The Permanental Polynomials of Certain Graphs*

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Abstract

In this paper we consider the computation of permanental polynomials of some graphs. By orienting even cycles oddly, explicit expressions for the permanental polynomials of some basic graphs including a path and a cycle are obtained in terms of roots. For hexagonal systems, based on reduction procedures, the permanental polynomials of hexagonal chains and a type of pericondensed hexagonal system are deduced from product of matrices of order 5. Meanwhile, the permanental polynomial of a general polygonal chain is also derived.

1 Introduction

This paper deals with the computation of permanental polynomials of some graphs. Suppose $G = (V, E)$ is a finite and simple graph on $n$ vertices. The permanental polynomial of $G$ is defined as

$$\pi(G, x) = \text{per}(xI - A(G)) = \sum_{k=0}^{n} b_k x^{n-k},$$

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where $I$ is the identity matrix of order $n$, $A(G)$ is the adjacency matrix of $G$ and the permanent $\text{per}(\tilde{A})$ of a matrix $\tilde{A} = (a_{ij})_{n \times n}$ is given as [18]

$$\text{per}(\tilde{A}) = \sum_{\sigma \in \Lambda_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

with $\Lambda_n$ denoting the set of all the permutations of $\{1, 2, \cdots, n\}$. For the permanent polynomial of a bipartite graph with an even number of vertices, one has $b_{2i+1} = 0$ and $b_{2i} = \sum_H \text{per}(A(H)) = \sum_H M^2(H)$, where the summation extends over all induced subgraphs $H$ with $2i$ vertices of $G$ and $M(H)$ is the number of perfect matchings of $H$ [4, 17, 24]. Obviously, the value of the last coefficient is the square of the number of perfect matchings of $G$.

As is well known, computing the permanent of a matrix is a #P-complete problem [23]. So computing the permanental polynomial of a graph is difficult. In the literature, Merris et al. [17] proved that the coefficient of the permanental polynomial satisfies that

$$(-1)^i b_i = \sum_H 2^{k(H)},$$

where the sum ranges over all subgraphs $H$ on $i$ vertices whose components are single edges or cycles, and $k(H)$ is the number of cycles. Based on this result, similarly to the technique of computing the characteristic polynomial of a graph in terms of subgraphs [20], Borowiecki and Józwiak [5] studied the relationship between the permanental polynomial of a dimultigraph (resp. a multigraph) and certain subgraphs. Recently, Belardo et al. extended these results to characteristic and permanental polynomials of weighted graphs and matrices in [2] and [3], respectively. For the permanental polynomials of chemical graphs, in [7] by generating all the coefficients of the permanental polynomial of fullerenes up through $C_{36}$, the zeros of these polynomials were dealt with by Cash. It was shown that of the independent zeros, ten are nearly constant within an isomer series of constant $N$, while the remaining $(N/2 - 10)$ zeros vary greatly with structure. This indicates that the permanental polynomial encodes a variety of structure information. To determine the coefficients of a permanental polynomial, Gutman and Cash [13] considered the relation between the permanental polynomial and the characteristic polynomial of hexagonal systems and fullerenes, and established a formula on a part of coefficients of these two polynomials. Later by focusing on the orientation graph of a bipartite graph containing no even subdivision of $K_{2,3}$, Yan and Zhang [25] proved that the permanental polynomial of such a bipartite graph can be computed by the characteristic polynomial of a skew adjacency matrix. Furthermore, in [26] we obtained that only the permanental polynomials of bipartite graphs containing no even subdivision of $K_{2,3}$ can be computed in this way, and a characterization of this kind of graphs is given. For more studies on the permanental polynomials in chemistry and mathematics, see [4, 6, 8, 9, 14, 15, 16, 22] and related references.
To overcome the difficulty of computing permanents, it is reasonable to convert the computation of permanental polynomials to the computation of matrices and determinants. Motivated by this idea, in this work, we first pay attentions to some basic graphs, such as a path and a cycle. Instead of computing the permanental polynomials directly, we assign orientations to graphs, and compute the characteristic polynomials of the corresponding skew adjacency matrices. Then we turn to some chemical graphs including some types of hexagonal systems, which is a natural graph representation of benzenoid hydrocarbons. The corresponding polynomials are produced by the product of matrices.

The organization of this paper is as follows. In Section 2 we give explicit expressions of the permanental polynomials of a path, an even cycle, an even $n$-sun graph and one subgraph of an $n$-sun graph. Under these formulas, the roots of the corresponding polynomials follow immediately. Applying the reduction procedures, in Section 3 we obtain the permanental polynomials of a general polygonal chain $G_n$ and a kind of pericondensed system $H_n$ by multiplications of matrices. According to this, the permanental polynomial of a hexagonal chain is provided. As special cases, explicit formulas on the permanental polynomials of a linear chain, zigzag chain and helix chain are obtained.

## 2 Explicit expressions for the permanental polynomials of some basic graphs

In this section, we will compute the permanental polynomials of a path, an even cycle, an even $n$-sun graph and the subgraph of an $n$-sun graph. Throughout this paper we denote by $P_n$ a path on $n$ vertices, $C_n$ a cycle on $n$ vertices and $S_n$ an $n$-sun graph. An $n$-sun graph is the graph on $2n$ vertices obtained by attaching a pendant edge to each vertex of a cycle $C_n$ [1]. Particularly, we call $C_n$ (resp. $S_n$) an even cycle (resp. even $n$-sun graph) if $n$ is even.

Using the results of matching polynomials on paths and cycles [11] and formula (1), it is easy to check that the permanental polynomials of a path and a cycle given as

$$
\pi(P_n, x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} x^{n-2k}
$$

and

$$
\pi(C_n, x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} \frac{n}{n-k} \binom{n-k}{k} x^{n-2k} + b(C_n),
$$

where $b(C_n) = 4$ if $n$ is even, and $b(C_n) = -2$ if $n$ is odd. Here by computing the characteristic polynomial of the skew adjacency matrix of an orientation graph, we will derive the explicit expressions of the permanental polynomials of a path and an even cycle in terms of roots. We begin by introducing a few definitions and lemmas.
A graph $G$ is an even subdivision of a graph $H$ if $G$ is obtained from $H$ by replacing the edges of $H$ by internally disjoint paths, each containing an even number of vertices and at least one edge.

