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Some Bounds on GA_1 Index of Graphs

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Abstract

Let G= (V, E) be a simple graph without isolated vertices. The GA_1 index of G is defined as the summation of $2\sqrt{d(u)d(v)}/(d(u)+d(v))$, where for every vertex x, d(x) denotes the degree of vertex **x.** In this note, some properties of GA_1 index are presented.

1. Introduction

Let G = (V, E) be a simple graph where V(G) and E(G) are the vertex and edge sets of the graph G, respectively. A topological index of a graph is a number invariant under its automorphisms. The simplest topological indices are the number of vertices and edges of the graph. The Wiener index (W) is one of the oldest topological indices introduced by Harold Wiener [1].

The Cartesian product $G_1 \times G_2$ of graphs G_1 and G_2 is a graph such that $V(G_1 \times G_2) = V(G_1) \times V(G_2)$, and any two vertices (a, b) and (u, v) are adjacent in $G_1 \times G_2$ if and only if either a = u and *b* is adjacent with *v*, or b = v and *a* is adjacent with u. If e = uv is an edge of G then GA_1 index [2] of *G* defined as the summation of $2\sqrt{d(u)d(v)}/(d(u)+d(v))$, where for every vertex x, d(x) denotes the degree of vertex x [2]. In this paper, we present some new bounds for GA_1 index. Throughout this paper, our notation is standard and taken from the standard graph theory books and [3-13]. In particular, papers [6,8,10] are concerned with geometric-arithmetic indices. For review on *GA* indices see [14].

2. Bounds on the GA_1 Index of Graphs

By the well-known relation of arithmetic-geometric mean, we can easily seen that $GA_1(G) \le m$, with equality if and only if G is a union of regular graphs.

Lemma 1. Let G be a simple graph. Then $GA_1(G) \le ½ Zg_1$ with equality if and only if G is a union of K_2.

Proof. For any positive real numbers a and b, $\frac{2\sqrt{ab}}{a+b} \le \frac{2ab}{a+b} \le \sqrt{ab} \le \frac{a+b}{2}$. Thus $\frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)} \le \frac{d(u)+d(v)}{2}$ and so $GA_1(G) \le \frac{1}{2} Zg_1$. The equality holds if and only if $d(u) = d(v) = 1$; for any edge $e = uv$. This condition satisfies if and only if G is a union of K_2, proving the lemma.

Lemma 2. If G is a simple graph without isolated vertices then $GA_1(G) \le \sqrt{mZg_2}$ with equality if and only if G is a union of K_2.

Proof. Since G is a graph without isolated vertex,

$$GA_1(G) = \sum_{uv\in E} \frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)} \le \sum_{uv\in E} \frac{2\sqrt{d(u)d(v)}}{1+1} \le \sum_{uv\in E} \sqrt{d(u)d(v)}.$$

For any edge e = uv, we define $a_{uv} = 1$ and $b_{uv} = \sqrt{d(u)d(v)}$. Apply Cauchy-Schwarz inequality to conclude that $\sum_{uv\in E} \sqrt{d(u)d(v)} \le \sqrt{\Sigma_{uv} 1 \Sigma_{uv} d(u)d(v)} = \sqrt{mZg_2}$ and equality holds if and only if $b_{uv} = \sqrt{d(u)d(v)} = 1$. Thus G is a union of K_2, proving the lemma.

Proposition 1. Suppose that G is a graph without isolated vertex. Then

a) $GA_1(G) \le \sqrt{Zg_2 + m(m-1)}$ with equality if and only if G is a union of K_2.

b) $GA_1(G) \le \left\lceil \frac{m-1}{2} \right\rceil + \sqrt{\left\lceil \frac{m-1}{2} \right\rceil^2 + Zg_2}$ with equality if and only if G is a union of an odd number of K_2.

Proof. a) We can see that,

$$[GA_1(G)]^2 = \sum_{uv\in E}\frac{4d(u)d(v)}{[d(u)+d(v)]^2} + 2\sum_{uv\neq xy}\frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)}\cdot\frac{2\sqrt{d(x)d(y)}}{d(x)+d(y)}$$

$$\leq \sum_{uv\in E}\frac{4d(u)d(v)}{(1+1)^2} + 2\sum_{uv\neq xy}(1)(1) = \sum_{uv\in E} d(u)d(v) + 2\frac{m(m-1)}{2}$$

$$= Zg_2 + m(m-1)\,.$$

Therefore, $GA_1(G) \leq \sqrt{Zg_2 + m(m-1)}$.

b) $$[GA_1(G)]^2 = \sum_{uv\in E}\frac{4d(u)d(v)}{[d(u)+d(v)]^2} + 2\sum_{uv\neq xy}\frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)}\cdot\frac{2\sqrt{d(x)d(y)}}{d(x)+d(y)}$$

$$\leq \sum_{uv\in E} d(u)d(v) + 2\left\lceil\frac{m-1}{2}\right\rceil\cdot GA_1(G)$$

$$\leq Zg_2 + 2\left\lceil\frac{m-1}{2}\right\rceil\cdot GA_1(G)$$

Therefore, $[GA_1(G) - \left\lceil\frac{m-1}{2}\right\rceil]^2 \leq \left\lceil\frac{m-1}{2}\right\rceil^2 + Zg_2$ and equality holds if and only if G is a union of an odd number of K_2.

