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Some New Results on Distance–Based Polynomials

A. Behmaram[1], H. Yousefi-Azari[1,•] and A.R.Ashrafi[2]

[1]School of Mathematics, Statistics and Computer Sciences, University of Tehran, Tehran, I. R. Iran
[2]Department of Mathematics, Faculty of Science, University of Kashan, Kashan, I. R. Iran

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Abstract

Suppose G is a graph and $d_G(x,y)$ denotes the length of a minimum path connecting vertices x and y of G. In this paper, some distance-based polynomials such as Hosoya, edge Wiener, Schultz and Gutman polynomials and their relationship are investigated. The Schultz polynomial of some graph operations is also computed. Finally, the mathematical properties of a new two variables polynomial are presented.

1. Introduction

Throughout this paper we consider only simple connected graphs with vertex and edge sets V(G) and E(G), respectively. A topological index Top(G) for a graph G, is a number with this property that for every graph H isomorphic to G, Top(H) = Top(G). Between topological indices of graphs those are "distance-based" is very important both in mathematics and in chemistry. To explain, we assume that $d_G(x,y)$ denotes the length of a minimum path connecting vertices x and y of G. A "distance-based topological index" is a topological index related to the distance

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•Corresponding author (email: hyousefi@ut.ac.ir).

function $d_G(-,-)$. The Wiener index W(G) is the first distance-based topological index introduced by the chemist Harold Wiener [28] for investigating boiling point of alkanes, see also [14]. This index is defined as the sum of all distances between vertices of the graph under consideration, see [4,5,7] for details. The definition of the Wiener index in terms of distances between vertices of a graph was first given by a Japanese chemist Haruo Hosoya [15].

We recall some definition that will be used in the paper. Suppose G and H are graphs. The Cartesian product $G \times H$ has the vertex set $V(G \times H) = V(G) \times V(H)$ and $(a,x)(b,y)$ is an edge of $G \times H$ if $a = b$ and $xy \in H$ or $ab \in E(G)$ and $x = y$. If $G_1, G_2, \ldots, G_n$ are graphs then we denote $G_1 \times G_2 \times \ldots \times G_n$ by $\otimes G_i$. In the case that $G_1 = G_2 = \ldots = G_n = G$, we denote $\otimes G_i$ by G^n.

Suppose G and H are graphs and $V(G) \cap V(H) = E(G) \cap E(H) = \emptyset$. Then the join $G + H$ is the graph union $G \cup H$ together with all edges joining V(G) and V(H). If $G = H + \ldots + H$ then G is denoted by nH. The composition G[H] is the graph with vertex set $V(G) \times V(H)$ and $u = (u_1,v_1)$ is adjacent with $v = (u_2,v_2)$ whenever u_1 is adjacent with u_2 or $u_1 = u_2$ and v_1 is adjacent with v_2, [16, p. 185]. The disjunction $G \vee H$ is the graph with vertex set $V(G) \times V(H)$ and (u_1,v_1) is adjacent with (u_2,v_2) whenever $u_1u_2 \in E(G)$ or $v_1v_2 \in E(H)$. The symmetric difference $G \oplus H$ is the graph with vertex set $V(G) \times V(H)$ and $E(G \oplus H) = \{(u_1,u_2)(v_1,v_2) \mid u_1v_1 \in E(G)$ or $u_2v_2 \in E(H)$ but not both$\}$.

The edge Wiener index of a graph G is defined as $W_e(G)=\sum_{e,f\in E(G)} d(e,f)$, where for edges $e = uv$ and $f = ab$ of G, $d(e,f) = \min\{d(u,a),d(u,b),d(v,a),d(v,b)\}$. The hyper wiener index is defined as $WW(G) = \frac{1}{2}\sum_{\{u,v\}\subseteq V(G)}(d^2(u,v) + d(u,v))$. The Schultz [26] and Gutman [12] indices of G are defined as $W_+(G) = \sum_{\{u,v\}\subseteq V(G)}(deg(u) + deg(v))\, d(u,v)$ and $W_*(G) = \sum_{\{u,v\}\subseteq V(G)} deg(u)deg(v)\, d(u,v)$, respectively. The first and second Zagreb indices have been introduced more than thirty years ago by Gutman and Trinajstić [9]. They are denoted by $M_1(G)$ and $M_2(G)$ and defined as $M_1(G) = \sum_{v\in v(G)} \deg(v)^2$ and $M_2(G) = \sum_{uv\in E(G)} \deg(u)\deg(v)$. We encourage the interested readers to consult [1-3, 6, 10-13, 27] and references therein for more information on this topic.