For a graph $G$, an even cycle $C$ is said to be nice if $G - V(C)$ has a perfect matching.

Let $G^e$ be an orientation of $G$. An even cycle is said to be oddly oriented in $G^e$ if the number of edges pointing in each direction is odd. Under an orientation $G^e$, the skew adjacency matrix $A(G^e) = (a'_{ij})_{n \times n}$ is defined as

$$a'_{ij} = \begin{cases} 
1 & \text{if } (v_i, v_j) \text{ is a directed edge from } v_i \text{ to } v_j, \\
-1 & \text{if } (v_j, v_i) \text{ is a directed edge from } v_j \text{ to } v_i, \\
0 & \text{if no edges connect } v_i \text{ and } v_j.
\end{cases}$$

**Theorem 2.1.** [24] Let $G$ be a bipartite graph containing no even subdivision of $K_{2,3}$. Then there exists an orientation $G^e$ of $G$ such that

$$\pi(G, x) = \det(xI - A(G^e)).$$

Moreover, each cycle of $G^e$ is oddly oriented.

**Lemma 2.2.** [21] Define $n \times n$ matrices $U_n$ and $U_n^{-1}$ with components $1 \leq k, k' \leq n$:

$$(U_n)_{k,k'} = \sqrt{\frac{2}{n+1}} i^k \sin \left( \frac{k k' \pi}{n+1} \right), \quad (U_n^{-1})_{k,k'} = \sqrt{\frac{2}{n+1}} (-i)^{k'} \sin \left( \frac{k k' \pi}{n+1} \right).$$

Let $Q_n$ be the $n \times n$ matrix

$$
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
-1 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
-1 & 0 & \cdots & \cdots & 0 \\
-1 & 0 & 1
\end{pmatrix}.
$$

Then the matrix $\tilde{Q}_n = U_n^{-1}Q_n U_n$ has the element $(\tilde{Q}_n)_{k,k'} = \delta_{k,k'} \cdot 2i \cos \frac{k \pi}{n+1}$ for $1 \leq k, k' \leq n$ and $i^2 = -1$.

**Theorem 2.3.** The permanental polynomial of a path $P_n$ is

$$\pi(P_n, x) = \prod_{i=1}^{n} \left( x + 2i \cos \frac{t \pi}{n+1} \right).$$
Proof. By Theorem 2.1 and according to the orientation $P_n^e$ of $P_n$ shown in Figure 1(a),
\[ \pi(P_n, x) = \det(xI - A(P_n^e)) = \det(xI_n + Q_n) \] holds. Following the result of Lemma 2.2, conjugate the matrix $(xI_n + Q_n)$ by $U_n$ to obtain $U_n^{-1}(xI_n + Q_n)U_n = \text{diag}(x + 2i \cos \frac{\pi}{n+1}, x + 2i \cos \frac{2\pi}{n+1}, \ldots, x + 2i \cos \frac{n\pi}{n+1})$. So $\pi(P_n, x) = \prod_{t=1}^{n}(x + 2i \cos \frac{\pi}{n+1})$ is got. \hfill \Box

Remark 2.4. Combining the formula for the characteristic polynomial of a path $P_n$ [20]
\[ \phi(P_n, x) = \prod_{t=1}^{n}(x - 2 \cos \frac{t\pi}{n+1}) = \prod_{t=1}^{n}(x + 2 \cos \frac{t\pi}{n+1}) \]
and the theorem presented in [4], the same result as (4) can be obtained.

Lemma 2.5. [25] Define $n \times n$ matrices $V_n$ and $V_n^{-1}$ with components $1 \leq t, j \leq n$:
\[ (V_n)_{t,j} = \sqrt{\frac{1}{n}} e^{i \frac{(2t-1)\pi}{n}}, \quad (V_n^{-1})_{t,j} = \sqrt{\frac{1}{n}} e^{-i \frac{(2t-1)\pi}{n}}. \]
Let $Y_n$ be the $n \times n$ matrix
\[ \begin{pmatrix}
0 & 1 & 1 \\
-1 & 0 & 1 \\
-1 & 0 & 1 \\
\vdots & \vdots & \vdots \\
-1 & 0 & 1 \\
-1 & 0 & 1
\end{pmatrix}. \]
Then the matrix $\tilde{Y}_n = V_n^{-1} Y_n V_n$ has the element $(\tilde{Y}_n)_{t,j} = \delta_{t,j} \cdot 2i \sin \frac{(2t-1)\pi}{n}$ for $1 \leq t, j \leq n$ and $i^2 = -1$.

Theorem 2.6. The permanental polynomial of an even cycle $C_n$ is
\[ \pi(C_n, x) = \prod_{t=1}^{n}(x + 2i \sin \frac{(2t-1)\pi}{n}). \quad (5) \]
Proof. An orientation $C_n^e$ of $C_n$ referring to Figure 1(b) is oddly oriented when $n$ is even. The matrix $xI_n - A(C_n^e)$ takes the form $R_n$, where
\[ R_n = \begin{pmatrix}
x & 1 & 1 \\
-1 & x & 1 \\
-1 & x & 1 \\
\vdots & \vdots & \vdots \\
-1 & x & 1 \\
-1 & -1 & x
\end{pmatrix}. \quad (6) \]
Conjugating $R_n = xI_n + Y_n$ by $V_n$, we obtain that $\det(R_n) = \det(V_n^{-1} R_n V_n) = \det(\text{diag}(x + 2i \sin \frac{\pi}{n}, x + 2i \sin \frac{3\pi}{n}, \ldots, x + 2i \sin \frac{(2n-1)\pi}{n}))$. So $\pi(C_n, x) = \prod_{t=1}^{n}(x + 2i \sin \frac{(2t-1)\pi}{n})$ holds. \hfill \Box
Remark 2.7. For \( n = 4k + 2 \), using the characteristic polynomial of a cycle \( C_n \) [20]

\[
\phi(C_n, x) = \prod_{t=1}^{n}(x - 2\cos \frac{2t\pi}{n}) = \prod_{t=1}^{n}(x + 2\sin \frac{(3n - 4t)\pi}{2n})
\]

and the theorem presented in [4], the same result as (5) can be also obtained. For other values of \( n \), \( \pi(C_n, x) \) cannot be get in this way.

