MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **65** (2011) 807-819

ISSN 0340 - 6253

Hosoya Polynomials of Pentachains

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(Received April 25, 2009)

Abstract

Using induction on the size of the graphs, Hosoya polynomials of several types of graphs consisting of concatenated pentagonal rings are obtained and studies in this paper.

1. Introduction

We follow the terminology of [1–4]. Let G be a finite connected graph. The Wiener index of $G = (V, E)$ is defined as :

$$W(G) = \sum_{u,v \in V} d(u,v),$$

where $d(u,v)$ is the minimum of the lengths of all u-v paths in G. Hosoya polynomial (also called Wiener polynomial [4]) of G is defined as :

$$H(G;x) = \sum_{u,v \in V} x^{d(u,v)},$$

It is clear that

$$H(G;x) = \sum_{k \geq 0} d(G,k) x^k,$$

where $d(G,k)$ is the number of pairs (u,v) of vertices of G such that $d(u,v) = k$.

Hosoya polynomial of a vertex v of G is defined as [4]

$$H(v,G;x) = \sum_{k \geq 0} d(v,G,k) x^k,$$

in which $d(v,G,k)$ is the number of all vertices $u \in V$, such that $d(v,u) = k$. It is clear that

$$H(G;x)=\frac{1}{2}\sum_{v\in V}H(v,G;x)+\frac{1}{2}p,$$

where p is the order of G .

The Wiener index of G can be obtained directly from the Hosoya polynomial of G as follows:

$$W(G)=\frac{d}{dx}H(G;x)\ \Big|_{x=1}.$$

The Wiener index is helpful to build a correlation model between the chemical structures of various chemical compounds. Indeed, Wiener index is the most celebrated topological index that identifies the characteristics chemical compounds.

In [5,6], Rao, N.P. and Prasanna, A.L., obtained formulas for Wiener indices of chemical graphs formed of concatenated 5-cycles (pentenes). It is clear that Hosoya polynomial $H(G;x)$ gives additional information about the distances in G. Thus, in this paper we obtain Hosoya polynomials of graphs constructed from chains of pentagons, and study some of their properties.

Given a collection of connected graphs, one may build bridge and chain graphs out of them. Mansour and Schork [7] the authors shown how the Wiener, hyper-Wiener, detour and hyper-detour indices for bridge and chain graphs are determined from the respective indices of the individual graphs (for other topological indices, see [7,8] and references therein).

Definition 1.1: The sequence $\{d(G,k)\}_{k=0}^{\delta}$ is called unimodal if, for some index h, $d(G,0)\le d(G,1)\le \ldots \le d(G,h)\ge d(G,h+1)\ge \ldots \ge d(G,\delta)$, and we call it is strong-unimodal if inequality holds without equality .

2. Straight Chaining of Pentagons

A straight chaining [5] is a graph consisting of m pentagonal rings, every two successive rings have a common edge, forming a chain denoted by $G(m,S)$ as shown in Figure 2.1. It is clear that the order of $G(m,S)$ is $3m+2$, the size is $4m+1$, and the diameter is $m+2$, $m\ge 2$.

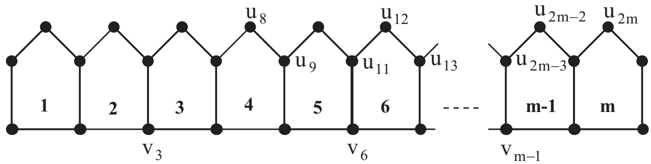


Figure 2.1: The graph $G(m,S)$

The Hosoya polynomial of the graph $G(m,S)$ is obtained in the next theorem.

