n, k_i) = C_{s,i}n^s + C_{s-1,i}n^{s-1} + \cdots + C_{1,i}n + C_{0,i}$$

is a polynomial obtained by interpolating the points $((n_0, k_i), I(G_{n_0,k_i}))$, $((n_1, k_i), I(G_{n_1,k_i}))$, ..., $((n_s, k_i), I(G_{n_s,k_i}))$. Then, $p(n, k_i) = I(G_{n,k_i})$, holds for $i = 0, 1, \ldots, s$. Finally, the coefficients of n^j in these $t + 1$ polynomials are determined by interpolating the points $\{(k_i, C_{j,i})\}_{0 \le i \le t}$.

3 Carbon nanocones $CNC_k[n]$

In this section we demonstrate the use of the (modified) interpolation method on the carbon-nanocone graphs $CNC_k[n]$. These graphs are defined for any $n \geq 1$ and any $k \geq 3$. The carbon nanocones $CNC_5[4]$ and $CNC_6[3]$ are shown in Fig. 1 and from these two examples the general construction should be clear: the parameter k defines the length of the inner cycle and n defines the number of layers of the graph. Note that in the particular case $k = 6$, the series $CNC_6[n]$ plays a special role among hexagonal graphs (alias benzenoid graphs) and is known as the coronene/circumcoronene series, and denoted $H_n = CNC_6[n]$. The first term of it is the 6-cycle which is the molecular graph of the benzene.

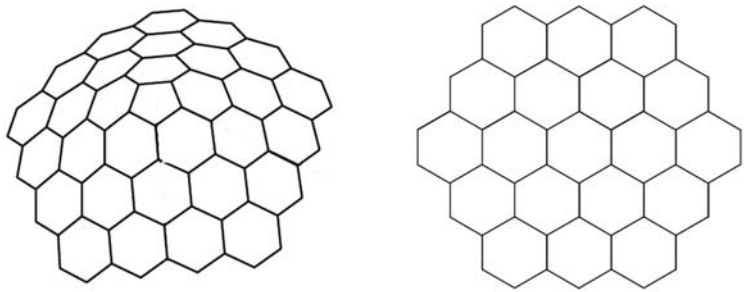

Figure 1: $CNC_5[4]$ and $CNC_6[3]$

3.1 Wiener index

Lemma 3.1 *Suppose $p(n,k)$ interpolates the Wiener index on $\{CNC_k[n]\}_{n\geq 5,k\geq 1}$. Then the largest powers in n and in k are at most 5 and 3, respectively.*

Proof. For a connected graph G, $W(G) \leq \binom{|V(G)|}{2}\mathrm{diam}(G)$. The numbers of vertices and edges of $CNC_k[n]$ are kn^2 and $\frac{1}{2}k(3n^2-n)$ respectively. Since $\mathrm{diam}(CNC_k[n]) = 4(n-1)+\lfloor\frac{k}{2}\rfloor$ holds for $n\geq 1$ and $k\geq 5$, it follows that

$$W(CNC_k[n]) \leq \binom{kn^2}{2}\left(4(n-1)+\frac{k}{2}\right) < 2n^5k^3,\ n\geq 5, k\geq 1\,.$$

The assertion now follows from Lemma 2.1. □

It turns out that the interpolation method does not yield to a polynomial. On the other hand, considering separately the cases when $k\geq 5$ is odd and when $k\geq 6$ is even, leads to two intrinsically different polynomials. When $n\geq 1$ and $k\geq 5$ is odd, we get

$$\begin{aligned} W\left(CNC_k[n]\right) &= \left(\frac{4}{3}k^2-\frac{38}{15}k\right)n^5+\left(\frac{1}{8}k^3-\frac{3}{2}k^2+\frac{35}{8}k\right)n^4+ \\ &\quad \left(\frac{1}{6}k^2-2k\right)n^3+\left(\frac{k}{30}\right)n\,, \end{aligned} \tag{1}$$

and for $n\geq 1$ and even $k\geq 6$ is even,

$$\begin{aligned} W\left(CNC_k[n]\right) &= \left(\frac{4}{3}k^2-\frac{38}{15}k\right)n^5+\left(\frac{1}{8}k^3-\frac{3}{2}k^2+\frac{9}{2}k\right)n^4+ \\ &\quad \left(\frac{1}{6}k^2-2k\right)n^3+\left(\frac{k}{30}\right)n\,. \end{aligned} \tag{2}$$

The first few polynomials for fixed ks are collected in Table 1.