![Figure 2: The \( n \)-sun graph \( S^n_c \) and the graph \( S^n_c - n\vec{1} \).](image)

Theorem 2.8. The permanental polynomial of an even \( n \)-sun graph \( S_n \) is

\[
\pi(S_n, x) = \prod_{t=1}^{n}(x^2 + x2i\sin \frac{(2t - 1)\pi}{n} + 1).
\]

Proof. Figure 2(a) gives an orientation \( S^n_c \) of an even \( n \)-sun graph \( S_n \) with the only cycle being oddly oriented. Following the labeling of vertices and the orientation graph \( S^n_c \) in Figure 2(a), the matrix \( xI - A(S^n_c) \) takes the form \( \begin{pmatrix} R_n & I_n \\ -I_n & xI_n \end{pmatrix} \). Since \( R_n(-I_n) = (-I_n)R_n \), it follows that det\( (xI - A(S^n_c)) \) = det\( (R_n(xI_n) - (-I_n)I_n) \) = det\( (D_n) \), where the matrix \( R_n \) takes the form shown in (6) and

\[
D_n = \begin{pmatrix}
    x^2 + 1 & x & & & x \\
    -x & x^2 + 1 & x & & \\
    & -x & x^2 + 1 & x & \\
    & & & \ddots & \ddots & \ddots \\
    & & & & -x & x^2 + 1 & x \\
    & & & & & -x & x^2 + 1
\end{pmatrix}
\]

By Lemma 2.5, conjugate \( D_n = (x^2 + 1)I_n + xY_n \) by \( V_n \) to obtain

\[
V_n^{-1}D_nV_n = \text{diag}(x^2 + 1 + x2i\sin \frac{\pi}{n}, x^2 + 1 + x2i\sin \frac{3\pi}{n}, \cdots, x^2 + 1 + x2i\sin \frac{(2n-1)\pi}{n}).
\]

Then according to Theorem 2.1, \( \pi(S_n, x) = \text{det}(xI - A(S^n_c)) = \text{det}(D_n) = \prod_{t=1}^{n}(x^2 + 1 + 2x \sin \frac{(2t-1)\pi}{n}) \) is obtained with \( i = \sqrt{-1} \). \( \square \)
For an edge $e$ of a graph $G$, $G-e$ is the graph resulting from the remove of $e$. Choosing an edge $e$ belonging to the cycle of $S_n$, the resulting graph $S_n-e$ takes the form shown in Figure 2(b). By Lemma 2.2 and the discussion in the proof of Theorem 2.8, we have

**Theorem 2.9.** Let $e$ be an edge belonging to the cycle of an $n$-sun graph. Then

$$\pi(S_n-e, x) = \prod_{t=1}^{n}(x^2 + x2i \cos \frac{t\pi}{n} + 1).$$

**Remark 2.10.** By the equations $x^2 + x2i \sin \frac{(2t-1)\pi}{n} + 1 = 0$ and $x^2 + x2i \cos \frac{t\pi}{n+1} + 1 = 0$, the roots of $\pi(S_n, x)$ and $\pi(S_n-e, x)$ can be obtained, respectively.

### 3 Recursive expressions for the permanental polynomials of some kinds of graphs

#### 3.1 Identities for permanental polynomials

In [5] Borowiecki and Jóźwiak proved an identity on the permanental polynomial, which is described in Theorem 3.1.

**Theorem 3.1.** [5] Let $e = (u, v)$ be an edge of a graph $G$ and $C_e(G)$ the set of cycles containing $e$. Then

$$\pi(G, x) = \pi(G - e, x) + \pi(G - u - v, x) + 2 \sum_{C \in C_e(G)} (-1)^{|V(C)|} \pi(G - V(C), x).$$

Formula (9) provides a general connection between the permanental polynomial of a graph and the permanental polynomials of its subgraphs. With the help of Theorems 2.3 and 2.6, a different expression comparing with (3) appears immediately.

**Theorem 3.2.** The permanental polynomial of a cycle $C_n$ on $n$ vertices is

$$\pi(C_n, x) = \begin{cases} \prod_{t=1}^{n}(x + 2i \sin \frac{(2t-1)\pi}{n}), & \text{if } n \text{ is even,} \\ \prod_{t=1}^{n}(x + 2i \cos \frac{t\pi}{n+1}) + \prod_{t=1}^{n-2}(x + 2i \cos \frac{t\pi}{n-1}) - 2, & \text{if } n \text{ is odd.} \end{cases}$$

Let $u_1$, $v_1$ (resp. $u_2$, $v_2$) be a pair of vertices of a graph $G_1$ (resp. $G_2$). Then the bridge graph $G_1 \circ G_2$ of $G_1$ and $G_2$ through $e_1$ and $e_2$ is the graph obtained by joining edges $e_1$ between $u_1$ and $u_2$ and $e_2$ between $v_1$ and $v_2$. See Figure 3 for an illustration.

**Corollary 3.3.** For the bridge graph $G = G_1 \circ G_2$ through $e_1 = (u_1, u_2)$ and $e_2 = (v_1, v_2)$, the following result holds.

$$\pi(G, x) = \pi(G_1, x) \pi(G_2, x) + \pi(G_1 - u_1, x) \pi(G_2 - u_2, x) + \pi(G_1 - v_1, x) \pi(G_2 - v_2, x) + \pi(G_1 - u_1 - v_1, x) \pi(G_2 - u_2 - v_2, x) + 2 \sum_{C \in C_e(G)} (-1)^{|V(C)|} \pi(G - V(C), x).$$

(10)
3.2 The permanental polynomial of a general polygonal chain

Inspired by the idea given in [19], we now deduce the permanental polynomial of a general polygonal chain by a recursive procedure. To derive our main results, we give some definitions and notations.

