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The Jones Polynomial for Polyhedral Links

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Abstract

In this paper, a general approach is presented to compute the Jones polynomial of polyhedral links introduced in [18]. We show that Jones polynomials of the whole family of polyhedral links based on a fixed polyhedron can be obtained in a unified way from the Tutte polynomial of the 1-skeleton of the polyhedron via special parametrizations. As applications, by using computer algebra (Maple) techniques, Jones polynomials of Platonic polyhedral links are obtained and used to detect the chirality of Platonic polyhedral links.

1 Introduction

The Jones polynomial [1, 2] was discovered in 1985. It is an invariant of oriented links up to ambient isotopy. The discovery of the Jones polynomial is a very exciting event in the study of invariants of links which provokes the discovery of the Homfly [3, 4] as well as many types of polynomials. The Jones polynomial has connections with statistical mechanics [5] and quantum field theory [6] and has been widely studied by mathematicians and physicists.

It's well known that Jones polynomial of links is, in general, hard to compute [7]. There exist several available software packages to compute the Jones polynomial and the Homfly polynomial. For example, Knotscape [8] computes various link invariants. But these software packages usually only handle knots and links with smaller crossing numbers. In recent years, the computer algebra (Maple) techniques were used to calculate Jones polynomials for various special link families with larger crossing numbers, see [9–12].

Knots and links occurs in real world. Chemists and molecular biologists have also synthesized many knotted and linked molecules, see [13] and references therein. As the potential structure for synthesizing new types of topologically complex molecules, in a series of papers [14–18], Qiu et al introduced several types of polyhedral links with highly symmetry. Some topological properties such as component number of polyhedral links are studied in the above references.

In this paper, we focus on one type of such polyhedral links introduced in [18], formed from a polyhedron by 'n-branched curve and k-twisted double-line covering'. To be precise, given a polyhedron P, to construct a polyhedral link, we need two types of basic building blocks. One is an n-branched curve designed to replace the vertex with degree n of the polyhedron. The other is an k-twisted double-line ($k = 1, 2, \cdots$), which is proposed to replace the edge of the polyhedron. Finally by connecting these two building blocks we obtain an alternating link and we shall call it a polyhedral link.

Note that actually we will obtain an infinite family of polyhedral links based on P as k increases. A polyhedral link having k-twisted double-line is called a T_k-polyhedral link. When $k = 2m$, we call a T_k-polyhedral link an even polyhedral link. When $k = 2m + 1$, we call a T_k-polyhedral link an odd polyhedral link. See Fig. 1 for an example.

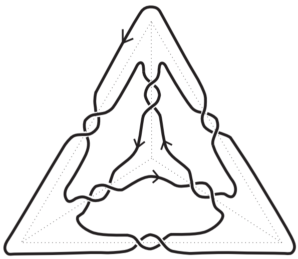

Fig. 1: T_2-tetrahedral link A_2. Note that it has 4 components which surround boundaries of 4 faces of the tetrahedron and are all given a clockwise orientation.

In [19], the present authors presented a general approach to compute the Homfly polynomial for even polyhedral links. However, this approach can not be used to deal with odd polyhedral links. In this paper, we shall give a general method to compute the Jones polynomial of a family of (even or odd) polyhedral links via the Tutte polynomial [20] of the 1-skeleton of the polyhedron. Note that the component number of a link can be deduced from its Jones polynomial [21].

In addition, the analysis of chirality problems is very important in stereochemistry [22]. In [18], linking number was used to determine the chirality of oriented polyhedral links. In general, this link invariant is relatively weak, since it can not be used to deal with oriented knots. Even for the multi-component links it does not always work. For example, the odd tetrahedral links all have linking number zero, the chirality of such links is still uncertain in [18]. It is well known that Jones polynomial can be used to detect the chirality of knots and links. For example, the trefoil knot is chiral, a fact difficult to prove before 1980s [23], can be easily proved by using the Jones polynomial [24]. As an application, we can use the Jones polynomial obtained to judge the chirality of Platonic polyhedral links.

2 Generalization and some notations

In this section we first generalize polyhedral links to plane graphs, then give some notations.

Note that the surface of a 3-polyhedron is topologically homeomorphic to the sphere S^2. Via the well-known stereographic projection, the graph consisting of vertices and edges of the polyhedron, i.e. the 1-skeleton, can be drawn on a plane with no edges intersected. Hence, the 1-skeleton of a polyhedron is a planar graph. However, a planar graph is not necessarily the 1-skeleton of some polyhedron. It is worth pointing out that the graphs in this paper allow loops and multiple edges in general.