In [25], Sagan et al. computed some exact formulas for the Hosoya polynomials of various graph operations. In [8, 17-24, 29], the authors computed exact formulas for the Wiener, edge Wiener, PI, vertex PI, first and second Zagreb, Szeged and edge Szeged indices of some graph operations.

In this paper the Schultz polynomial of some graph operations are investigated. We also obtain some new relations between Schultz, Gutman Schultz, Wiener, and edge Wiener polynomials of graphs. We begin with the following crucial lemma that will be used later.

Lemma 1. Let G and H be graphs. Then we have:

(a) $|V(G\times H)| = |V(G\vee H)| = |V(G[H])| = |V(G\oplus H)| = |V(G)||V(H)|$; $|E(G\times H)| = |E(G)||V(H)| + |V(G)||E(H)|$; $|E(G+H)| = |E(G)|+|E(H)| + |V(G)||V(H)|$; $|E(G[H])| = |E(G)||V(H)|^2 + |E(H)|V(G)|$; $|E(G\vee H)| = |E(G)||V(H)|^2 + |E(H)|V(G)|^2 - 2|E(G)||E(H)|$; $|E(G\oplus H)| = |E(G)||V(H)|^2 + |E(H)|V(G)|^2 - 4|E(G)||E(H)|$.

(b) $G \times H$ is connected if and only if G and H are connected.

(c) If (a,c) and (b,d) are vertices of $G \times H$ then $d_{G\times H}((a,c),(b,d)) = d_G(a,b) + d_H(c,d)$.

(d) The Cartesian product, join, composition, disjunction and symmetric difference of graphs are associative and all of them are commutative except from composition.

(e) $d_{G+H}(u,v) = \begin{cases} 0 & u = v \\ 1 & uv \in E(G) \text{ or } uv \in E(H) \\ & (u \in V(G)\ \&\ v \in V(H)) \\ 2 & \text{otherwise} \end{cases}$

(f) $d_{G[H]}((a,b),(c,d)) = \begin{cases} d_G(a,c) & a \neq c \\ 0 & a = c, b = d \\ 1 & a = c, bd \in E(H) \\ 2 & a = c, bd \notin E(H \end{cases}$,

(g) $d_{G\vee H}((a,b),(c,d)) = \begin{cases} 0 & a = c, b = d \\ 1 & ac \in E(G) \text{ or } bd \in E(H), \\ 2 & \text{otherwise} \end{cases}$

(h) $d_{G\oplus H}((a,b),(c,d)) = \begin{cases} 0 & a = c, b = d \\ 1 & ac \in E(G) \text{ or } bd \in E(H) \text{ but not both}, \\ 2 & \text{otherwise} \end{cases}$

(i) $deg_{G\times H}(a,b) = deg_G(a) + deg_H(b)$,

(j) $deg_{G[H]}(a,b) = |V(H)|deg_G(a) + deg_H(b)$,

(k) $deg_{G+H}(a) = \begin{cases} deg_G(a) + |V(H)| & a \in V(G) \\ deg_H(a) + |V(G)| & a \in V(H) \end{cases}$,

(l) $deg_{G \vee H}((a,b)) = |V(H)| deg_G(a) + |V(G)| deg_H(b) - deg_G(a) deg_H(b)$,

(m) $deg_{G \oplus H}((a,b)) = |V(H)| deg_G(a) + |V(G)| deg_H(b) - 2 deg_G(a) deg_H(b)$.

Proof. The parts (a-e) are consequences of definition and some well-known results of the book of Imrich and Klavzar [16]. For the proof of (f−m) we refer reader to [21]. ■

Throughout this paper, our notation is standard and taken mainly from the standard book of graph theory. The notations K_n, S_n and P_n stand for the complete, star and path on n vertices.

2. Preliminary Results

Hosoya [14] introduced a distance–based graph polynomial as H(G,x) = $\sum_{k \geq 0} d(G,k) x^k$ called Wiener polynomial where d(G,k)=|{(u,v): d(u,v)=k}|, but recently most of authors prefer the name "Hosoya polynomial". An equivalent from of this polynomial is H(G,x) = $\sum_{\{a,b\} \subseteq V(G)} x^{d(a,b)}$. Since the Hosoya polynomial at x = 1 is equal to the Wiener index, one can consider it as a polynomial version of the Wiener index. In a similar way as Hosoya polynomial, one can define the edge Hosoya polynomial as $H_e(G,x) = \sum_{\{f,g\} \subseteq E(G)} x^{d_e(f,g)}$.