Theorem 2.1: If $G(m,S)$ is the straight chaining of pentagons of order $3m+2$, $m \geq 5$, then

$$H(G(m,S);x) = (3m+2)+(4m+1)x+(7m-2)x^2+(7m-8)x^3+(7m-14)x^4$$

$$+(8m-22)x^5+\sum_{k=6}^{m+1}\left[9(m-k)+21\right]x^k+4x^{m+2}. \qquad .....(2.1)$$

Proof: We use mathematical induction on m, $m \geq 5$. For $m=5$ we find, by direct calculation,

$$H(G(5,S);x) = 17+21x+33x^2+27x^3+21x^4+18x^5+12x^6+4x^7.$$

Thus, (2.1) is true for $m=5$. We assume that (2.1) is true for $m=r\geq 5$ and prove that it is true for $m = r+1$. It is clear from Fig.2.1, that

$$H(G(r+1,S);x) = H(G(r,S);x)+F(x),$$

where

$$F(x)=\sum_{i=1}^{3}H(w_i,G(r+1,S);x)-(2x+x^2),$$

in which $w_1 = v_{r+2}$, $w_2 = u_{2r+3}$, and $w_3 = u_{2r+2}$ (see Fig.2.1 , putting $m = r+1$).

Then, we find that

$$H(w_1,G(r+1,S);x) = 1+2x+3\sum_{k=2}^{r+1}x^k+2x^{r+2},$$

$$H(w_2,G(r+1,S);x) = 1+2\sum_{k=1}^{4}x^k+3\sum_{k=5}^{r+2}x^k+2x^{r+3},$$

and

$$H(w_3,G(r+1,S);x) = 1+2x+3x^2+2\sum_{k=3}^{5}x^k+3\sum_{k=6}^{r+2}x^k+2x^{r+3}.$$

Thus

$$H(G(r+1,S);x) = (3r+2)+(4r+1)x+(7r-2)x^2+(7r-8)x^3+(7r-14)x^4$$

$$+(8r-22)x^5+\sum_{k=6}^{r+1}[9(r-k)+21]x^k+4x^{r+2}+3+4x+7x^2++7x^3$$

$$+7x^4+8x^5+9\sum_{k=6}^{r+1}x^k+8x^{r+2}+4x^{r+3}$$

$$=[3(r+1)+2]+[4(r+1)+1]x+[7(r+1)-2]x^2+[7(r+1)-8]x^3$$

$$+[7(r+1)-14]x^4+[8(r+1)-22]x^5+\sum_{k=6}^{r+2}[9(r+1-k)+21]\,x^k+4x^{r+3}.$$

Hence , (2.1) is true for all m, $m\geq 5$.

This completes the proof . ■

Remark 2.2: Hosoya polynomials of $G(m,S)$ for $1\leq m\leq 4$, are obtained as follows :

- $H(G(1,S);x)=5+5x+5x^2$,
- $H(G(2,S);x)=8+9x+12x^2+6x^3+x^4$,
- $H(G(3,S);x)=11+13x+19x^2+13x^3+7x^4+3x^5$,
- $H(G(4,S);x)=14+17x+26x^2+20x^3+14x^4+10x^5+4x^6$.

Thus we have obtained $H(G(m,S);x)$ for all positive integers m.

We notice from the formula (2.1) that the sequence $\{d(G(m,S),k)\}_{k=0}^{m+2}$ is strong-unimodal, at index h=2, for $m\geq 2,\ m\neq 8,11$, because $d(G(m,S),0)<d(G(m,S),1)<d(G(m,S),2)>d(G(m,S),3)>\ ...>d(G(m,S),m+2)$, and $d(G(8,S),4)=d(G(8,S),5)$, $d(G(11,S),5)=d(G(11,S),6)$. It is not palindromic. Also we notice that the sequence $\{d(G(m,S),k)\}_{k=2}^{m+2}$ is decreasing for $m\geq 2,\ m\neq 8,11$.

Finally, we obtain the Wiener index of the straight chaining of pentagons $G(m,S)$, by taking the derivative of $H(G(m,S);x)$ with respect to x then putting x=1. Thus, we get $W(G(m,S))=\frac{1}{2}\left[3m^3+21m^2-6m+14\right]$. This is the same result obtained in [5].

3. Alternate Chaining of Pentagons

Definition 3.1: An alternate chaining [5] $G(m,A)$ is a graph consisting of m pentagonal rings, every two successive rings have a common edge, forming a chain as shown in Figure 3.1.