The interpolation method can also be applied for $k=3$ and $k=4$. For $k=4$ we obtain

$$W(CNC_4[n]) = \frac{58}{2}n^5-\frac{8}{3}n^3-\frac{14}{15}n\,,$$

a result different than we would obtain by substituting $k=4$ into equation (2). The situation is even more surprising for $k=3$. In this case it turned out that three cases depending on the parity of n must be considered:

$$\begin{aligned} W(CNC_3[n]) &= \frac{163}{30}n^5-\frac{20}{9}n^3-\frac{1}{10}n, \quad n\equiv 0 \pmod 3, \\ W(CNC_3[n]) &= \frac{163}{30}n^5-\frac{20}{9}n^3-\frac{1}{10}n-\frac{1}{9} \quad n\equiv 1 \pmod 3, \\ W(CNC_3[n]) &= \frac{163}{30}n^5-\frac{20}{9}n^3-\frac{1}{10}n+\frac{1}{9}, \quad n\equiv 2 \pmod 3\,, \end{aligned}$$

k	$W(CNC_k[n])$
5	$\frac{62}{3}n^5 - \frac{35}{6}n^3 + \frac{1}{6}n$
6	$\frac{164}{5}n^5 - 6n^3 + \frac{1}{5}n$
7	$\frac{238}{5}n^5 - \frac{35}{6}n^3 + \frac{7}{30}n$
8	$\frac{976}{15}n^5 + 4n^4 - \frac{16}{3}n^3 + \frac{4}{15}n$
9	$\frac{426}{5}n^5 + 9n^4 - \frac{9}{2}n^3 + \frac{3}{10}n$
10	$108n^5 + 20n^4 - \frac{10}{3}n^3 + \frac{1}{3}n$
11	$\frac{2002}{15}n^5 + 33n^4 - \frac{11}{6}n^3 + \frac{11}{30}n$
12	$\frac{808}{5}n^5 + 54n^4 + \frac{2}{5}n$
13	$\frac{962}{5}n^5 + 78n^4 + \frac{13}{6}n^3 + \frac{13}{30}n$
14	$\frac{3388}{15}n^5 + 112n^4 + \frac{14}{3}n^3 + \frac{7}{15}n$
15	$262n^5 + 150n^4 + \frac{15}{2}n^3 + \frac{1}{2}n$

Table 1: The Wiener index of $CNC_k[n]$, $5 \leq k \leq 15$

so that we can write

$$W(CNC_3[n]) \;=\; \text{round}\left(\frac{163}{30}n^5 - \frac{20}{9}n^3 - \frac{1}{10}n\right).$$

where the function round is the usual rounding function.

3.2 Edge Wiener index

We continue with the edge Wiener index W_e of $CNC_k[n]$. It is well known, cf. [9] that it is equal to $W(L(CNC_k[n]))$, where $L(G)$ denotes the line graph of G. Since $|L(CNC_k[n])| = \frac{1}{2}k(3n^2 - n)$ and $\text{diam}(L(CNC_k[n])) \leq \text{diam}(CNC_k[n]) + 1 \leq 4(n-1) + \left\lfloor \frac{k}{2} \right\rfloor$, we infer that the degree of n and k for an interpolation polynomial for the edge Wiener index of $CNC_k[n]$ are again bounded by 5 and 3, respectively. In particular,

$$W_e(CNC_k[n]) \leq \binom{\frac{1}{2}k(3n^2-n)}{2}\left(4(n-1) + \left\lfloor \frac{k}{2} \right\rfloor\right) < 9n^5k^3\,.$$

