The Energy of Kneser Graphs

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Abstract

In this note, by proving two combinatorial identities, we compute the energy of Kneser graphs.

Let \( \Gamma \) be a graph with \( n \) vertices and eigenvalues \( \lambda_1, \ldots, \lambda_n \). The energy of \( \Gamma \) is defined as \( E(\Gamma) = |\lambda_1| + \cdots + |\lambda_n| \). A graph \( \Gamma \) is said to be hyperenergetic if \( E(\Gamma) > 2n - 2 \). The concept of hyperenergeticity was first introduced by Gutman in [3]. Hyperenergetic graphs are important because molecular graphs with maximum energy pertain to maximality stable \( \pi \)-electron systems.

The Kneser graph \( K(v, k) \) is the graph with \( k \)-subsets of a fixed \( v \)-set as its vertices, with two vertices adjacent if they are disjoint. By [2, Theorem 9.4.3], if \( v \geq 2k + 1 \) then the eigenvalues of \( K(v, k) \) are \((-1)^j \binom{v-k-j}{j} \) with multiplicity \( \binom{v}{j} - \binom{v}{j-1} \), \( j = 0, 1, \ldots, k \). Then the energy of \( K(v, k) \) is

\[
E(K(v, k)) = \sum_{j=0}^{k} \left( \binom{v}{j} - \binom{v}{j-1} \right) \binom{v-k-j}{j}.
\]

Akbari [1] proved that \( K(v, k) \) is hyperenergetic for any integers \( v \) and \( k \geq 2 \) with \( v \geq 2k + 1 \). In this note, we shall compute the energy of Kneser graphs.

We start with two combinatorial identities.

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Lemma 1 For any odd number \( n \) and any integer \( k \) with \( 2k > n > 0 \), we have
\[
\sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{2k-n}{k-j} = 0.
\]

Proof. Note that
\[
\sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{2k-n}{k-j} = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{n-j} \binom{2k-n}{k-n+j} = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \binom{2k-n}{k-j}.
\]
Since \( n \) is odd number, the desired result follows.

Lemma 2 For any integer \( r \) and positive integer \( k \), we have
\[
\sum_{j=0}^{k} (-1)^{k-j} \binom{r}{j} \binom{2k-r}{k-j} = \frac{(r-1)(r-3) \cdots (r-2k+1)2^k}{k!}.
\]

Proof. Consider the polynomial
\[
g(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{x}{j} \binom{2k-x}{k-j} - \frac{(x-1)(x-3) \cdots (x-2k+1)2^k}{k!}.
\]
The degree of \( g(x) \) is at most \( k \). Lemma 1 implies that \( 1, 3, \ldots, 2k-1 \) are \( k \) distinct roots of \( g(x) \). Since \( g(0) = 0 \), we have \( g(x) = 0 \). Hence \( g(r) = 0 \), as desired.

Theorem 1 For \( v \geq 2k+1 \), the energy of \( K(v, k) \) is
\[
E(K(v, k)) = \frac{(v-1)(v-3) \cdots (v-2k+1)2^k}{k!}.
\]

Proof. Since
\[
\sum_{j=0}^{k} \binom{v}{j} \left( \binom{v}{j-1} \right) \left( v-k-j \right) = \sum_{j=0}^{k} \binom{v}{j} \binom{v-k-j}{k-j} - \sum_{j=1}^{k} \binom{v}{j-1} \binom{v-k-j}{k-j} = \sum_{j=0}^{k} \binom{v}{j} \left( \binom{v-k-j-1}{k-j} + \binom{v-k-j-1}{k-j-1} \right) - \sum_{j=0}^{k-1} \binom{v}{j} \binom{v-k-j-1}{k-j-1} = \sum_{j=0}^{k} \binom{v}{j} \binom{v-k-j-1}{k-j} = \sum_{j=0}^{k} (-1)^{k-j} \binom{v}{j} \binom{2k-v}{k-j},
\]
by Lemma 2 the desired result follows.
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References