In [12], Ivan Gutman introduced a polynomial version of the Schultz and modified Schultz indices of graphs as follows:

$$H_1(G, x) = \sum_{\{u,v\} \subseteq V(G)} (\deg(u) + \deg(V)) x^{d(u,v)},$$

$$H_2(G, x) = \sum_{\{u,v\} \subseteq V(G)} (\deg(u) \deg(V)) x^{d(u,v)}.$$

He proved that if G is a n-vertex tree then the polynomials $H_1(G,x)$ and H(G,x) are related as $H_1(G,x) = 2(1 + 1/x)H(G,x) - 2(1 + n/x)$. Recently, most of authors prefer to change the name "modified Schultz index" by "Gutman index". For this reason, we name H_2, the Gutman polynomial. These polynomials have the following mathematical properties:

Lemma 2. Let G be a graph. Then the following statements are hold:

a) $H_1(G,1) = 2(|V(G)| - 1)\,|E(G)|$,

b) $H_1'(G,1) = W_+(G)$, $H_2'(G,1) = W_*(G)$,

c) If G is r-regular then $H_1(G,x) = 2rH(G,x)$,

d) $H_1'(G,0) = M_1(G)$,

e) $H_2'(G,0) = M_2(G)$.

Proof. The proof is straightforward and so omitted. ■

Lemma 3. The following statements are hold:

a) $H_1(K_n,x) = n(n-1)^2x$

b) $H_1(K_{m,n},x) = mn(m+n)x + (2n\binom{m}{2} + 2m\binom{n}{2})x^2$

c) $H_1(W_n,x) = (n-1)[(n+5)x + 2(n-4)x^2]$

d) $H_1(P_n,x) = \sum_{i=1}^{n-1}[4(n-i)-2]x^i$

e) $H_1(C_{2n},x) = 8n(\frac{x^n-1}{x-1}) + 4nx^n$

f) $H_1(C_{2n+1},x) = (8n+4)\frac{x^{n+1}-1}{x-1}$

g) $H_1(Q_n,x) = 2n((1+x)^n - 1)$.

In the end of this section, relationship between coefficients of H(G,x) and $H_1(G,x)$ is presented.

Lemma 4. Suppose a_i is the coefficient of x^i in H(G,x) and b_i is the coefficient of x^i in $H_1(G,x)$. Then we have:

1. $a_1 = |E(G)|$ and $b_1 = M_1(G)$,
2. If the girth of G is greater than 4, then $a_2 = \sum_{i=1}^{p}\binom{d_i}{2}$, where d_i's are degree of vertices and p is the number of vertices.

Proof. The part (1) is trivial. To prove part (2), we assume that u and v are two vertices with d(u,v) = 2. But the number of paths of length two is equal to $\sum_{i=1}^{p}\binom{d_i}{2}$. and if G is triangular and rectangular free, then the number of pairs which distance is 2 is equal the number of path of length 2, proving the lemma. ■

3. Main Results

In this section, the main properties of edge Hosoya polynomial are achieved.

Theorem 1. Let G be a connected graph. Then $H_e(G,x)=\frac{1}{x}H(L(G),x)$.

Proof. If the edge e_i and e_j have common vertices then $d(e_i,e_j) = 0$ and $d_{L(G)}(e_i,e_j) = 1$. Clearly, $d_{L(G)}(e_i,e_j) = d(e_i,e_j) + 1$. Therefore, $H(L(G),x) = xH_e(G,x)$ and this complete the proof. ■

By derivation of this polynomial at x=1, we have $W_e(G) = W(L(G)) - \binom{|E(G)|}{2}$.