Now we generalize polyhedral links to plane graphs. Let G be a plane graph with edge set $\{e_1, e_2, \cdots, e_q\}$. We use n-branched curve to cover the vertex of degree n of G and m_i-twisted double-line to cover the edge e_i for each $i = 1, 2, \cdots, q$. See Fig. 2. We shall denote by $L(G)$ the link thus constructed from G. Let k be a fixed nonnegative integer. If $m_i = k$ for each $i = 1, 2, \cdots, q$, we denote $L(G)$ by $L_k(G)$.

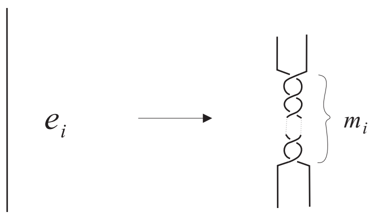

Fig. 2: Covering the edge e_i with m_i-twisted double-line.

Example 2.1 *Let B_n be a 'bouquet of n circles', i.e. a graph with one vertex and n loops. Then $L_2(B_1)$ is the Hopf link, $L_3(B_1)$ is the trefoil knot, $L_4(B_1)$ is the two-component $(4,2)$-torus link, and $L_3(B_2)$ is the connected sum of two trefoil knots. Molecules in the form of these knots and links were actually all synthesized in the past four decades, see [13, 25].*

Example 2.2 *Let Θ be the the theta graph, i.e. the graph with two distinct vertices joined by three parallel edges. Then $L_k(\Theta)$ is the well-known pretzel link $P(k,k,k)$ in knot theory.*

We shall compute the Jones polynomial of $L_k(G)$ in the next sections, which clearly include the polyhedral link as special case. Now we give some terminologies and notations in graph theory. An edge e in a graph G is called a loop if it connects a vertex to itself. An edge e in a graph G is called an isthmus or bridge if its removal disconnects the graph. The graph $G-e$ is obtained from G by deleting the edge e, and the graph G/e is obtained from G by contracting e, that is, by deleting e and identifying its two adjacent vertices. Throughout the paper, we shall use p, q and c to denote numbers of vertices, edges and connected components of G, respectively.

3 The Jones, chain and Tutte polynomials

Let L be an oriented link, we shall denote by $V_L(t)$ the Jones polynomial of L. It is a Laurent polynomial in the variable $\sqrt{t}$, determined by the following three axioms:

(i) Jones polynomial is an ambient isotopic invariant of oriented links.

(ii)

$$V_O(t) = 1\,, \tag{1}$$

where O is an unknot.

(iii) (Skein relation)

$$t^{-1}V_{L_+}(t) - tV_{L_-}(t) = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_{L_0}(t)\,, \tag{2}$$

where L_+, L_- and L_0 are link diagrams which are identical except near one crossing where they are as in Fig. 3 and are called a skein triple.

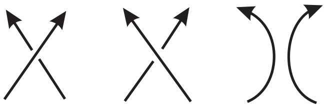

Fig. 3: L_+ (positive), L_- (negative) and L_0 in Eq. (2).

In principle, we can obtain the Jones polynomial of any oriented link by using the above definition. But this calculation is not convenient since it involves the isotopic deformations of links and it is also infeasible for links with large numbers of crossings. In the following we give two graph polynomial: the chain and Tutte polynomials, which will be used for us to simplify the computation of the Jones polynomial.

To study the chromatic polynomial [26] for homeomorphism class of graphs, Read and Whitehead introduced a multilinear polynomial of a graph in 1999, the chain polynomial [27] , which is associated with a graph whose edges have been labeled with elements of a commutative ring with unity 1. Let G be a labeled graph, we shall use $Ch[G]$ to denote the chain polynomial of G. We usually identify the edges with the labels assigned to them.

The chain polynomial of a labeled graph can be defined by the following recursive rules [28]:

(i) If G is edgeless, then

$$Ch[G] = 1 \ . \tag{3}$$

(ii) Otherwise, let a be an edge of G.

(a) If the edge a is a loop of G, then

$$Ch[G] = (a - w)Ch[G - a] \, , \tag{4}$$

(b) If the edge a is not a loop, then

$$Ch[G] = (a - 1)Ch[G - a] + Ch[G/a] \ . \tag{5}$$

The following lemma [27] will be used in the next section.

Lemma 3.1 *If M consists of two graphs, A and B, having at most one vertex in common, then $Ch(M) = Ch(A)Ch(B)$.*

The Tutte polynomial for graphs was constructed by Tutte in 1954 [20], building on his work seven years earlier [29]. The Tutte polynomial is a considerable generalization of the chromatic polynomial. Originally Tutte called his polynomial the dichromate of the graph, but now one usually call it the Tutte polynomial.

Let G be a graph, we shall denote by $T_G(x, y)$ the Tutte polynomial of G. There are several equivalent definitions of the Tutte polynomial [24]. We choose the following definition [24] for our own purpose.