A general polygonal chain is a polygonal system satisfying (a) each of the two end polygons has exactly one adjacent polygon and any other polygon has two adjacent polygons; (b) the intersection of any two adjacent polygons is a path whose internal vertices are of degree two; (c) no three polygons have a vertex in common.

For simple, a general polygonal chain with \( n \) polygons (each polygon has at least six vertices) is denoted by \( G_n \). For the \( i \)-th polygon in \( G_n \), \( i \in \{1, 2, 3, \ldots, n\} \), two root vertices \( u_i \) and \( v_i \) are prescribed, which are joined by a path \( P_{ri} \) of internal vertices of degree two. In the \( i + 1 \)-th polygon, the edge with \( u_i \) as an endvertex is marked by \( e_{i+1} \). The path in the \( i \)-th polygon joining \( u'_i \) (the neighbor of \( u_{i-1} \)) and \( v'_i \) (the neighbor of \( v_{i-1} \)) is denoted by \( P_{ri}, \) and the path connecting \( u'_i \) (resp. \( v'_i \)) and \( u_i \) (resp. \( v_i \)) is denoted by \( P_{si} \) (resp. \( P_{ti} \)). See Figure 4. In addition, we denote by \( \mathcal{C}_{V(H)}(G) \) the set of cycles of \( G \) containing the vertices of the subgraph \( H \) and \( \mathcal{C}_e(G) \) the set of cycles in \( G \) including the edge \( e \).

**Theorem 3.4.** For a general polygonal chain \( G_{n+1} \), let \( \alpha(G_n) \) be the column vector \((\pi(G_n, x), \pi(G_n - u_n, x), \pi(G_n - v_n, x), \pi(G_n - V(P_n), x), \sum_{C \in \mathcal{C}_{V(P_n)}(G_n)} (-1)^{|V(C)|} \pi(G_n - C))\).
\[ V(C, x)^T. \] Then the permanental polynomial of \( G_{n+1} \) satisfies the recurrence

\[ \alpha(G_{n+1}) = A_{n+1} \cdot \alpha(G_n), \] (11)

where \( A_n \) is a 5 \times 5 matrix whose \( i \)-th row vector is \( l^1_i \) for \( 1 \leq i \leq 5 \). Explicitly,

\[
\begin{align*}
l^1_n &= (\pi(P_n, x), \pi(P_{n-1}, x), \pi(P_{n-1}, x), \pi(P_{n-2}, x)\pi(P_{n-1}, x) + 2(-1)^{r_n+l_{n-1}}, \\
l^2_n &= (\pi(P_{n-1}, x)\pi(P_{n-1}, x), \pi(P_{n-2}, x)\pi(P_{n-1}, x), \pi(P_{n-1}, x)\pi(P_{n-2}, x)\pi(P_{n-1}, x) \\
l^3_n &= (\pi(P_{n-1}, x)\pi(P_{n-1}, x), \pi(P_{n-1}, x)\pi(P_{n-2}, x)\pi(P_{n-1}, x), \pi(P_{n-1}, x)\pi(P_{n-2}, x)\pi(P_{n-1}, x) \\
l^4_n &= (\pi(P_{n-1}, x)\pi(P_{n-1}, x), \pi(P_{n-2}, x)\pi(P_{n-1}, x), \pi(P_{n-1}, x)\pi(P_{n-2}, x)\pi(P_{n-1}, x) \\
l^5_n &= (0, 0, 0, (-1)^{r_{n+1}+l_{n-1}}, (-1)^{r_{n+1}+l_{n-1}}\pi(P_{n-1}, x)).
\end{align*}
\]

Consequently,

\[ \alpha(G_n) = A_n \cdot A_{n-1} \cdots A_1 \cdot \alpha(G_0), \] (12)

where \( \alpha(G_0) = (x^2 + 1, x, x, 1, 0)^T \).

**Proof.** We can see that the general polygonal chain \( G_{n+1} \) is the bridge graph of \( G_n \) and the path \( P_{r_{n+1}} \). Then by Corollary 3.3,

\[
\begin{align*}
\pi(G_{n+1}, x) &= \pi(G_n, x)\pi(P_{r_{n+1}}, x) + \pi(G_n - u_n, x)\pi(P_{r_{n+1}, 1}, x) \\
&\quad + \pi(G_n - v_n, x)\pi(P_{r_{n}+1, 1}, x) + \pi(G_n - P_{r_{n}}, x)\pi(P_{r_{n+1}, 2}, x)\pi(P_{r_{n}-2}, x) \\
&\quad + 2 \sum_{C \in \mathcal{C}_{n+1}(G_{n+1})} (-1)^{|V(C)|} \pi(G_{n+1} - V(C), x).
\end{align*}
\] (13)
As the internal vertices of \( P_{r_n+1} \) are of degree two, a cycle containing \( e_{n+1} \) in \( G_{n+1} \) must pass through the path \( P_{r_n+1} \). So it follows that

\[
\sum_{C \in \mathcal{E}_{e_{n+1}}(G_{n+1})} (-1)^{|V(C)|} \pi(G_{n+1} - V(C), x) = (-1)^{r_n+1+l_n} \pi(G_n - V_{P_{n-1}}, x) \\
+ (-1)^{r_n+1+l_n} \pi(P_{n-2}, x) \sum_{C \in \mathcal{E}_{V(P_{n-1})}(G_n)} (-1)^{|V(C)|} \pi(G_n - V(C), x).
\]

From (13) and (14), we have

\[
\pi(G_{n+1}, x) = \pi(G_n, x) \pi(P_{r_n+1}, x) + \pi(G_n - u_{n+1}, x) \pi(P_{r_n+1-1}, x) + \pi(G_n - v_{n+1}, x) \pi(P_{r_n+1-1}, x) \\
+ \pi(G_n - V_{P_{n-1}}, x) [\pi(P_{r_n+1-2}, x) \pi(P_{n-2}, x) + 2(-1)^{r_n+1+l_n}] \\
+ 2(-1)^{r_n+1+l_n} \pi(P_{n-2}, x) \sum_{C \in \mathcal{E}_{V(P_{n-1})}(G_n)} (-1)^{|V(C)|} \pi(G_n - V(C), x).
\]

Thus, \( \pi(G_{n+1}, x) = t_{n+1}^1 \cdot \alpha(G_n) \).