Theorem 2. Let T be a tree. Then the Schultz, Hosoya, Gutman and edge Wiener polynomials are related as follows:

$$H_1(T,x) - H_2(T,x) = H(T,x) - H_e(T,x)$$

Proof. Let $A_{(u,v)} = \{uw \mid d(v,w) = d(u,v) + 1\} \times \{vw' \mid d(u,w') = d(u,v) + 1\}$. Then we have $E(T) \times E(T) = \cup A_{(u,v)}$ and $|A_{(u,v)}| = (\deg(u) - 1)(\deg(v) - 1)$ and for every $(ab,cd) \in A_{(a,c)}$, $d(ab,cd) = d(a,c)$. So the sets $A_{(u,v)}$, constitute a partition of $E(T) \times E(T)$. Therefore,

$$H_e(T,x)=\frac{1}{2}\sum_{(e,f)\in E(T)\times E(T)} x^{d(e,f)} = \frac{1}{2}\sum_{(u,v)\in V(T)\times V(T)}\sum_{(e,f)\in A_{(u,v)}} x^{d(e,f)}$$

$$= \frac{1}{2}\sum_{(u,v)\in V(T)\times V(T)} |A(u,v)| x^{d(u,v)}$$

$$= \frac{1}{2}\sum_{(u,v)\in V(T)\times V(T)} (\deg(u)-1)(\deg(v)-1)x^{d(u,v)}$$

$$= \frac{1}{2}\Big(\sum_{(u,v)\in V(T)\times V(T)} \deg(u)\deg(v)x^{d(u,v)}$$

$$-\sum_{(u,v)\in V(T)\times V(T)}(\deg(u)+\deg(v))x^{d(u,v)}+\sum_{(u,v)\in V(T)\times V(T)} x^{d(u,v)}\Big)$$

$$= H_2(T,x) - H_1(T,x) + H(T,x).$$

This completes the proof. ■

Theorem 3. Suppose G_1 and G_2 are connected graphs, $V_1 = V(G_1)$, $V_2 = V(G_2)$, $E_1 = E(G_1)$, $E_2 = E(G_2)$ and $V = V_1 \times V_2$. Then we have:

a) $H_1(G_1 \times G_2,x) = 2H(G_1,x)H_1(G_2,x) + 2H(G_2,x)H_1(G_1,x) + |V_1|H_1(G_2,x) + |V_2|H_1(G_1,x) + 4|E_1|H(G_2,x) + 4|E_2|H(G_1,x)$

b) $H_1(G_1 + G_2,x) = (M_1(G) + M_2(G) + 4|E_1||V_2| + 4|E_2||V_1| + |V_1||V_2|(|V_1| + |V_2|)x + (|V_1||V_2|(|V_1| + |V_2|) - 2|E(G_1+G_2)| - M_1(G_1) – M_1(G_2) + 2(|E_1| - |E_2|)(|V_1| - |V_2|)) x^2$

c) $H_1(G_1[G_2],x) = |V_2|^3H_1(G_1,x) + 4|V_2|\,|E_2|H(G_1,x) + 2(|V_2| - 1)(|V_2|^2|E_1| + |V_1||E_2|)x^2 + (4|V_2||E_1||E_2| + |V_1|M_1(G_2))(x - x^2)$

d) $H_1(G_1 \vee G_2,x)=(2|E|(|V|-1)-M_1(G))x^2+M_1(G)x$

e) $H_1(G_1 \oplus G_2,x) = (2|E|(|V| - 1) - M_1(G))x^2 + M_1(G)x$.

Proof. In each part, it is assumed that G is the graph obtained from G_1, G_2 and the corresponding graph operations. The part (a) proved in [6]. To prove (b), we apply Lemma 1. The distance between two vertices of G is 1 or 2. Therefore, it is enough to compute the coefficient of x and x^2 in the Schultz polynomial.

i) If $d(u,v) = 1$ then u and v are adjacent in G_1 or G_2 or $u \in E_1$ and $v \in E_2$ or $v \in E_1$ and $u \in E_2$. By considering all cases, we have:

$$Coefficient\ of\ x = \sum_{u\in v_1}\sum_{v\in v_2}(deg_{G_1}(u) + deg_{G_2}(v) + |V_1| + |V_2|)$$

$$+ \sum_{uv\in E(G_1)}(deg_{G_1}(u) + deg_{G_1}(v) + 2|V_2|)$$

$$+ \sum_{uv\in E(G_2)}(deg_{G_2}(u) + deg_{G_2}(v) + 2|V_1|)$$

$$= M_1(G_1)+M_1(G_2)+4|E_1||V_2|+4|E_2||V_1|+|V_1||V_2|(|V_1| + |V_2|)$$

ii) If $d(u,v) = 2$ then both of u and v are in G_1 or G_2. Therefore,

Coefficient of x^2*(when* $u, v \in G_1$*)* $= \frac{1}{2}\sum_{u\in V_1}\sum_{v\in V_1}\left(deg_{G_1}(u) + deg_{G_1}(v) + 2|V_2|\right) - \frac{1}{2}\sum_{v\in V_1}(2deg_{G_1}(v) + 2|V_2|) - (M_1(G_1) + 2|V_2||E_1|) = 2|E_1||V_1| + |V_2||V_1|^2-2|E_1|-|V_2||V_1| - M_1(G_1) - 2|V_2||E_1|$.