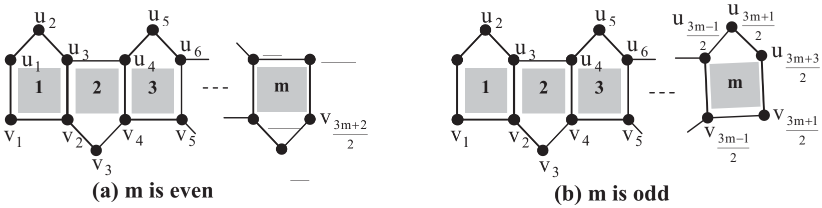


Figure 3.1: The graph $G(m,A)$

The order of $G(m,A)$ is $3m+2$, the size is $4m+1$, and the diameter is $\left\lfloor \frac{3m+2}{2} \right\rfloor$.

The Hosoya polynomial of the graph $G(m,A)$ is obtained in the next theorem.

Theorem 3.2: If $G(m,A)$ is an alternate chaining of order $3m+2$, $m \geq 2$, then

$$H(G(m,A);x) = (3m+2)+(4m+1)x+(7m-2)x^2+(7m-8)x^3$$

$$+\sum_{k=4}^{\delta}(6m-4k+5)x^k , \qquad .....(3.1)$$

where $\delta = \left\lfloor \frac{3m+2}{2} \right\rfloor$.

Proof: The relation (3.1) will be proved by mathematical inductional on m. By direct calculation, one can easily show that

$$H(G(1,A);x) = 5+5x+5x^2 ,$$

$$H(G(2,A);x) = 8+9x+12x^2+6x^3+x^4 ,$$

$$H(G(3,A);x) = 11+13x+19x^2+13x^3+7x^4+3x^5 .$$

Therefore , the relation (3.1) is true for m = 1, 2, and 3.

Thus, we assume that (3.1) is true for $m = r \geq 3$ and consider $G(r+1,A)$.

It is clear from Figure 3.1 (putting $m = r+1$), that

$$H(G(r+1,A);x) = H(G(r,A);x)+F(x) ,$$

where

$$F(x) = \sum_{i=1}^{3} H(w_i, G(r+1,A);x)-(2x+x^2) ,$$

in which $w_1 = v_{(3r+3)/2}$, $w_2 = v_{(3r+5)/2}$, and $w_3 = u_{(3r+5)/2}$ for odd r, and

$w_1 = v_{(3r+4)/2}$, $w_2 = u_{(3r+6)/2}$, and $w_3 = u_{(3r+4)/2}$ for even r.

If r is odd, then

$$H(w_1, G(r+1, A); x) = 1 + 2x + +3x^2 + 3x^3 + 2\sum_{k=4}^{\frac{3r+1}{2}} x^k + x^{\frac{3r+3}{2}},$$

$$H(w_2, G(r+1, A); x) = 1 + 2\sum_{k=1}^{\frac{3r+3}{2}} x^k + x^{\frac{3r+5}{2}},$$

and

$$H(w_3, G(r+1, A); x) = 1 + 2x + +3x^2 + 2\sum_{k=3}^{\frac{3r+3}{2}} x^k .$$

Thus

$$H(G(r+1, A); x) = (3r+2) + (4r+1)x + (7r-2)x^2 + (7r-8)x^3 + \sum_{k=4}^{\frac{3r+1}{2}} (6r-4k+5)x^k$$

$$+3 + 4x + 7x^2 + +7x^3 + 6\sum_{k=4}^{\frac{3r+1}{2}} x^k + 5x^{\frac{3r+3}{2}} + x^{\frac{3r+5}{2}}$$

$$= [3(r+1)+2] + [4(r+1)+1]x + [7(r+1)-2]x^2 + [7(r+1)-8]x^3$$

$$+\sum_{k=4}^{\frac{3r+5}{2}} \left[6(r+1)-4k+5\right] x^k .$$

If r is even, then

$$H(w_1, G(r+1, A); x) = 1 + 2x + +3x^2 + 2x^3 + 2\sum_{k=4}^{\frac{3r+2}{2}} x^k + x^{\frac{3r+4}{2}},$$

$$H(w_2, G(r+1, A); x) = 1 + 2\sum_{k=1}^{\frac{3r+4}{2}} x^k ,$$

and

$$H(w_3, G(r+1, A); x) = 1 + 2x + +3x^2 + 3x^3 + 2\sum_{k=4}^{\frac{3r+2}{2}} x^k .$$