(i) If G is edgeless, then

$$T_G(x, y) = 1 \ . \tag{6}$$

(ii) Otherwise, let e be an edge of G.

(a) If e is a bridge of G, then

$$T_G(x, y) = xT_{G/e}(x, y) \, ; \tag{7}$$

(b) If e is a loop, then

$$T_G(x, y) = yT_{G-e}(x, y) \, ; \tag{8}$$

(c) If e is neither a bridge nor a loop, then

$$T_G(x, y) = T_{G/e}(x, y) + T_{G-e}(x, y) \ . \tag{9}$$

4 Computing the Jones polynomial by the Tutte polynomial

Let L be an oriented link (diagram). The writhe $w(L)$ of L is defined to be the sum of signs of the crossings of L. Let $[L]$ be the one-variable Kauffman bracket polynomial in A of L (with the orientations assigned to L ignored). For the detail of the Kauffman bracket polynomial, see [30] or [31]. In [30], Kauffman constructed a state model for the Jones polynomial using his bracket polynomial, i.e.

Lemma 4.1 *Let*

$$f_L(A) = \left(-A^3\right)^{-w(L)} [L] \ .$$

The polynomial f_L in A is an invariant of ambient isotopy, which gives a model for the original Jones polynomial by the substitution

$$V_L(t) = f_L(t^{-1/4}) \ .$$

By Lemma 4.1, to obtain the Jones polynomial of an oriented link, one need to compute both the writhe and the Kauffman bracket polynomial of any diagram of the link. It is clear that the writhe is easily calculated. But the Kauffman bracket polynomial $[L]$ is the sum of 2^n terms if L has n crossings. Hence the main difficulty in computing the Jones polynomial is in the computation of the Kauffman bracket polynomial.

Now we consider the computation of the Kauffman bracket polynomial of polyhedral links. In general, let $L_k(G)$ be the link obtained from G by covering each edge by k-twisted double-line. In the remaining part of the section, we shall build a relation between the Kauffman bracket polynomial of $L_k(G)$ and the Tutte polynomial of the graph G.

Let $L(G)$ be the link obtained from G by covering the edge e_i by the m_i-twisted double-line for $i = 1, 2, \cdots, q$. Let G^l be the labeled graph obtained from G by labeling the edge e_i by a_i for $i = 1, 2, \cdots, q$. In [32], the present authors obtained a relation between $[L(G)]$ and the chain polynomial $Ch[G^l]$ of the labeled graph G^l.

Lemma 4.2 *If we replace w by $-A^4 - 1 - A^{-4}$, and replace a_i by $(-A^{-4})^{m_i}$ for each $i = 1, 2, \cdots, q$ in $Ch[G^l]$, then*

$$[L(G)] = \frac{A^{\sum_{i=1}^{q} m_i}}{(-A^2 - A^{-2})^{q-p+1}} Ch\left[G^l\right] \ .$$

Lemma 4.3 *Let G^a be a uniform labeled graph, i.e. all edges of G are labeled with the same label a. Then the chain polynomial of G^a is related to the Tutte polynomial of G by*

$$Ch[G^a] = (a-1)^{q-p+c}\, T_G\left(a, \frac{a-w}{a-1}\right) \ . \tag{10}$$

Proof. We shall prove Lemma 4.3 by induction on the number of edges of G. If G is an edgeless graph, then the chain polynomial of G is 1. The right hand of (10) equals to

$$(a-1)^{q-p+c}\, T_G\left(a, \frac{a-w}{a-1}\right) = 1 \ .$$

Lemma 4.3 holds. If $E(G) \neq \emptyset$, suppose that e is an edge of G. There are three cases.

(1) If e is a bridge, by Eq. (5), Lemma 3.1, induction hypothesis and Eq. (7), we have

$$\begin{aligned}
Ch[G^a] &= (a-1)Ch[G^a - a] + Ch[G^a/a] \\
&= a\ Ch[G^a/a] \\
&= a(a-1)^{(q-1)-(p-1)+c}\ T_{G/a}\left(a, \frac{a-w}{a-1}\right) \\
&= (a-1)^{q-p+c}\ T_G\left(a, \frac{a-w}{a-1}\right) .
\end{aligned}$$

(2) If e is a loop, by Eq. (4), induction hypothesis and Eq. (8), we have

$$\begin{aligned}
Ch[G^a] &= (a-w)Ch[G^a - a] \\
&= (a-w)(a-1)^{(q-1)-p+c}\ T_{G-a}\left(a, \frac{a-w}{a-1}\right) \\
&= (a-w)(a-1)^{(q-1)-p+c}\ \left(\frac{a-1}{a-w}\right)\ T_G\left(a, \frac{a-w}{a-1}\right) \\
&= (a-1)^{q-p+c}\ T_G\left(a, \frac{a-w}{a-1}\right) .
\end{aligned}$$