Similarly, we also have \( \pi(G_{n+1} - u_{n+1}, x) = t_{n+1}^2 \cdot \alpha(G_n) \), \( \pi(G_{n+1} - v_{n+1}, x) = t_{n+1}^3 \cdot \alpha(G_n) \)

and \( \pi(G_{n+1} - u_{n+1} - v_{n+1}, x) = t_{n+1}^4 \cdot \alpha(G_n) \).

Since

\[
\sum_{C \in \mathcal{E}_{V(P_{n+1})}(G_{n+1})} (-1)^{|V(C)|} \pi(G_{n+1} - V(C), x) = (-1)^{r_n+1+l_n} \pi(G_n - V_{P_n}, x) \\
+ (-1)^{r_n+1+l_n} \pi(P_{n-2}, x) \sum_{C \in \mathcal{E}_{V(P_{n})}(G_n)} (-1)^{|V(C)|} \pi(G_n - V(C), x),
\]

we get that

\[
\sum_{C \in \mathcal{E}_{V(P_{n+1})}(G_{n+1})} (-1)^{|V(C)|} \pi(G_{n+1} - V(C), x) = t_{n+1}^5 \cdot \alpha_n.
\]

By now

\[
\alpha(G_{n+1}) = A_{n+1} \cdot \alpha(G_n)
\]

is established.

For a general polygonal chain, the starting step \( G_0 \) is an edge \((u_0, v_0)\). Given this, we obtain that \( \alpha(G_0) = (x^2 + 1, x, 1, 0)^T \). So equation (12) follows.

\[\square\]

**Remark 3.5.** It needs to point out that \( \pi(P_0, x) = 1 \) and \( \pi(P_{-1}, x) = 0 \).

As a special case of a general polygonal chain, a *polygonal chain* is a connected series of polygons arranged in a linear form satisfying the intersection of two adjacent polygons is an edge and no three polygons have a vertex in common. For a polygonal chain with at least six vertices on each polygon (denoted by \( G'_n \)), we have the following corollary.
Corollary 3.6. For a polygonal chain $G_n'$, let $\alpha(G_n')$ be the column vector $(\pi(G_n', x), \pi(G_n' - u_n, x), \pi(G_n' - v_n, x), \pi(G_n' - u_n - v_n, x), \sum_{C \in e(u_n, v_n)G_n} (-1)^{|V(C)|} \pi(G_n' - V(C), x))^T$. Then

$$\alpha(G_{n+1}') = A_{n+1}' \cdot \alpha(G_n'),$$

where $A_n'$ is the matrix obtained from $A_n$ with $l_{n-1} = 2$ and $\pi(P_{l_{n-1}}, x) = 1$.

Furthermore,

$$\alpha(G_n') = A_n' \cdot A_{n-1}' \cdots A_1' \cdot \alpha(G_0')$$

with $\alpha(G_0') = (x^2 + 1, x, x, 0)^T$.

Figure 5: (a) a hexagonal chain $F_5$, (b) a linear chain $L_5$, (c) a zigzag chain $Z_5$ and (d) a helix chain $T_5$.

A hexagonal system is a finite connected plane graph without cut vertices in which each interior region is surrounded by a regular hexagon of side length one. A catacondensed hexagonal system corresponds to those hexagonal system with no internal vertices, and a pericondensed hexagonal system has at least one internal vertex. A hexagonal chain is a catacondensed hexagonal system satisfying each hexagon has at most two adjacent hexagons and only each of two end hexagons has one adjacent hexagon (refer to Figure 5(a)). For more about hexagonal systems, see [12] and related references.

As an important special case, a hexagonal chain is a polygonal chain with all polygons being hexagons. Before presenting the permanental polynomial of a hexagonal chain, we introduce three types of hexagons. A hexagon $h_i$ in a hexagonal chain is of type-I if the minimum length of the path joining $u_{i-1}$ and $u_i$ in $h_i$ is two, and of type-II (resp. type-III) if the minimum length of the corresponding path is three (resp. one). As shown
in Figure 5(a), the hexagons $h_1$, $h_2$ and $h_5$ are of type-I, $h_3$ is of type-III and $h_4$ is of type-II. According to the matrix $A'_n$ in Corollary 3.6, we define three matrices $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ as follows

\[
\Gamma_1 = \begin{pmatrix}
 x^4 + 3x^2 + 1 & x^3 + 2x & x^2 + 2x & x^2 + 3 & 2 \\
 x^3 + x & x^2 + 1 & x^2 & x & 0 \\
 x^3 + x & x^2 & x^2 + 1 & x & 0 \\
 x^2 & x & x & 1 & 0 \\
 0 & 0 & 0 & 1 & 1
\end{pmatrix},
\]

\[
\Gamma_2 = \begin{pmatrix}
 x^4 + 3x^2 + 1 & x^3 + 2x & x^2 + 2x & x^2 + 3 & 2 \\
 x^3 + x & x^2 & x^2 + 1 & x & 0 \\
 x^3 + 2x & x^2 + 1 & 0 & 0 & 0 \\
 x^2 + 1 & x & 0 & 0 & 0 \\
 0 & 0 & 0 & 1 & 1
\end{pmatrix},
\]

and

\[
\Gamma_3 = \begin{pmatrix}
 x^4 + 3x^2 + 1 & x^3 + 2x & x^2 + 2x & x^2 + 3 & 2 \\
 x^3 + 2x & 0 & x^2 + 1 & 0 & 0 \\
 x^3 + x & x^2 + 1 & x & 0 & 0 \\
 x^2 + 1 & 0 & x & 0 & 0 \\
 0 & 0 & 0 & 1 & 1
\end{pmatrix}.
\]

**Corollary 3.7.** Let $F_n$ be a hexagonal chain with hexagons $h_1,h_2,\cdots,h_n$. Then

\[
\alpha(F_n) = W(h_n) \cdot W(h_{n-1}) \cdots W(h_1) \cdot \alpha(F_0),
\]

where

\[
W(h_i) = \begin{cases}
 \Gamma_1, & \text{if } h_i \text{ is of type-I,} \\
 \Gamma_2, & \text{if } h_i \text{ is of type-II,} \\
 \Gamma_3, & \text{if } h_i \text{ is of type-III,}
\end{cases}
\]

(17)

and $\alpha(F_0) = (x^2 + 1, x, x, 1, 0)^T$.