Thus

$$H(G(r+1,A);x)=(3r+2)+(4r+1)x+(7r-2)x^2+(7r-8)x^3+\sum_{k=4}^{\frac{3r+2}{2}}(6r-4k+5)x^k$$

$$+3+4x+7x^2++7x^3+6\sum_{k=4}^{\frac{3r+2}{2}}x^k+3x^{\frac{3r+4}{2}}$$

$$=[3(r+1)+2]+[4(r+1)+1]x+[7(r+1)-2]x^2+[7(r+1)-8]x^3$$

$$+\sum_{k=4}^{\frac{3r+4}{2}}\left[6(r+1)-4k+5\right]x^k .$$

Therefore, the relation (3.1) is true for all $m,\ m\geq 2$.

This completes the proof . ■

We notice that the sequence $\{d(G(m,A),k)\}_{k=0}^{\delta}$, $\delta=\left\lfloor\frac{3m+2}{2}\right\rfloor$ is strong- unimodal at index h=2, since $d(G(m,A),0)<d(G(m,A),1)<d(G(m,A),2)>d(G(m,A),3)>\ ...\ >d(G(m,A),\delta)$, and its decreasing for all $2\leq k\leq\left\lfloor\frac{3m+2}{2}\right\rfloor$.

Finally, we can obtain the Wiener index of the graph $G(m,A)$, by taking the derivative of $H(G(m,A);x)$ with respect to x, and then putting x=1. Then, we get

$$W(G(m,A))=\begin{cases}\frac{1}{8}[18m^3+45m^2+58m] & ,\ m\ \ is\ \ even\ ,\\[2ex] \frac{1}{8}[18m^3+45m^2+58m-1] & ,\ m\ \ is\ \ odd\ .\end{cases}$$

This is the same result obtained in [5].

Now, we find the average distance of the pentachain graphs $G(m,S)$ and $G(m,A)$.

$$\overline{D}(G(m,S))=\binom{3m+2}{2}^{-1}W(G(m,S))=\frac{3m^3+21m^2-6m+14}{(3m+2)(3m+1)}\ ,$$

and

$$\overline{D}(G(m,A))=\binom{3m+2}{2}^{-1}W(G(m,A))=\frac{18m^3+45m^2+58m-1}{4(3m+2)(3m+1)},\ \ m\ \ is\ \ odd\ ,$$

$$\overline{D}(G(m,A)) = \binom{3m+2}{2}^{-1} W(G(m,A)) = \frac{18m^3+45m^2+58m}{4(3m+2)(3m+1)},\ m\ is\ even.$$

Hence, we deduce that for all values of $m \geq 2$, $\overline{D}(G(m,S)) \leq \overline{D}(G(m,A))$**,** and the equality holds at m=2,3.

4. Double Row Pentachains

In this section, we obtain Hosoya polynomials of the pentachains in two rows. Following N.P.Rao and A.L.Prasanna [6], we denote the graphs consisting of two rows of straight chains with m 5-cycles in the two rows combined as shown in Figure 4.1, by $G(m,S_1)$ and $G(m,S_2)$.

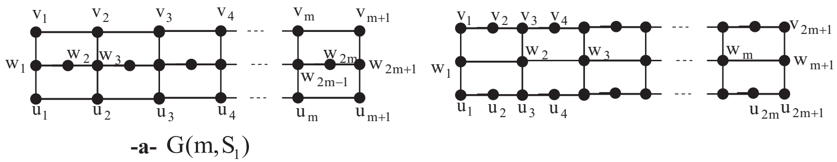


Figure 4.1.

From Figure 4.1., we notice that order of the graphs $G(m,S_1)$ and $G(m,S_2)$ are 4m+3 and 5m+3 respectively, the size of them are 6m+2 and 7m+2 respectively, and the diameter of both are m+2.