(3) If e is a neither a bridge nor a loop, by Eq. (5), induction hypothesis and Eq. (9), we have

$$\begin{aligned}
Ch[G^a] &= (a-1)Ch[G^a - a] + Ch[G^a/a] \\
&= (a-1)(a-1)^{(q-1)-p+c}\ T_{G-a}\left(a, \frac{a-w}{a-1}\right) + \\
&\quad (a-1)^{(q-1)-(p-1)+c}\ T_{G/a}\left(a, \frac{a-w}{a-1}\right) \\
&= (a-1)^{q-p+c}\ \left[T_{G-a}\left(a, \frac{a-w}{a-1}\right) + T_{G/a}\left(a, \frac{a-w}{a-1}\right)\right] \\
&= (a-1)^{q-p+c}\ T_G\left(a, \frac{a-w}{a-1}\right) .
\end{aligned}$$

Hence, Lemma 4.3 also holds. This complete the proof of Lemma 4.3.

□

Theorem 4.4 *Let G be a connected plane graph, having p vertices and q edges, which includes the 1-skeleton P of a polyhedron. Then*

$$[L_k(G)] = A^{qk}\left[\frac{(-A^{-4})^k - 1}{-A^2 - A^{-2}}\right]^{q-p+1} T_G\left(\left(-A^{-4}\right)^k, \frac{(-A^{-4})^k + A^4 + 1 + A^{-4}}{(-A^{-4})^k - 1}\right),$$

and therefore,

$$V_{L_k(G)}(t) = \left(-A^3\right)^{-w(L_k(G))}[L_k(G)]|_{A=t^{-1/4}}.$$

Proof. It follows directly from Lemmas 4.1- 4.3.□

5 Jones polynomials of Platonic polyhedral links

As applications, we use computer algebra (Maple) techniques to compute Jones polynomials of Platonic polyhedral links. For convenience, we shall denote by A_k, B_k, C_k, D_k and E_k the T_k-tetrahedral link, T_k-hexahedral link, the T_k-octahedral link, the T_k-dodecahedral link and the T_k-icosahedral link, respectively.

We first consider orientations of Platonic polyhedral links. In [18], by assigning the same direction to each component of polyhedral links, the authors calculated the writhe and linking number (It is actually half of the writhe.). For the even Platonic polyhedral link, its each component corresponds to a face of the polyhedron. Thus, for a planar embedding of the polyhedron, we can assign the same orientation (clockwise or anticlockwise) to each component of the corresponding even polyhedral link. See Fig. 1 for an example. Let $L_{2m}(P)$ be the even polyhedral link based on the polyhedron P with $q(P)$ edges. Then $w(L_{2m}(P)) = -2mq(P)$. We list writhes of five even Platonic polyhedral links in Table 1 as well as numbers of vertices and edges of polyhedra.

Table 1 Numbers of vertices p and edges q of Platonic polyhedra and writhes of even Platonic polyhedral links. (From [18].) Plane graphs of Platonic polyhedra (1-skeletons) are drawn in dashed lines in Fig. 4.

Platonic polyhedra	p	q	Even Platonic polyhedral links	The writhe w
Tetrahedron	4	6	A_{2m}	$-12m$
Hexahedron	8	12	B_{2m}	$-24m$
Octahedron	6	12	C_{2m}	$-24m$
Dodecahedron	20	30	D_{2m}	$-60m$
Icosahedron	12	30	E_{2m}	$-60m$

However, we can not define what is the same orientation to each component of the odd Platonic polyhedral link. In [33], Doll and Hoste introduced a nomenclature for the orientations of multi-component links by choosing a reference ordering of the components and a reference orientations of components. For example, for a 2-component link L, the corresponding reference oriented link will be denoted by $L{+}{+}$. If we reverse the orientation of the first component, we obtain the oriented link $L{-}{+}$, and so on. For five odd Platonic polyhedral links, the corresponding reference oriented links are given in Fig. 4. If L has n components, then L has 2^n different orientations in general. Note that if we reverse orientations of all components, the writhe will keep unchanged. Thus, in order to compute writhes, we need to consider at most 2^{n-1} different orientations for a n-component link. Actually, by symmetry of link diagrams, the number usually can be further reduced, as you shall see in Table 2. For example, C_{2m+1} has (at most) 4 different orientations for us to consider, i.e. $++++, +++- = ++-+ = +-++, ++-- = +--+ = +-+-, +---$. Let $L_{2m+1}(P)$ be the odd polyhedral link based on the polyhedron P. Then it is not difficult to see that $w(L_{2m+1}(P)) = (2m+1)w(L_1)$. We list writhes of five odd Platonic polyhedral links under all different orientations we need to consider in Table 2.

