MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **68** (2012) 957-958

ISSN 0340 - 6253

BOOK REVIEW

Applications of Combinatorial Matrix Theory to Laplacian Matrices of Graphs

by

Jason J. Molitierno

CRC Press, Boca Raton, 2012, XIV+405pp. ISBN 978-1-4398-6337-4

Ever since the seminal work of Fiedler in 1973, researchers were interested in the eigenvalues, especially the second smallest eigenvalue of the Laplacian matrix of a graph, the so-called *algebraic connectivity*, and its corresponding normalized eigenvector, the so-called *Fiedler vector*. The algebraic connectivity and the Fiedler vector have found numerous applications, the most popular one probably being its use as the main ingredient of spectral clustering methods.

The book consists of preface and eight chapters. The book assumes prior knowledge of linear algebra at an undergraduate level, and in Chapter 1 further, mostly spectral, concepts are introduced and discussed: the spectral radius, the Perron-Frobenius theorem, the Geršgorin circles, doubly stochastic matrices and the Moore-Penrose and group inverses.

In Chapter 2 a short overview of the main graph theory concepts is given, together with a detailed account of threshold graphs, important examples of graphs having integers only in the Laplacian spectrum.

Chapter 3 motivates the study of Laplacian matrices by deriving them from continuous Laplacians, used in differential equations to study the energy flow through a region, and applying the discrete form of the Laplacian to minimize the energy of the representations of graph vertices in R^k. It also contains the detailed proof of the celebrated Matrix-Tree theorem on the number of spanning trees in a graph, together with a short study of Laplacian eigenvalues of random graphs, small world networks and scale-free networks.

Chapter 4 surveys a number of results on the Laplacian spectrum: the effect of taking the unions, joins, products and complements of graphs; upper bounds on the spectral radius; the distribution of Laplacian eigenvalues less than, equal to and greater than one; as well as the proof of the Grone-Merris conjecture and the study of the Laplacian spectrum of threshold graphs.

Chapter 5 is devoted to the study of the algebraic connectivity and its relation to the structure of a graph, while Chapters 6 and 7 are devoted to the study of the Fiedler vector in trees (Chapter 6) and general graphs (Chapter 7). This study is performed through the study of bottleneck matrices, which are, for each vertex u, defined as the inverse of the submatrix of L obtained by deleting the row and the column corresponding to u, and whose spectral radius is closely related to the algebraic connectivity.

Since the Laplacian matrix is always singular, it cannot have usual inverse, but only generalized inverses. Among these, the group inverse relies heavily on the bottleneck matrices, and it is the topic of Chapter 8. The properties of the group inverse turn out to sharpen many of the earlier results from the book. It is also used to define the Zenger function, which is a recent sharp lower bound on the algebraic connectivity.

The book can certainly be used as a convenient reference book by experienced researchers, and as an introduction to the theory of graph Laplacians by graduate students. However, a very large number of examples and observations, worked out in full details, besides detailed proofs and a handful of exercises, make it worthwhile for use in undergraduate courses and suggest that this is the main audience for the book.

Dragan Stevanović