It again turned out that there is no uniform polynomial, but distinguishing two cases based on the parity of k two essentially different polynomials are obtained. Let $n \geq 1$, then if $k \geq 5$ is odd,

$$\begin{aligned} W_e\left(CNC_k[n]\right) \;=\; & \left(3k^2 - \frac{57}{10}k\right)n^5 + \left(\frac{9}{32}k^3 - \frac{47}{8}k^2 + \frac{467}{32}k\right)n^4 - \\ & \left(\frac{3}{16}k^3 - \frac{7}{2}k^2 + \frac{173}{16}k\right)n^3 + \\ & \left(\frac{1}{32}k^3 - \frac{5}{8}k^2 + \frac{51}{32}k\right)n^2 + \left(\frac{k}{5}\right)n\,, \end{aligned} \tag{3}$$

and if $k \geq 6$ is even,

$$\begin{aligned} W_e\left(CNC_k[n]\right) &= \left(3k^2-\frac{57}{10}k\right)n^5+\left(\frac{9}{32}k^3-\frac{47}{8}k^2+\frac{119}{8}k\right)n^4- \\ &\left(\frac{3}{16}k^3-\frac{7}{2}k^2+\frac{47}{4}k\right)n^3+ \\ &\left(\frac{1}{32}k^3-\frac{5}{8}k^2+\frac{19}{8}k\right)n^2+\left(\frac{k}{5}\right)n\,. \end{aligned} \tag{4}$$

Just as for the Wiener index, the cases $k = 4$ and $k = 3$ must be treated separately. In the first case, we must (contrary to the situation with the Wiener index) distinguish cases based on the parity of n:

$$\begin{aligned} W_e(CNC_4[n]) &= \frac{261}{10}n^5-\frac{87}{4}n^4+\frac{37}{6}n^3-2n^2-\frac{19}{15}n+\frac{3}{4}, \quad n \text{ is odd}, \\ W_e(CNC_4[n]) &= \frac{261}{10}n^5-\frac{87}{4}n^4+\frac{37}{6}n^3-2n^2-\frac{19}{15}n, \quad n \text{ is even}, \end{aligned}$$

so that we can write

$$W_e(CNC_4[n]) = \left\lceil \frac{261}{10}n^5-\frac{87}{4}n^4+\frac{37}{6}n^3-2n^2-\frac{19}{15}n \right\rceil,$$

When $k = 3$, the situation is even more tricky, now we need to consider cases modulo 6:

$$\begin{aligned} W_e(CNC_3[n]) &= \frac{489}{40}n^5-\frac{163}{16}n^4+\frac{77}{24}n^3-2n^2-\frac{1}{10}n, \quad n \equiv 0 \pmod 6, \\ W_e(CNC_3[n]) &= \frac{489}{40}n^5-\frac{163}{16}n^4+\frac{77}{24}n^3-2n^2-\frac{1}{10}n-\frac{7}{48}, \quad n \equiv 1 \pmod 6, \\ W_e(CNC_3[n]) &= \frac{489}{40}n^5-\frac{163}{16}n^4+\frac{77}{24}n^3-2n^2-\frac{1}{10}n+\frac{1}{3}, \quad n \equiv 2 \pmod 6, \\ W_e(CNC_3[n]) &= \frac{489}{40}n^5-\frac{163}{16}n^4+\frac{77}{24}n^3-2n^2-\frac{1}{10}n+\frac{3}{16}, \quad n \equiv 3 \pmod 6, \\ W_e(CNC_3[n]) &= \frac{489}{40}n^5-\frac{163}{16}n^4+\frac{77}{24}n^3-2n^2-\frac{1}{10}n-\frac{1}{3}, \quad n \equiv 4 \pmod 6, \\ W_e(CNC_3[n]) &= \frac{489}{40}n^5-\frac{163}{16}n^4+\frac{77}{24}n^3-2n^2-\frac{1}{10}n+\frac{25}{48}, \quad n \equiv 5 \pmod 6\,. \end{aligned}$$

The above six cases can be condensed as follows:

$$W_e(CNC_3[n]) = \begin{cases} \left\lceil \frac{489}{40}n^5-\frac{163}{16}n^4+\frac{77}{24}n^3-2n^2-\frac{1}{10}n \right\rceil; & n \equiv 5 \pmod 6, \\ \text{round}\left(\frac{489}{40}n^5-\frac{163}{16}n^4+\frac{77}{24}n^3-2n^2-\frac{1}{10}n\right); & \text{otherwise}\,. \end{cases}$$