Then we consider the Tutte polynomials of polyhedra. Computing the Tutte polynomials is, in general, very difficult. Fortunately, the Maple software has a function called TuttePolynomial in the GraphTheory package, which can be used for us to calculate the Tutte polynomial of small graphs, including the 1-skeletons of the five Platonic polyhedra. We refer the reader to [19] for Tutte polynomials of skeletons of the five Platonic polyhedra. Then according to Theorem 4.4 and by using Maple, we obtain the Kauffman bracket polynomial of the five Platonic polyhedra listed as follows.

1.
$$\begin{aligned}
&[A_k] = -\frac{A^{6k}}{A^6(A^4+1)^3}(A^{12}(-A^{-4})^{6k}\\
&+(4A^8+4A^{12}+4A^{16})(-A^{-4})^{3k}\\
&+(3A^8+3A^{12}+3A^{16})(-A^{-4})^{2k}\\
&+(6A^4+6A^8+12A^{12}+6A^{16}+6A^{20})(-A^{-4})^k\\
&+1+2A^8+2A^{16}+A^{24})
\end{aligned}$$

2.
$$\begin{aligned}
&[B_k] = -\frac{A^{6k}}{A^{14}(A^4+1)^3((-A^{-4})^k-1)^2}(A^{20}(-A^{-4})^{12k}\\
&+(-A^{-4})^{8k}(6A^{16}+6A^{20}+6A^{24})
\end{aligned}$$

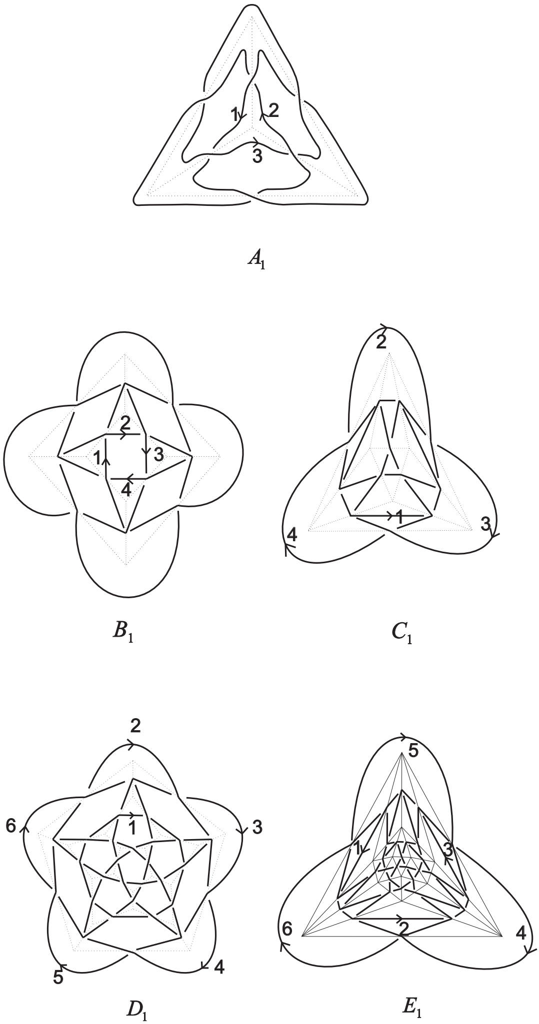

Fig. 4: Reference ordering and orientations of components of odd Platonic polyhedral links. The dashed lines represent the 1-skeleton of the polyhedron. Note that $w(A_1) = 0$,$w(B_1) = 4$,$w(C_1) = 12$,$w(D_1) = 10$ and $w(E_1) = 6$.

Table 2 Writhes of odd Platonic polyhedral links. In the last column, 'c' denotes 'chiral'.