Coefficient of x^2*(when* $u, v \in G_2$*)* $= 2|E_2||V_2| + |V_1||V_2|^2 - 2|E_2| - |V_1||V_2| - M_1(G_2) - 2|V_1||E_2|$.

Thus by summation, we have:

Coefficient of x^2 = $|V_1||V_2|(|V_1| + |V_2|) - 2|E(G_1 + G_2)| - M_1(G_1) – M_1(G_2) + 2(|E_1| - |E_2|)(|V_1| - |V_2|)$, which proves the theorem. ■

For part (c), we assume that $u = (u_1,v_1)$ and $v = (u_2,v_2)$. Then for computing Schultz polynomial of $G_1[G_2]$, two cases are considered.

i) Suppose $u_1 \neq u_2$ and $A = \sum_{v_1,v_2} \sum_{u_1 \neq u_2}[(|V_2|\ \text{deg}u_1 + \text{deg}v_1) + (|V_2|\text{deg}u_2 + \text{deg}v_2)]x^{d(u_1,u_2)}$. Then

$$A = \sum_{v_1,v_2} \sum_{u_1 \neq u_2}[(|V_2|(\text{deg}(u_1) + \text{deg}(u_2))\ x^{d(u_1,u_2)} + (\text{deg}(v_1)+ \text{deg}(v_2))\ x^{d(u_1,u_2)}$$

$$= \sum_{v_1,v_2} |V_2|H_1(G_1,x) + (\text{deg}v_1+\text{deg}v_2)H(G_1,x)$$

$$= |V_2|^3H_1(G_1,x) + 4|V_2|\ |E_2|H(G_1,x)\ .$$

ii) Suppose $u_1 = u_2$ and $B = \sum_{u_1} \sum_{v_1v_2 \in E_2}(2|V_2|deg(u_1) + deg(v_1) + deg(v_2)x + \sum_{u_1} \sum_{v_1v_2 \notin E_2}(2|V_2|\text{deg}\ (u_1) + deg(v_1) + \text{deg}\ (v_2)x^2$. Then

$$B = (4|V_2||E_1||E_2| + |V_1|M_1(G_2))x + 2(|V_2| – 1)(|V_2|^2|E_1| + |V_1||E_2|)x^2$$

$$– (4|V_2||E_1||E_2| + |V_1|M_1(G_2))\ x^2 .$$

It is easy to see that, the Schultz polynomial of G is equal to A + B, which is obtained from (i) and (ii).

To prove part (d), apply again Lemma 1. Suppose u and v are vertices of $G_1 \vee G_2$ then $d(u,v) = 2$, if $uv \notin E$. So,

$$H_1(G_1 \vee G_2,x) = \sum_{\{u,v\} \subseteq V(G)}(\text{deg}(u) + \text{deg}\ (V))x^{d(u,v)}$$

$$= \sum_{\{u,v\} \subseteq V(G1 \vee G2)}(\text{deg}(u) + \text{deg}\ (v))x^2 – \sum_{uv \subseteq E(G1 \vee G2)}(\text{deg}(u) + \text{deg}\ (v))x^2$$

$$+ \sum_{uv \subseteq E(G1 \vee G2)}(\text{deg}(u) + \text{deg}\ (v))x^1$$

$$= 2|E|(|V| – 1)x^2 – M_1(G)x^2 + M_1(G)x\ .$$

The proof for part (e) is similar to (d) and is omitted. ■

The Gutman polynomials of some graph operation are computed in [6].

Corollary 1. The Schultz index of some graph operation is computed as follows:

a) $W_+(G_1\times G_2) = |V_1|^2{}_+(G_2) + |V_2|^2W_+(G_1) + 4|E_1||V_1|W(G_2) + 4|E_2||V_2|W(G_1)$
b) $W_+(G_1+G_2) = |V|(|V| - 1)^2 - |V_1|(|V_1| - 1)^2 - |V_2|(|V_2| - 1)^2 + 4|E_1|(|V_1| - 1) + 4|E_2|(|V_2| - 1) - M_1(G_1) - M_1(G_2)$
c) $W_+(G_1[G_2]) = |V_2|W_+(G_1) + 4|V_2||E_2|W(G) - |V_1|M_1(G_2) + 4(|V_2| - 1)|V_2|^2|E_2| + |V_1||E_2| - 4|V_2||E_2||E_1|$
d) $W_+(G_1 \vee G_2) = 4|E(G_1 \vee G_2) |(|V(G_1 \vee G_2)| - 1) - M_1(G_1 \vee G_2)$.