3.3 Reverse Wiener index

For the reverse Wiener index of $CNC_k[n]$ the situation is similar. For $k = 3$ we get

$$RW(CNC_3[n]) = \frac{121}{15}n^5-9n^4-\frac{41}{18}n^3+3n^2+\frac{1}{10}n, \quad n \equiv 0 \pmod 3,$$

$$\begin{aligned} RW(CNC_3[n]) &= \frac{121}{15}n^5 - 9n^4 - \frac{41}{18}n^3 + 3n^2 + \frac{1}{10}n + \frac{1}{9}, \quad n \equiv 1 \pmod 3, \\ RW(CNC_3[n]) &= \frac{121}{15}n^5 - 9n^4 - \frac{41}{18}n^3 + 3n^2 + \frac{1}{10}n - \frac{1}{9}, \quad n \equiv 2 \pmod 3, \end{aligned}$$

and hence

$$RW(CNC_3[n]) = \text{round}\left(\frac{121}{15}n^5 - 9n^4 - \frac{41}{18}n^3 + 3n^2 + \frac{1}{10}n\right).$$

For $k = 4$ we get the polynomial

$$RW(CNC_4[n]) = \frac{102}{5}n^5 - 16n^4 - \frac{16}{3}n^3 + 4n^2 + \frac{14}{15}n.$$

Then, for any $n \geq 1$ and odd $k \geq 5$,

$$\begin{aligned} RW\left(CNC_k[n]\right) &= \left(\frac{2}{3}k^2 + \frac{38}{15}k\right)n^5 + \left(\frac{1}{8}k^3 - \frac{3}{4}k^2 - \frac{35}{8}k\right)n^4 - \\ &\quad \left(\frac{1}{6}k^2\right)n^3 - \left(\frac{1}{4}k^2 - \frac{9}{4}k\right)n^2 - \left(\frac{k}{30}\right)n, \end{aligned} \tag{5}$$

and for any $n \geq 1$ and even $k \geq 6$,

$$\begin{aligned} RW\left(CNC_k[n]\right) &= \left(\frac{2}{3}k^2 + \frac{38}{15}k\right)n^5 + \left(\frac{1}{8}k^3 - \frac{1}{2}k^2 - \frac{9}{2}k\right)n^4 - \\ &\quad \left(\frac{1}{6}k^2\right)n^3 - \left(\frac{1}{4}k^2 - 2k\right)n^2 - \left(\frac{k}{30}\right)n. \end{aligned} \tag{6}$$

3.4 Szeged index

For any edge uv of $CNC_k[n]$ we have $n_u n_v \leq \frac{1}{4}\left|V(CNC_k[n])\right|^2$, therefore

$$Sz(G) = \sum_{uv \in E(G)} n_u n_v \leq \frac{1}{4}n^4(3n^2 - n)k^3 \leq n^6k^3.$$

It follows that the degrees of n and k in the interpolation polynomial for the Szeged index of $CNC_k[n]$ are bounded by 6 and 3, respectively. For $n \geq 1$ and odd $k \geq 5$,

$$\begin{aligned} Sz\left(CNC_k[n]]\right) &= \left(\frac{9}{4}k^2 - \frac{9}{2}k\right)n^6 + \left(\frac{1}{4}k^3 - \frac{7}{2}k^2 + \frac{37}{4}k\right)n^5 + \\ &\quad \left(\frac{3}{4}k^2 - \frac{19}{4}k\right)n^4 + \left(\frac{k}{4}\right)n^2, \end{aligned} \tag{7}$$

while for $n \geq 1$ and even $k \geq 6$,

$$\begin{aligned} Sz\left(CNC_k[n]\right) &= \left(\frac{9}{4}k^2 - \frac{9}{2}k\right)n^6 + \left(\frac{1}{4}k^3 - 3k^2 + 9k\right)n^5 \\ &+ \left(\frac{3}{4}k^2 - \frac{19}{4}k\right)n^4 + \left(\frac{k}{4}\right)n^2. \end{aligned} \tag{8}$$

Let $k = 4$, then if n is odd,

$$Sz\left(CNC_4[n]\right) = \frac{557}{30}n^6 - \frac{1}{3}n^4 - \frac{37}{30}n^2 - 1\,,$$

and if n is even,

$$Sz\left(CNC_4[n]\right) = \frac{557}{30}n^6 - \frac{1}{3}n^4 - \frac{41}{15}n^2.$$

For $k = 3$ we were trying to obtain some relation combining the values up to $n = 30$, but we could obtain no polynomial expression. Hence the existence of polynomial(s) for $Sz(CNC_3[n])$ is an open problem. Perhaps one could deduce them by computing some more values $n \geq 31$.