Odd Platonic polyhedral links	$w(L_1)$	$w(L_{2m+1})$	Chirality
$A_{2m+1}\ +++$	0	0	c ($m \geq 1$)
$A_{2m+1}\ ++-$	0	0	c ($m \geq 1$)
$B_{2m+1}\ ++++$	4	$4(2m+1)$	c
$B_{2m+1}\ +++-$	0	0	c
$B_{2m+1}\ ++--$	4	$4(2m+1)$	c
$B_{2m+1}\ +-+-$	-12	$-12(2m+1)$	c
$C_{2m+1}\ ++++$	12	$12(2m+1)$	c
$C_{2m+1}\ +++-$	0	0	c
$C_{2m+1}\ ++--$	-4	$-4(2m+1)$	c
$C_{2m+1}\ +---$	0	0	c
$D_{2m+1}\ ++++++$	10	$10(2m+1)$	c
$D_{2m+1}\ +++++-$	6	$6(2m+1)$	c
$D_{2m+1}\ ++++--$	10	$10(2m+1)$	c
$D_{2m+1}\ +++-+-$	-6	$-6(2m+1)$	c
$D_{2m+1}\ +++---$	6	$6(2m+1)$	c
$D_{2m+1}\ ++-+--$	-10	$-10(2m+1)$	c
$D_{2m+1}\ ++----$	-6	$-6(2m+1)$	c
$D_{2m+1}\ +-----$	-10	$-10(2m+1)$	c
$E_{2m+1}\ ++++++$	6	$6(2m+1)$	c
$E_{2m+1}\ +++++-$	-6	$-6(2m+1)$	c
$E_{2m+1}\ ++++--$	-10	$-10(2m+1)$	c
$E_{2m+1}\ +++---$	-6	$-6(2m+1)$	c
$E_{2m+1}\ ++-+++$	10	$10(2m+1)$	c
$E_{2m+1}\ ++-++-$	-10	$-10(2m+1)$	c
$E_{2m+1}\ ++-+-+$	6	$6(2m+1)$	c
$E_{2m+1}\ ++-+--$	-6	$-6(2m+1)$	c
$E_{2m+1}\ ++---+$	10	$10(2m+1)$	c
$E_{2m+1}\ ++----$	6	$6(2m+1)$	c
$E_{2m+1}\ +--+++$	6	$6(2m+1)$	c
$E_{2m+1}\ +--++-$	-6	$-6(2m+1)$	c
$E_{2m+1}\ +---++$	10	$10(2m+1)$	c
$E_{2m+1}\ +--+--$	-10	$-10(2m+1)$	c
$E_{2m+1}\ +---+-$	6	$6(2m+1)$	c
$E_{2m+1}\ +-----$	10	$10(2m+1)$	c

$$+(-A^{-4})^{6k}(16A^{16}+16A^{20}+16A^{24})$$
$$+(-A^{-4})^{5k}(12A^{12}+12A^{16}+24A^{20}+12A^{24}+12A^{28})$$
$$+(-A^{-4})^{4k}(27A^{12}+36A^{16}+63A^{20}+36A^{24}+27A^{28})$$
$$+(-A^{-4})^{3k}(8A^{8}+32A^{12}+48A^{16}+64A^{20}+48A^{24}$$
$$+32A^{28}+8A^{32})$$
$$+(-A^{-4})^{2k}(42A^{8}+18A^{12}+102A^{16}+36A^{20}+102A^{24}$$
$$+18A^{28}+42A^{32})$$
$$+(-A^{-4})^{k}(12A^{4}-12A^{8}+36A^{12}-24A^{16}+48A^{20}$$
$$-24A^{24}+36A^{28}-12A^{32}+12A^{36})$$
$$+1-2A^{4}+7A^{8}-5A^{12}+14A^{16}-6A^{20}+14A^{24}$$
$$-5A^{28}+7A^{32}-2A^{36}+A^{40}).$$

3.

$$[C_k]=-\frac{A^{6k}}{A^{22}(A^4+1)^3((-A^{-4})^k-1)^4}(A^{28}(-A^{-4})^{12k}$$
$$+(-A^{-4})^{9k}(8A^{24}+8A^{28}+8A^{32})$$
$$+(-A^{-4})^{8k}(15A^{24}+15A^{28}+15A^{32})$$
$$+(-A^{-4})^{7k}(12A^{20}+36A^{24}+48A^{28}+36A^{32}+12A^{36})$$
$$+(-A^{-4})^{6k}(100A^{20}+132A^{24}+232A^{28}+132A^{32}+100A^{36})$$
$$+(-A^{-4})^{5k}(48A^{16}+180A^{20}+300A^{24}+384A^{28}+300A^{32}$$
$$+180A^{36}+48A^{40})$$
$$+(-A^{-4})^{4k}(6A^{12}+297A^{16}+282A^{20}+873A^{24}+567A^{28}$$
$$+873A^{32}+282A^{36}+297A^{40}+6A^{44})$$
$$+(-A^{-4})^{3k}(196A^{12}+92A^{16}+812A^{20}+416A^{24}+1240A^{28}$$
$$+416A^{32}+812A^{36}+92A^{40}+196A^{44})$$
$$+(-A^{-4})^{2k}(66A^{8}+36A^{12}+432A^{16}+240A^{20}+930A^{24}$$
$$+408A^{28}+930A^{32}+240A^{36}+432A^{40}+36A^{44}$$
$$+66A^{48})$$
$$+(-A^{-4})^{k}(12A^{4}+12A^{8}+108A^{12}+72A^{16}+312A^{20}$$
$$+168A^{24}+432A^{28}+168A^{32}+312A^{36}$$

$$+72A^{40} + 108A^{44} + 12A^{48} + 12A^{52})$$
$$+1 + 2A^4 + 13A^8 + 18A^{12} + 60A^{16} + 64A^{20} + 125A^{24}$$
$$+97A^{28} + 125A^{32} + 64A^{36} + 60A^{40} + 18A^{44} + 13A^{48}$$
$$+2A^{52} + A^{56}).$$