Proof. This follows from Theorem 3 and Lemma 2.

Corollary2. The first Zagreb index of three graph operations is computed as follow:

a) $M_1(G_1\times G_2) = |V_1|M_1(G_2) + |V_2|M_1(G_1) + 8|E_1||E_2|$
b) $M_1(G_1{}_+G_2) = M_1(G_1) + M_1(G_2) + |V_1||V_2|^2 + |V_2||V_1|^2 + 4|E_1||V_2| + 4|E_2||V_1|$
c) $M_1(G_1[G_2]) = |V_2|^3M_1(G_1) + |V_1|M_1(G_2) + 8|V_2||E_2||E_1|$.

Proof. This is follows from Theorem 3 and Lemma 2(d).

4. A New Two Variables Polynomial

In this section, a two variable polynomial is introduced by which it is possible to compute the Wiener, hyper Wiener, Schultz and first Zagreb index of graphs. An exact formula for this polynomial under Cartesian product of graphs is obtained. This polynomial is defined as:

$$W(G, x, y)= \sum_{\{u,v\}\subseteq V(G)} x^{\deg(u)+\deg(v)}y^{d(u,v)}.$$

We obtain the following result immediately from its definition:

1. $\frac{\partial W(G,1,y)}{\partial y}|_{y=1}= w(G)$
2. $\frac{\partial\partial W(G,x,y)}{\partial y\partial x}|_{y=1,x=1}=W_+(G)$
3. $\frac{\partial\partial W(G,x,y)}{\partial y\partial x}|_{y=0,x=1}=M_1(G)$

4. $\frac{1}{2}(\frac{\partial\partial W(G,1,y)}{\partial y^2}|_{y=1})+\frac{\partial W(G,1,y)}{\partial y}|_{y=1}$=ww(G) , where ww(G) is the hyper wiener index of G.
5. If G is regular graph then we have: $W(G,x,y)=x^{2r}H(G,y)$.

Theorem 4. Suppose G_1 and G_2 are graphs. Then,

$$W(G_1\times G_2,x,y) = \sum_{u\in V(G_1)} x^{2\deg(u)} W(G_2,x,y) + \sum_{u\in V(G_2)} x^{2\deg(u)} W(G_1,x,y) + 2W(G_1,x,y)W(G_2,x,y)$$

Proof. By definition,

$$W(G_1\times G_2,x,y)=\sum_{\{u,v\}\subseteq V(G1\times G2)} x^{\deg(u)+\deg(v)} y^{d(u,v)}$$

Suppose $u = (u_1,v_1)$ and $v = (u_2,v_2)$, where $u_1,u_2\in V(G_1)$ and $v_1,v_2\in V(G_2)$ .Then we have:

$$W(G_1\times G_2,x,y)=\sum_{(u,v)\in V(G_1\times G_2)} x^{\deg(u_1)+\deg(u_2)+\deg(v_1)+\deg(v_2)} y^{d(u_1,u_2)+d(v_1,v_2)}$$

$$=\sum_{u_1=u_2} x^{2\deg(u_1)} \sum_{\{v_1,v_2\}\subseteq V(G_2)} x^{\deg(v_1)+\deg(v_2)} y^{d(v_1,v_2)}$$

$$+\sum_{v_1=v_2} x^{2\deg(v_1)} \sum_{\{u_1,u_2\}\subseteq V(G_1)} x^{\deg(u_1)+\deg(u_2)} y^{d(u_1,u_2)}$$

$$+2\sum_{u_1\neq u_2} x^{\deg(u_1)+\deg(u_2)} y^{d(u_1,u_2)} \sum_{v_1\neq v_2} x^{\deg(v_1)+\deg(v_2)} y^{d(v_1,v_2)}$$

$$=\sum_{u\in V(G_1)} x^{2\deg(u)} W(G_2,x,y) + \sum_{u\in V(G_2)} x^{2\deg(u)} W(G_1,x,y)$$

$$+2W(G_1,x,y)W(G_2,x,y)$$

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