In the next theorems, we given the Hosoya Polynomial of the graphs $G(m,S_1)$ and $G(m,S_2)$.

Theorem 4.1: For $\mathrm{m} \geq 5$, we have

$$H(G(m,S_1);x) = (4m+3)+(6m+2)x+(13m-2)x^2+14(m-1)x^3$$

$$+(14m-27)x^4+(15m-42)x^5+4\sum_{k=6}^{m+1}[4(m-k)+9]\,x^k+6x^{m+2}$$

Proof: First we can partition the vertex set of $G(m,S_1)$ into three subsets V, U, and W, such that $V=\{v_1,v_2,v_3,\ \dots\ ,v_{m+1}\}$, $U=\{u_1,u_2,u_3,\ \dots\ ,u_{m+1}\}$, and $W=\{w_1,w_2,w_3,\ \dots\ ,w_{2m},w_{2m+1}\}$. From Figure 4.1. (a), we find that the Hosoya polynomial of $G(m,S_1)$ can be obtained in terms of $H(G(m,S);x)$ as follows:

$$H(G(m,S_1);x) = 2H(G(m,S);x) - \sum_{\forall w,w' \in W} x^{d(w,w')} + \sum_{\substack{\forall v \in V \\ \forall u \in U}} x^{d(v,u)} .$$

$$= 2(3m+2) + 2(4m+1)x + 2(7m-2)x^2 + 2(7m-8)x^3$$

$$+2(7m-14)x^4 + 2(8m-22)x^5 + 2\sum_{k=6}^{m+1}[9(m-k)+21]x^k + 8x^{m+2}$$

$$-\left[\sum_{k=0}^{4}(2m+1-k)x^k + (3m-6)x^5 + 4\sum_{k=6}^{m+2}(m+3-k)x^k\right]$$

$$+(m+1)x^2 + 2\sum_{k=3}^{m+2}(m+3-k)x^k .$$

$$= (4m+3) + (6m+2)x + (13m-2)x^2 + 14(m-1)x^3 + (14m-27)x^4$$

$$+(15m-42)x^5 + 4\sum_{k=6}^{m+1}[4(m-k)+9]x^k + 6x^{m+2} . \quad \blacksquare$$

Remark 4.2:

- $H(G(1,S_1);x) = 7 + 8x + 11x^2 + 2x^3$,
- $H(G(2,S_1);x) = 11 + 14x + 24x^2 + 14x^3 + 3x^4$,
- $H(G(3,S_1);x) = 15 + 20x + 37x^2 + 28x^3 + 15x^4 + 5x^5$,
- $H(G(4,S_1);x) = 19 + 26x + 37x^2 + 50x^3 + 42x^4 + 18x^5 + 6x^6$.

We notice that the sequence $\{d(G(m,S_1),k)\}_{k=0}^{m+2}$ is strong unimodal for all $1 \le m \le 11$ at index *h*=2, and for all $m = 13,14$ at index *h*=3, but is not unimodal for all $m \ge 16$.

Finally the Wiener index of $G(m,S_1)$ for $m \ge 2$, is given by :

$$W(G(m,S_1)) = \frac{d}{dx} H(G(m,S_1;x))\Big|_{x=1}$$

$$W(G(m,S_1)) = \frac{1}{3}[8m^3 + 54m^2 + 19m + 30] . \quad \blacksquare$$

Now, we find the Hosoya polynomial of $G(m,S_2)$ in the next theorem:

Theorem 4.3: For $m \ge 5$, we have

$$H(G(m,S_2);x) = (5m+3) + (7m+2)x + (14m-2)x^2 + (19m-14)x^3 + 11(2m-3)x^4$$

$$+(23m-56)x^5 + 5\sum_{k=6}^{m+1}[5(m-k)+13]x^k + 16x^{m+2} .$$

Proof: Let $V=\{v_1, v_2, v_3, \ldots, v_{2m}, v_{2m+1}\}$, $U=\{u_1, u_2, u_3, \ldots, u_{2m}, u_{2m+1}\}$, and $W=\{w_1, w_2, w_3, \ldots, w_{m+1}\}$. From Figure 4.1. (b), we obtain :