4. The expressions of formulae for $[D_k]$ and $[E_k]$ are too long. We only write the first and last several terms down here.

$$\begin{aligned}
[D(k)] \quad = \quad & -\frac{A^{6k}}{A^{38}(A^4+1)^3((-A^{-4})^k-1)^8}(A^{44}(-A^{-4})^{30k} \\
& +(-A^{-4})^{25k}(12A^{40} + 12A^{44} + 12A^{48}) \\
& +(-A^{-4})^{22k}(30A^{40} + 30A^{44} + 30A^{48}) \\
& +(-A^{-4})^{21k}(30A^{36} + 50A^{40} + 80A^{44} + 50A^{48} + 30A^{52}) \\
& +(-A^{-4})^{20k}(36A^{36} + 108A^{40} + 144A^{44} + 108A^{48} + 36A^{52}) \\
& +(-A^{-4})^{19k}(60A^{36} + 180A^{40} + 240A^{44} + 180A^{48} + 60A^{52}) \\
& +\cdots \\
& +(-A^{-4})^{2k}(375A^8 - 2250A^{12} + 10755A^{16} - 33600A^{20} \\
& \qquad\qquad +82890A^{24} - 157665A^{28} + 254865A^{32} \\
& \qquad\qquad -345345A^{36} + 416835A^{40} - 438060A^{44} \\
& \qquad\qquad +416835A^{48} - 345345A^{52} + 254865A^{56} \\
& \qquad\qquad -157665A^{60} + 82890A^{64} - 33600A^{68} \\
& \qquad\qquad +10755A^{72} - 2250A^{76} + 375A^{80}) \\
& +(-A^{-4})^{1k}(30A^4 - 210A^8 + 1110A^{12} - 4080A^{16} + 11880A^{20} \\
& \qquad\qquad -26820A^{24} + 50040A^{28} - 78090A^{32} + 105900A^{36} \\
& \qquad\qquad -125700A^{40} + 133320A^{44} - 125700A^{48} \\
& \qquad\qquad +105900A^{52} - 78090A^{56} + 50040A^{60} - 26820A^{64} \\
& \qquad\qquad +11880A^{68} - 4080A^{72} + 1110A^{76} - 210A^{80} \\
& \qquad\qquad +30A^{84}) \\
& +1 - 8A^4 + 46A^8 - 190A^{12} + 633A^{16} - 1676A^{20} + 3627A^{24} \\
& -6526A^{28} + 10044A^{32} - 13436A^{36} + 15919A^{40} - 16808A^{44} \\
& +15919A^{48} - 13436A^{52} + 10044A^{56} - 6526A^{60} + 3627A^{64}
\end{aligned}$$

$$-1676A^{68}+633A^{72}-190A^{76}+46A^{80}-8A^{84}+A^{88}).$$

5.

$$\begin{aligned}[E(k)] \;\; = \;\; & -\frac{A^{6k}}{A^{70}(A^4+1)^3((-A^{-4})^k-1)^{16}}(A^{76}(-A^{-4})^{30k} \\ & +(-A^{-4})^{27k}(20A^{72}+20A^{76}+20A^{80}) \\ & +\cdots \\ & +(-A^{-4})^{1k}(30A^4+\cdots+30A^{148}) \\ & +1+8A^4+\cdots+8A^{148}+A^{152}.\end{aligned}$$

Using the information in Tables 1 and 2, we can obtain the formula of the Jones polynomial of the five Platonic polyhedral links. However, the form of the formula is not a polynomial. By applying computer algebra (MAPLE) techniques, we can transform it into a polynomial. For example, it is easy for us to write a Maple program to calculate $V_{A_k}(t)$ for each k. We list some numerical results in Table 3. Note that in Table 3, the number in the curly bracket is the minimum degree of the Jones polynomial and the next sequence of numbers gives the coefficients of the polynomial, beginning with the coefficient of the minimum degree term. For example, $\{-3\}(-1,3,-8,16,-8,5,-4,1)$ denotes the polynomial $-t^{-3}+3t^{-2}-8t^{-1}+16-8t+5t^2-4t^3+t^4$.

Table 3 The Jones polynomial of A_k for k from 1 to 5.