Following Corollary 3.7, for the hexagonal chain $F_5$ in Figure 5(a), we have that $\alpha(F_5) = \Gamma_1 \cdot \Gamma_2 \cdot \Gamma_3 \cdot \Gamma_1^2 \cdot (x^2 + 1, x, x, 1, 0)^T$. By a simple computation with MAPLE, it gives that

\[
\pi(F_5, x) = x^{22} + 26x^{20} + 287x^{18} + 1770x^{16} + 6757x^{14} + 16708x^{12} + 27173x^{10} + 28855x^8 + 19391x^6 + 7720x^4 + 1592x^2 + 121.
\]

Now we focus on some special hexagonal chains. If all the hexagons in a hexagonal chain are of type-I, then we call such a hexagonal chain a linear chain (see Figure 5(b)).

**Corollary 3.8.** Let $L_n$ be a linear chain with $n$ hexagons. Then

\[
\alpha(L_n) = \Gamma_1^n \cdot \alpha(L_0)
\]

with $\alpha(L_0) = (x^2 + 1, x, x, 1, 0)^T$. 

Table 1: The permanental polynomials of linear chain $L_n$ for $n = 1, 2, 3, 4, 5$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$x^6 + 6x^4 + 9x^2 + 4$</td>
</tr>
<tr>
<td>2</td>
<td>$x^{10} + 11x^8 + 41x^6 + 65x^4 + 43x^2 + 9$</td>
</tr>
<tr>
<td>3</td>
<td>$x^{14} + 16x^{12} + 98x^{10} + 296x^8 + 473x^6 + 392x^4 + 148x^2 + 16$</td>
</tr>
<tr>
<td>4</td>
<td>$x^{18} + 21x^{16} + 180x^{14} + 822x^{12} + 2192x^{10} + 3510x^8 + 3321x^6 + 1731x^4 + 415x^2 + 25$</td>
</tr>
<tr>
<td>5</td>
<td>$x^{22} + 26x^{20} + 287x^{18} + 1768x^{16} + 6725x^{14} + 16498x^{12} + 26429x^{10} + 27292x^8 + 17399x^6 + 6230x^4 + 1009x^2 + 36$</td>
</tr>
</tbody>
</table>

We compute the permanental polynomials of $L_n$ for $n = 1, 2, 3, 4, 5$ as exhibited in Table 1. The permanental polynomials in Tables 1-4 are all determined with MAPLE using the built-in MatrixVector-Multiply function.

If the hexagons in a hexagonal chain appear with type-II and type-III alternately, then it is said to be a zigzag chain. An illustration is given in Figure 5(c).

**Corollary 3.9.** For the zigzag chain $Z_n$ with hexagons $h_1, h_2, \cdots, h_n$,

$$
\alpha(Z_n) = \begin{cases} 
(\Gamma_2 \cdot \Gamma_3)^{x^{2-1}} \cdot \Gamma_2 \cdot \alpha(Z_0), & \text{if } h_1 \text{ and } h_n \text{ are both of type-II}, \\
(\Gamma_3 \cdot \Gamma_2)^{x^{2-2}} \cdot \Gamma_3 \cdot \alpha(Z_0), & \text{if } h_1 \text{ and } h_n \text{ are both of type-III}, \\
(\Gamma_3 \cdot \Gamma_2)^{x^{2-3}} \cdot \alpha(Z_0), & \text{if } h_1 \text{ is of type-II and } h_n \text{ is of type-III}, \\
(\Gamma_2 \cdot \Gamma_3)^{x^{2-4}} \cdot \alpha(Z_0), & \text{if } h_1 \text{ is of type-III and } h_n \text{ is of type-II}, 
\end{cases}
$$

where $\alpha(Z_0) = (x^2 + 1, x, x, 1, 0)^T$.

For the zigzag chains $Z_n$ with the starting hexagon $h_1$ of type-III, we show their permanental polynomials for $n = 1, 2, 3, 4, 5$ in Table 2.

Table 2: The permanental polynomials of zigzag chain $Z_n$ for $n = 1, 2, 3, 4, 5$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$x^6 + 6x^4 + 9x^2 + 4$</td>
</tr>
<tr>
<td>2</td>
<td>$x^{10} + 11x^8 + 41x^6 + 65x^4 + 43x^2 + 9$</td>
</tr>
<tr>
<td>3</td>
<td>$x^{14} + 16x^{12} + 98x^{10} + 297x^8 + 479x^6 + 407x^4 + 166x^2 + 25$</td>
</tr>
<tr>
<td>4</td>
<td>$x^{18} + 21x^{16} + 180x^{14} + 824x^{12} + 2214x^{10} + 3605x^8 + 3533x^6 + 1990x^4 + 577x^2 + 64$</td>
</tr>
<tr>
<td>5</td>
<td>$x^{22} + 26x^{20} + 287x^{18} + 1771x^{16} + 6773x^{14} + 16812x^{12} + 27538x^{10} + 29618x^8 + 20364x^6 + 8453x^4 + 1886x^2 + 169$</td>
</tr>
</tbody>
</table>

We call a hexagonal chain whose hexagons are all of type-II a helix chain [10]. An example is provided in Figure 5(d). The same analysis as above, we obtain that