$$H(G(m,S_2);x)=2H(G(m,S);x)-\sum_{\forall w,w'\in W} x^{d(w,w')}+\sum_{\substack{\forall v\in V\\ \forall u\in U}} x^{d(v,u)}.$$

$$=2(3m+2)+2(4m+1)x+2(7m-2)x^2+2(7m-8)x^3+2(7m-14)x^4$$

$$+2(8m-22)x^5+2\sum_{k=6}^{m+1}[9(m-k)+21]x^k+8x^{m+2}-\sum_{k=0}^{m}(m+1-k)x^k$$

$$+(m+1)x^2+6mx^3+(9m-8)x^4+8\sum_{k=5}^{m+2}(m+3-k)x^k.$$

$$=(5m+3)+(7m+2)x+(14m-2)x^2+(19m-14)x^3+11(2m-3)x^4$$

$$+(23m-56)x^5+5\sum_{k=6}^{m+1}[5(m-k)+13]x^k+16x^{m+2}. \quad \blacksquare$$

By taking the derivative of $H(G(m,S_2);x)$ with respect to x and then putting $x=1$, we get Wiener index :

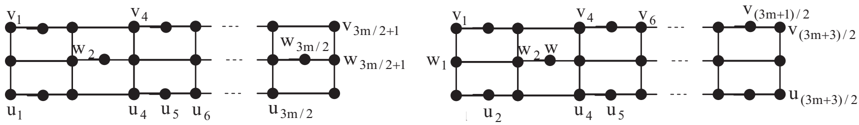
$$W(G(m,S_2))=\frac{1}{6}[25m^3+195m^2+2m+96],\ m\geq 2.$$

Remark 4.4:

- $H(G(1,S_2);x)=8+9x+12x^2+6x^3+x^4$,
- $H(G(2,S_2);x)=13+16x+26x^2+24x^3+12x^4$,
- $H(G(3,S_2);x)=18+23x+40x^2+43x^3+33x^4+14x^5$,
- $H(G(4,S_2);x)=23+30x+54x^2+62x^3+55x^4+36x^5+16x^6$.

The sequence $\{d(G(m,S_2),k)\}_{k=0}^{m+2}$ is strong unimodal at index h=3, for all $3\leq m\leq 6$, and index h=4, for all $7\leq m\leq 14$, but is not unimodal for all $15\leq m\leq 22$. Finally, the sequence $\{d(G(m,S_2),k)\}_{k=0}^{m+2}$ is strong-unimodal at index h=6, for all $m\geq 24$.

Now, we denote by the graph consisting of two rows of alternate chaining with m 5-cycles in each of the two rows, combined as shown in Figure 4.2 , by $G(m,A^*)$.



(a) *m* is even

Figure 4.2: The graph $G(m, A^*)$

From Figure 4.2., we notice that order of the graphs $G(m,A^*)$ is $\left\lceil \frac{9m+6}{2} \right\rceil$, size $\left\lceil \frac{13m+4}{2} \right\rceil$, and the diameter $\left\lfloor \frac{3m+3}{2} \right\rfloor$.

Also, in the next theorems, we given the Hosoya polynomials of the graphs $G(m,A^*)$ which dependent on $H(G(mA);X)$ and $H(P_p;X)$, where $p = \left\lfloor \frac{3m+2}{2} \right\rfloor$.

Theorem 4.5: (a) For m is even, $m \geq 4$, we have

$$H(G(m,A^*);x) = \frac{3}{2}(3m+2) + \frac{1}{2}(13m+4)x + \frac{1}{2}(27m-4)x^2 + \frac{1}{2}(33m-28)x^3$$

$$+(15m-23)x^4 + \frac{1}{2}(29m-68)x^5 + \frac{1}{2}\sum_{k=6}^{3m/2}[9(3m-2k)+26]x^k$$

$$+4x^{3m/2+1}.$$

(b) For m is odd**,** $m \geq 4$, we have

$$H(G(m,A^*);x) = \frac{1}{2}(9m+7) + \frac{1}{2}(13m+5)x + \frac{1}{2}(27m-3)x^2 + \frac{1}{2}(33m-23)x^3$$