Tetrahedral links		The Jones polynomials
A_1	$\{-3\}$	$(-1,3,-2,4,-2,3,-1)$
A_2	$\{-27/2\}$	$(-1,3,-8,10,-17,17,-20,17,-15,10,-6,3,-1)$
A_3	$\{-6\}$	$(-1,3,-8,16,-23,35,-41,47,-50,47,-41,36,-28,$ $21,-15,10,-6,3,-1)$
A_4	$\{-51/2\}$	$(-1,3,-8,16,-29,41,-59,71,-83,89,-92,89,-83,$ $74,-66,55,-45,36,-28,21,-15,10,-6,3,-1)$
A_5	$\{-9\}$	$(-1,3,-8,16,-29,47,-65,89,-107,125,-137,143,$ $-146,143,-137,128,-116,105,-91,78,-66,55,$ $-45,36,-28,21,-15,10,-6,3,-1)$

6 Chirality analysis of Platonic polyhedral links

It is well known that the writhe of a connected reduced alternating oriented diagram is an invariant of the oriented link it represents [25, 34, 35]. Thus, if the writhe of a connected reduced alternating oriented diagram is not zero, then the oriented link it represents is chiral. It is clear that the Platonic polyhedral link diagrams we construct are all connected reduced alternating. Hence, there are only five oriented links in Table 2, i.e. $A_{2m+1} + ++$,$A_{2m+1} + +-$,$B_{2m+1} + + + -$,$C_{2m+1} + + + -$,$C_{2m+1} + - - -$ whose chirality can not be detected by using writhes.

It is also well known that the Jones polynomial of an achiral oriented link is symmetric [21, 24, 36], that is, if the Jones polynomial of an oriented link is not symmetric then it is topologically chiral. Note that if the writhe of an oriented link is zero, to prove that its Jones polynomial is not symmetric, it suffices to show that its Kauffman bracket polynomial is not symmetric. Now we compute the highest and lowest degrees of Kauffman bracket polynomials of Platonic polyhedral links. According to the expression of $[A_k]$, we know that the highest degree and the lowest degree of $[A_k]$ are $6k + 6$ and $-18k + 6$, respectively. Similarly, the highest degree and the lowest degree of $[B_k]$ are $6k + 14$ and $-34k + 6$, respectively. And the highest degree and the lowest degree of $[C_k]$ are $6k + 22$ and $-26k + 6$, respectively. Finally, we consider $[D_k]$ and $[E_k]$. It is not difficult to obtain that he highest degree and the lowest degree of $[D_k]$ are $6k+38$ and $-82k+6$, respectively. And the highest degree and the lowest degree of $[E_k]$ are $6k+70$ and $-50k+6$, respectively.

Simple calculations show that the sum of the highest and the lowest degree of the Kauffman polynomial of a T_k-Platonic polyhedral link is equal to zero if and only if the T_k-Platonic polyhedral link is T_1-tetrahedral link A_1. Thus, the Kauffman bracket polynomial of any Platonic polyhedral link is not symmetric except (possibly) the T_1-tetrahedral link. Note that $V_{T_1}(t)$ (see Table 3) is symmetric. Thus, $[A_{2m+1}] + ++$ $(m \geq 1)$, $[A_{2m+1}] + +-$ $(m \geq 1)$, $B_{2m+1} + + + -$, $C_{2m+1} + + + -$ and $C_{2m+1} + - - -$ are all topologically chiral.

Furthermore, note that the self-writhe [24] of all oriented Platonic polyhedral links are all zero (this means that the linking number is equal to half of the writhe). Thus, the asymmetry of their Kauffman bracket polynomials also imply that as the unoriented link, they are also chiral. We point out that the unoriented link A_1 (the 6^3_2 in Dale Rolfsen's Tabulation [37]) is achiral, see [38] and [39]. In addition, $[A_1] + ++$ and $[A_1] + +-$ (they are equivalent [40]) are achiral, see [39].

7 Concluding remarks

In this paper, we present a general approach to compute the Jones polynomial of polyhedral links. The key step of the approach is to convert the chain polynomial of a uniformly labeled graph to the Tutte polynomial of the unlabeled graph. The mathematics in this paper is not difficult. However, it converted the computation of the Jones polynomial of all polyhedral links to that of the Tutte polynomial of the polyhedron in a unified way, which can be obtained by the Maple program. Except the Platonic polyhedral links this approach can be applied to any polyhedral links or more generally, links constructed from any plane graphs. All different orientations of odd Platonic polyhedral links are considered and their writhes are computed, which provides infinite families of oriented links with writhes zero. We succeed in detecting the chirality of Platonic polyhedral links by using the Jones polynomial, which can not be determined by using writhes and linking numbers.

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