3.5 The case $k = 6$

Recall that the graphs $H_n = CNC_n[6]$ form the coronene/circumcoronene series. Using other methods (mostly the cut method, see [19]), closed expressions for several topological indices of H_n were previously obtained. Setting $k = 6$ into (2) (cf. Table 1) we get

$$W\left(H_n\right) = \frac{164}{5}n^5 - 6n^3 + \frac{1}{5}n\,,$$

a result independently obtained in [15] and [26]. Setting $k = 6$ into (8) we get

$$Sz\left(H_n\right) = 54n^6 - \frac{3}{2}n^4 + \frac{3}{2}n^2\,,$$

a result first obtained in [14]. Similarly, inserting $k = 6$ into (4) we get

$$W_e\left(H_n\right) = \frac{369}{5}n^5 - \frac{123}{2}n^4 + 15n^3 - \frac{3}{2}n^2 + \frac{6}{5}n\,,$$

a result very recently presented in [29]. (Note that there the index is shifter by one with respect to our notation.) Finally, inserting $k = 6$ into (6) we get

$$RW\left(H_n\right) = \frac{196}{5}n^5 - 18n^4 - 6n^3 + 3n^2 - \frac{1}{5}n\,,$$

a result that seems to be new.

4 Concluding remarks

We have also applied the interpolation method to some other classes of two-parametric families of graphs. We just briefly present them for the hexagonal-parallelogram graphs $P(n, k)$. These graphs consists of a hexagons arranged is a parallelogram fashion, see

Figure 2: Hexagonal-parallelogram graph $P(4,5)$

Fig. 2 where the hexagonal-parallelogram graph $P(4,5)$ is shown. The general definition of these graphs should be clear from this example.

It can be verified that diam$(P(n,k) = 2(n+k)-1$, and that and the number of vertices and edges of $P(n,k)$ are $2(n+1)(k+1)-2$ and $3nk+2(n+k)-1$, respectively. Then we get:

$$\begin{aligned} W\left(P(n,k)\right) &= \frac{4}{3}\left(k^2+2k+1\right)n^3+\frac{2}{3}\left(k^3+9k^2+8k\right)n^2+ \\ &\quad \frac{1}{3}\left(k^4+8k^3+16k^2+2k-1\right)n-\frac{1}{15}\left(k^5-20k^3+k\right), \end{aligned}$$

$$\begin{aligned} W_e\left(P(n,k)\right) &= \left(3k^2+4k+\frac{4}{3}\right)n^3+\left(\frac{2}{3}k^3+9k^2+3k-2\right)n^2+ \\ &\quad \left(\frac{3}{4}k^4+4k^3+\frac{13}{4}k^2-\frac{3}{2}k+\frac{2}{3}\right)n- \\ &\quad \left(\frac{3}{20}k^5-\frac{5}{4}k^3+2k^2-\frac{9}{10}k\right), \end{aligned}$$

$$\begin{aligned} RW\left(P(n,k)\right) &= \frac{8}{3}\left(k^2+2k+1\right)n^3+\frac{1}{3}\left(10k^3+24k^2+2k-12\right)n^2- \\ &\quad \frac{1}{3}\left(k^4-16k^3-2k^2+23k-4\right)n+ \\ &\quad \frac{1}{15}\left(k^5+40k^3-60k^2+19k\right), \end{aligned}$$

$$\begin{aligned} Sz\left(P(n,k)\right) &= \left(2k^3+6k^2+\frac{16}{3}k+\frac{4}{3}\right)n^3+\left(6k^3+12k^2+6k\right)n^2+ \\ &\quad \left(\frac{17}{3}k^3+7k^2+2k-\frac{1}{3}\right)n+\left(+\frac{1}{6}k^4-k^3-\frac{1}{6}k\right). \end{aligned}$$

The above formula for the Wiener index was first obtained in [20].

We conclude the paper by pointing out that, if necessary, the method described in this paper can be applied also to multi-parametric families of graphs, that is, families of graphs defined with three or more parameters.

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