**Corollary 3.10.** For the helix chain $T_n$,

$$
\alpha(T_n) = \Gamma_2^n \cdot \alpha(T_0),
$$

where $\alpha(T_0) = (x^2 + 1, x, x, 1, 0)^T$. 
We list the permanental polynomials of $T_n$ for $n = 1, 2, 3, 4, 5$ in Table 3.

| $\pi(T_1, x)$ | $x^6 + 6x^4 + 9x^2 + 4$ |
| $\pi(T_2, x)$ | $x^{10} + 11x^8 + 41x^6 + 65x^4 + 43x^2 + 9$ |
| $\pi(T_3, x)$ | $x^{14} + 16x^{12} + 98x^{10} + 297x^8 + 479x^6 + 407x^4 + 166x^2 + 25$ |
| $\pi(T_4, x)$ | $x^{18} + 21x^{16} + 180x^{14} + 824x^{12} + 2213x^{10} + 3599x^8 + 3518x^6 + 1972x^4 + 568x^2 + 64$ |
| $\pi(T_5, x)$ | $x^{22} + 26x^{20} + 287x^{18} + 1771x^{16} + 6771x^{14} + 16791x^{12} + 27450x^{10} + 29427x^8 + 20138x^6 + 8318x^4 + 1856x^2 + 169$ |

### 3.3 The permanental polynomial of a pericondensed system

Figure 6 illustrates a pericondensed system denoted by $H_n$. As the labeling of vertices, $H_{n+1}$ is the bridge graph of $H_n$ and $H_1$ through $e_n^1 = (u_n^1, u_{n+1}^1)$ and $e_n^2 = (v_n^2, v_{n+1}^2)$. Let $H_n^*$ be the graph obtained from $H_n$ by adding an edge $e_n^*$ joining $u_n^2$ and $v_n^2$. Now, we devote ourselves to computing the permanental polynomial of $H_n$ in a recursive technique.

![Figure 6: The pericondensed systems $H_1$, $H_2$ and $H_{n+1}$.](image)

**Theorem 3.11.** Let $\beta(H_n)$ be the column vector $(\pi(H_n, x), \pi(H_n - u_n^2, x), \pi(H_n - v_n^2, x), \pi(H_n - u_n^2 - v_n^2, x), \sum_{C \in \mathcal{C}_n^*(H_n)} (-1)^{|V(C)|} \pi(H_n^* - V(C), x))^T$. Then

$$\beta(H_{n+1}) = B \cdot \beta(H_n),$$

where $B$ is the $5 \times 5$ matrix with row vectors $y_1, y_2, y_3, y_4$ and $y_5$. More precisely,

$y_1 = (\omega_1, \omega_2, \omega_3, -2(\pi(P_1, x) + 2\pi(P_3, x) + x))$,

$y_2 = (\omega_2, \pi(C_6, x)\pi(P_2, x), \pi(P_3, x)\pi(P_2, x), -2\pi(P_2, x) - 2\pi(P_1, x)\pi(P_2, x))$,

$y_3 = (\omega_2, \pi(P_2, x), \pi(C_6, x)\pi(P_2, x), \pi(P_3, x)\pi(P_2, x), -2\pi(P_2, x) - 2\pi(P_1, x)\pi(P_2, x))$,

$y_4 = (\omega_3, \pi(P_5, x)\pi(P_2, x), \pi(P_3, x)\pi(P_2, x), \pi^3(P_2, x), -2\pi^2(P_2, x))$ and $y_5 = (-2\pi(P_7, x) + 2\pi(P_3, x) + x), -2\pi(P_2, x) + 2\pi(P_2, x)\pi(P_4, x)), -2(\pi(P_2, x) + \pi(P_2, x)\pi(P_4, x)), -2\pi(P_2, x) + 2\pi(P_2, x)\pi(P_4, x))$,

where $\omega_1 = x^{10} + 11x^8 + 41x^6 + 65x^4 + 43x^2 + 9$, $\omega_2 = x^9 + 9x^7 + 26x^5 + 29x^3 + 11x$ and $\omega_3 = x^8 + 7x^6 + 14x^4 + 8x^2$.
Thus

\[ \beta(H_n) = B^{n-1} \cdot \beta(H_1) \]

with \( \beta(H_1) = (\omega_1, \omega_2, \omega_3, -\pi(P_7, x) - 2\pi(P_3, x) - x)^T \).

**Proof.** The result of Corollary 3.3 derives

\[
\pi(H_{n+1}, x) = \pi(H_n, x) \pi(H_1, x) + \pi(H_n - u_n^1, x) \pi(H_1 - u_1^1, x) \\
+ \pi(H_n - v_n^2, x) \pi(H_1 - v_1^1, x) + \pi(H_n - u_n^2 - v_n^2, x) \pi(H_1 - u_1^1 - v_1^1, x) \\
+ 2 \sum_{C \in \mathcal{C}^{n+1}(H_{n+1})} (-1)^{|V(C)|} \pi(H_{n+1} - V(C), x),
\]

\[(19)\]

A cycle \( C \) using \( e_{n+1}^1 \) in \( H_{n+1} \) must contain \( e_{n+1}^2 \), \( C' - e_n^* \) \( (C' \text{ is a cycle belonging to } \mathcal{C}^{n+1}(H_n)) \) and the path induced by the bold edges as shown in Figure 7. So it implies

![Figure 7: The cases of a part of a cycle containing \( e_{n+1}^1 \) in \( H_{n+1} \).](image)

\[
\sum_{C \in \mathcal{C}^{n+1}(H_{n+1})} (-1)^{|V(C)|} \pi(H_{n+1} - V(C), x) \\
= [\pi(P_7, x) + 2\pi(P_3, x) + x] - \sum_{C \in \mathcal{C}^{n+1}(H_n)} (-1)^{|V(C)|} \pi(H_n^* - V(C), x),
\]

\[(20)\]

A series of computation leads to

\[ \pi(H_1, x) = \omega, \pi(H_1 - u_1^1, x) = \pi(H_1 - v_1^1, x) = \omega_2 \text{ and } \pi(H_1 - u_1^1 - v_1^1, x) = \omega_3. \]

\[(21)\]

Based on these results, we get that \( \pi(H_{n+1}, x) = y_1 \cdot \beta(H_n) \).