$$+(15m-22)x^4 + \frac{1}{2}(29m-63)x^5 + \frac{1}{2}\sum_{k=6}^{(3m-1)/2}[9(3m-2k)+29]x^k$$

$$+10x^{(3m+1)/2} + 2x^{(3m+3)/2}.$$

Proof (a): Let $V = \{v_1, v_2, v_3, \ldots, v_{\frac{3m}{2}+1}\}$, $U = \{u_1, u_2, u_3, \ldots, u_{\frac{3m}{2}+1}\}$, and $W = \{w_1, w_2, w_3, \ldots, w_{\frac{3m}{2}+1}\}$. Thus

$$H(G(m,A^*);x) = 2H(G(m,A);x) - \sum_{\forall w,w' \in w} x^{d(w,w')} + \sum_{\substack{\forall v \in V \\ \forall u \in U}} x^{d(v,u)}.$$

$$= 2(3m+2) + 2(4m+1)x + 2(7m-2)x^2 + 2(7m-8)x^3$$

$$+2\sum_{k=4}^{3m/2+1}(6m-4k+5)x^k-\sum_{k=0}^{3m/2}(\frac{3m}{2}+1-k)\,x^k+(m+1)x^2+4mx^3$$

$$+\frac{1}{2}(9m-8)x^4+(4m-8)x^5+\sum_{k=6}^{3m/2+1}(3m+4-2k)\,x^k\ .$$

$$=\frac{3}{2}(3m+2)+\frac{1}{2}(13m+4)x+\frac{1}{2}(27m-4)x^2+\frac{1}{2}(33m-28)x^3$$

$$+(15m-23)x^4+\frac{1}{2}(29m-68)x^5+\frac{1}{2}\sum_{k=6}^{3m/2}[9(3m-2k)+26]\,x^k$$

$$+4x^{3m/2+1}.$$

(b): Let $V=\left\{v_1,v_2,v_3,\ \ldots\ ,\ v_{\frac{3m+3}{2}}\right\}$, $U=\left\{u_1,u_2,u_3,\ \ldots\ ,\ u_{\frac{3m+3}{2}}\right\}$, and $W=\left\{w_1,w_2,w_3,\ \ldots\ ,\ w_{\frac{3m+1}{2}}\right\}$. Then

$$H(G(m,A^*);x)=2H(G(m,A);x)-\sum_{\forall\, w,w'\in W}x^{d(w,w')}+\sum_{\substack{\forall\, v\in V\\ \forall\, u\in U}}x^{d(v,u)}\ .$$

$$=2(3m+2)+2(4m+1)x+2(7m-2)x^2+2(7m-8)x^3$$

$$+2\sum_{k=4}^{(3m+1)/2}(6m-4k+5)x^k-\sum_{k=0}^{(3m-1)/2}(\frac{3m+1}{2}-k)\,x^k+(m+1)x^2$$

$$+(4m+2)x^3+\frac{1}{2}(9m-7)x^4+(4m-6)x^5+\sum_{k=6}^{3(m+1)/2}(3m+5-2k)\,x^k\ .$$

$$=\frac{1}{2}(9m+7)+\frac{1}{2}(13m+5)x+\frac{1}{2}(27m-3)x^2+\frac{1}{2}(33m-23)x^3$$

$$+(15m-22)x^4+\frac{1}{2}(29m-63)x^5+\frac{1}{2}\sum_{k=6}^{(3m-1)/2}[9(3m-2k)+29]\,x^k$$

$$+10x^{(3m+1)/2}+2x^{(3m+3)/2}\ .\qquad\blacksquare$$

Corollary 4.6: The Wiener index of $\mathrm{G(m,A^*)}$ is given by:

$$W(G(m,A^*))=\frac{1}{16}[81m^3+234m^2+424m-32],\ \ m\ is\, even\,,$$

$$W(G(m,A^*))=\frac{1}{16}[81m^3+261m^2+487m+51],\ \ m\ is\, odd\ .\qquad\blacksquare$$

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