Proposition 2. Suppose that G is a simple graph with no isolated vertex, then $GA_1(G) \geq \frac{2m}{n}$.

Proof. It is clear that $\frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)} \geq \frac{2\sqrt{d(u).1}}{d(u)+(n-1)}$. Let $f(x) = \frac{2\sqrt{x}}{x+(n-1)}$; for $x \geq 1$, then $f'(x) = \frac{\frac{1}{\sqrt{x}}(x+n-1) - 2\sqrt{x}}{x+n-1} = \frac{-x+n-1}{\sqrt{x}(x+n-1)}$. If $x\in[1, n-1]$ then $f'(x) \geq 0$ and so $\frac{2\sqrt{d(u)}}{d(u)+n-1} \geq \frac{2\sqrt{1}}{1+n-1} = \frac{2}{n}$. Hence $GA_1(G) \geq \frac{2m}{n}$.

Proposition 3. If G is a simple graph with no isolated vertices then $GA_1(G) \geq \sqrt{\frac{Zg_2}{(n-1)^2} + \frac{4m(m-1)}{n^2}}$.

Proof. By Proposition 1,

$$[GA_1(G)]^2 = \sum_{uv\in E}\frac{4d(u)d(v)}{[d(u)+d(v)]^2} + 2\sum_{uv\neq xy}\frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)}\cdot\frac{2\sqrt{d(x)d(y)}}{d(x)+d(y)}$$

$$\geq \sum_{uv\in E}\frac{4d(u)d(v)}{[(n-1)+(n-1)]^2} + 2\sum_{uv\neq xy}\frac{2}{n}\cdot\frac{2}{n}$$

$$= \sum_{uv\in E}\frac{d(u)d(v)}{(n-1)^2} + 2\left(\frac{4}{n^2}\right)\left(\frac{m(m-1)}{2}\right)$$

$$= \frac{Zg_2}{(n-1)^2} + \frac{4m(m-1)}{n^2}.$$

Therefore $GA_1(G) \geq \sqrt{\dfrac{Zg_2}{(n-1)^2} + \dfrac{4m(m-1)}{n^2}}$.

From our calculations given above, it is possible to compute the GA_1 index of some nanostructures covered by C_4. Notice that if G is a regular graph then $\dfrac{2\sqrt{d(u)d(v)}}{d(u)+d(v)} = 1$. Thus $GA_1(G)=|E(G)|$.

I) If $G=P_n\times P_m$, then

$$GA_1(P_n\times P_m)= \sum_{d(u)=d(v)=4}\frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)} + \sum_{d(u)=3,d(v)=4}\frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)}$$

$$+ \sum_{d(u)=d(v)=3}\frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)} + \sum_{d(u)=2,d(v)=3}\frac{2\sqrt{d(u)d(v)}}{d(u)d(v)}$$

$$= \sum_{d(u)=d(v)=4}\frac{2\sqrt{4\cdot 4}}{4+4} + \sum_{d(u)=3,d(v)=4}\frac{2\sqrt{3\cdot 4}}{3+4} + \sum_{d(u)=d(v)=3}\frac{2\sqrt{3\cdot 3}}{3+3}$$

$$+ \sum_{d(u)=2,d(v)=3}\frac{2\sqrt{2\cdot 3}}{2+3}$$

$$= |E(P_{n-2}\times P_{m-2})|(1) + (2(n-2)+2(m-2))\,\frac{2\sqrt{12}}{7} + (2(n-3)+2(m-3))(1)$$

$$+ \frac{8.2\sqrt{6}}{5} = 2mn - 3(m+n) + 8(m+n-4)\frac{\sqrt{3}}{7} + \frac{16\sqrt{6}}{5}.$$

II) If $G = C_m \times P_n$ then $GA(G)= \sum_{uv\in A} \frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)} + \sum_{uv\in B} \frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)} = (2mn-2m) + \frac{2m\cdot 2\sqrt{3\cdot 4}}{3+4} = 2m(n-1) + \frac{8m\sqrt{3}}{7}$, where $A=\{uv \mid d(u) = d(v) = 3 \text{ or } d(u) = d(v) = 4\}$ and $B = \{uv \mid d(u) = 3 \text{ and } d(v) = 4\}$.

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