The same analysis as above, \( \pi(H_{n+1} - u_{n+1}^2, x) = y_2 \cdot \beta(H_n), \pi(H_{n+1} - v_{n+1}^2, x) = y_3 \cdot \beta(H_n), \text{ and } \pi(H_{n+1} - u_{n+1}^2 - v_{n+1}^2, x) = y_4 \cdot \beta(H_n) \) are got.
Now we consider the cycle containing $c_{n+1}^*$ in $H_{n+1}^*$. The following formula is derived.

$$\sum_{C \in \mathcal{E}_{n+1}^* (H_{n+1}^*)} (-1)^{|V(C)|} \pi(H_{n+1}^* - V(C), x)$$

$$= -\pi(H_{n+1}^*, x) - \pi(H_1, x) - \pi(H_2^2, x) - x\pi(H_n, x)$$

$$- \pi(P_2, x) \sum_{C \in \mathcal{E}_{n+1}^* (H_2^2)} (-1)^{|V(C)|} \pi(H_{n+1}^* - V(C), x), \quad (22)$$

where the graph $H_{n+1}'$ is the one obtained from $H_n$ by deleting the path of length three joining $u_n^2$ and $v_n^2$, and $H_1$ (resp. $H_2^2$) is the coalescence of $H_n$ and $P_4$ with $u_n^2$ (resp. $v_n^2$) as the coalesced vertex. Refer to Figure 8.

![Figure 8](image)

Figure 8: (a) $H_1$, (b) $H_2^2$ and (c) $H_{n+1}'$.

By Theorem 3.1,

$$\pi(H_n^1, x) = \pi(H_n, x)\pi(P_3, x) + \pi(H_n - u_n^2, x)\pi(P_2, x),$$

$$\pi(H_n^2, x) = \pi(H_n, x)\pi(P_3, x) + \pi(H_n - v_n^2, x)\pi(P_2, x). \quad (23)$$

Using Corollary 3.3 to $\pi(H_{n+1}', x)$, we have

$$\pi(H_{n+1}', x) = \pi(H_n, x)\pi(P_7, x) + \pi(H_n - u_n^2, x)\pi(P_3, x)\pi(P_4, x)$$

$$+ \pi(H_n - v_n^2, x)\pi(P_2, x)\pi(P_4, x) + \pi(H_n - u_n^2 - v_n^2, x)\pi^2(P_2, x) - 2\pi^2(P_2, x) \sum_{C \in \mathcal{E}_{n+1}^* (H_2^2)} (-1)^{|V(C)|} \pi(H_{n+1}^* - V(C), x). \quad (24)$$

The substitution of (23) and (24) into (22) yields

$$\sum_{C \in \mathcal{E}_{n+1}^* (H_{n+1}^*)} (-1)^{|V(C)|} \pi(H_{n+1}^* - V(C), x) =$$

$$- [x + 2\pi(P_3, x) + \pi(P_7, x)]\pi(H_n, x) - [\pi(P_2, x) + \pi(P_2, x)\pi(P_4, x)]\pi(H_n - u_n^2, x)$$

$$- [\pi(P_2, x) + \pi(P_2, x)\pi^2(P_4, x)]\pi(H_n - v_n^2, x) - x\pi^2(P_2, x)\pi(H_n - u_n^2 - v_n^2, x)$$

$$+ [2\pi^2(P_2, x) - \pi(P_2, x)] \sum_{C \in \mathcal{E}_{n+1}^* (H_2^2)} (-1)^{|V(C)|} \pi(H_{n+1}^* - V(C), x).$$

According to this, $\sum_{C \in \mathcal{E}_{n+1}^* (H_{n+1}^*)} (-1)^{|V(C)|} \pi(H_{n+1}^* - V(C), x) = y_5 \cdot \beta(H_n)$ is obtained.

By the discussions above, $\beta(H_{n+1}) = B \cdot \beta(H_n)$ follows. On the other hand, by (21) and a direct calculation, it gives that $\beta(H_1) = (\omega_1, \omega_2, \omega_3, -\pi(P_7, x) - 2\pi(P_3, x) - x)^T$. Thus equation (18) is established.

In Table 4 we list some $\pi(H_n, x)$ explicitly by the method provided in Theorem 3.11.
Table 4: The permanental polynomials of $H_n$ for $n = 1, 2, 3, 4$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Permanental Polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\pi(H_1, x) = x^{10} + 11x^8 + 41x^6 + 65x^4 + 43x^2 + 9$</td>
</tr>
<tr>
<td>2</td>
<td>$\pi(H_2, x) = x^{20} + 24x^{18} + 240x^{16} + 1314x^{14} + 4350x^{12} + 9066x^{10} + 11993x^8 + 9882x^6 + 4791x^4 + 1178x^2 + 81$</td>
</tr>
<tr>
<td>3</td>
<td>$\pi(H_3, x) = x^{30} + 37x^{28} + 608x^{26} + 5878x^{24} + 37338x^{22} + 164826x^{20} + 521531x^{18} + 1202331x^{16} + 2032192x^{14} + 2512170x^{12} + 2244727x^{10} + 1414603x^8 + 600378x^6 + 156878x^4 + 20677x^2 + 729$</td>
</tr>
<tr>
<td>4</td>
<td>$\pi(H_4, x) = x^{40} + 50x^{38} + 1145x^{36} + 15954x^{34} + 151566x^{32} + 1042672x^{30} + 5384511x^{28} + 21354630x^{26} + 65992566x^{24} + 160313204x^{22} + 307464174x^{20} + 465761312x^{18} + 555380333x^{16} + 517220574x^{14} + 371141426x^{12} + 200779952x^{10} + 79079947x^8 + 21400680x^6 + 3585810x^4 + 298948x^2 + 6561$</td>
</tr>
</tbody>
</table>

References


