Extending a Theorem by Fiedler and Applications to Graph Energy

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Abstract

We use a lemma due to Fiedler to obtain eigenspaces of some graphs and apply these results to graph energy (= the sum of absolute values of the graph eigenvalues = the sum of singular values of the adjacency matrix). We obtain some new upper and lower bounds for graph energy and find new examples of graphs whose energy exceeds the number of vertices.

1 INTRODUCTION

Let A be an $n \times n$ matrix. The scalars $\lambda$ and vectors $v \neq 0$ satisfying $Av = \lambda v$ we call eigenvalues and eigenvectors of A, respectively, and any such pair $(\lambda, v)$ is called an eigenpair for A.

The set of distinct eigenvalues (including multiplicities), denoted by $\sigma(A)$, is called the spectrum of A. The eigenvectors of the adjacency matrix of a graph $G$, together with the eigenvalues, provide a useful tool in the investigation of the structure of the
graph. In this work, having a result presented in [1] as motivation, we obtain, for graphs with a special type of adjacency matrix, a lower bound for the energy of these graphs. In 1974 Fiedler obtained the following result [1].

Let \( A, B \) be \( m \times m \) and \( n \times n \) symmetric matrices with corresponding eigenpairs \( (\alpha_i, u_i), \ i = 1, \ldots, m, (\beta_{\ell}, v_{\ell}), \ \ell = 1, \ldots, n \), respectively.

**Lemma 1.1.** [1] Let \( A, B \) be \( m \times m \) and \( n \times n \) symmetric matrices with corresponding eigenpairs \( (\alpha_i, u_i), \ i = 1, \ldots, m, (\beta_{\ell}, v_{\ell}), \ i = 1, \ldots, n \), respectively. Suppose that \( \|u_1\| = 1 = \|v_1\| \). Then, for any \( \rho \), the matrix

\[
C = \begin{pmatrix}
A & \rho u_1 v_1^T \\
\rho v_1 u_1^T & B
\end{pmatrix}
\]

has eigenvalues \( \alpha_2, \ldots, \alpha_n, \beta_2, \ldots, \beta_m, \gamma_1, \gamma_2 \), where \( \gamma_1, \gamma_2 \) are eigenvalues of

\[
\hat{C} = \begin{pmatrix}
\alpha_1 & \rho \\
\rho & \beta_1
\end{pmatrix}.
\]

We now offer a generalization of Fiedler’s lemma.

Suppose now that \( u_1, \ldots, u_m \) (resp. \( v_1, \ldots, v_n \)) constitute an orthonormal system of eigenvectors of \( A \) (resp. \( B \)). Let \( (u_1|, \ldots, |u_m) \) and \( (v_1|, \ldots, |v_n) \) be the matrices whose columns consist of the ordinates of the eigenvectors \( u_i \) and \( v_i \) associated to the eigenvalues \( \alpha_i \) and \( \beta_i \), respectively.

In fact,

\[
(u_1|, \ldots, |u_m) = \begin{pmatrix}
\begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\end{pmatrix}
\begin{pmatrix}
u_{11} & u_{12} & u_{1m} \\
u_{21} & u_{22} & u_{2m} \\
\vdots & \vdots & \vdots \\
u_{m1} & u_{m2} & u_{mm}
\end{pmatrix}
\]

where \( u_p = \begin{pmatrix}
u_{1p} \\
\vdots \\
u_{mp}
\end{pmatrix} \) for \( p = 1, \ldots, m \), and

\[
(v_1|, \ldots, |v_n) = \begin{pmatrix}
\begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\end{pmatrix}
\begin{pmatrix}
v_{11} & v_{12} & v_{1n} \\
v_{21} & v_{22} & v_{2n} \\
\vdots & \vdots & \vdots \\
v_{n1} & v_{n2} & v_{nn}
\end{pmatrix}
\]

where \( v_p = \begin{pmatrix}
v_{1p} \\
\vdots \\
v_{np}
\end{pmatrix} \) for \( p = 1, \ldots, n \).
Lemma 1.2. Let $k \leq \min\{m, n\}$ and $U = (u_1, \ldots, u_k)$, $V = (v_1, \ldots, v_k)$. Then, for any $\rho$, the matrix
\[
C = \begin{pmatrix}
A & \rho UV^T \\
\rho VU^T & B
\end{pmatrix}
\]
has eigenvalues $\alpha_{k+1}, \ldots, \alpha_m, \beta_{k+1}, \ldots, \beta_n; \gamma_{1j}, \gamma_{2j}, \ j = 1, 2, \ldots, k,$ where for $s = 1, 2$, $\gamma_{sj}$ is an eigenvalue of
\[
\hat{C}_j = \begin{pmatrix}
\alpha_j & \rho \\
\rho & \beta_j
\end{pmatrix}.
\]

Proof. For $j = 1, 2, \ldots, k$, let $(\gamma_{sj}, \hat{w}_{sj})$, $s = 1, 2$, be an eigenpair of the matrix
\[
\hat{C}_j = \begin{pmatrix}
\alpha_j & \rho \\
\rho & \beta_j
\end{pmatrix}
\]
where $\hat{w}_{sj} = (w_{1sj}, w_{2sj})^T$. Then, for $j = 1, 2, \ldots, k$, $s = 1, 2$, we have,
\[
\begin{pmatrix}
\alpha_j & \rho \\
\rho & \beta_j
\end{pmatrix} \begin{pmatrix} w_{1sj} \\ w_{2sj} \end{pmatrix} = \gamma_{sj} \begin{pmatrix} w_{1sj} \\ w_{2sj} \end{pmatrix}.
\]

Let $\begin{pmatrix} w_{1sj} u_j \\ w_{2sj} v_j \end{pmatrix}$ be an $(m + n)$-vector. Then,
\[
\begin{pmatrix}
A & \rho UV^T \\
\rho VU^T & B
\end{pmatrix} \begin{pmatrix} w_{1sj} u_j \\ w_{2sj} v_j \end{pmatrix} = \begin{pmatrix} w_{1sj} A u_j + \rho w_{2sj} U^T v_j \\ \rho w_{1sj} V^T u_j + w_{2sj} B v_j \end{pmatrix}
\]
as
\[
w_{1sj} A u_j + \rho w_{2sj} U^T v_j = w_{1sj} \alpha_j \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} + \rho w_{2sj} U \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix}
\]
and
\[
\rho w_{1sj} V^T u_j + w_{2sj} B v_j = \rho w_{1sj} V \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ v_n \end{pmatrix} + w_{2sj} \beta_j \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}.
\]

Therefore,
\[
\begin{pmatrix}
A & \rho UV^T \\
\rho VU^T & B
\end{pmatrix} \begin{pmatrix} w_{1sj} u_j \\ w_{2sj} v_j \end{pmatrix} = \begin{pmatrix} (w_{1sj} A u_j + \rho w_{2sj} U^T v_j) \\ (\rho w_{1sj} V^T u_j + w_{2sj} B v_j) \end{pmatrix}
\]
\[
= \begin{pmatrix} (w_{1sj} \alpha_j + \rho w_{2sj}) u_j \\ (\rho w_{1sj} + w_{2sj} \beta_j) v_j \end{pmatrix} = \begin{pmatrix} \gamma_{sj} w_{1sj} u_j \\ \gamma_{sj} w_{2sj} v_j \end{pmatrix} = \gamma_{sj} \begin{pmatrix} w_{1sj} u_j \\ w_{2sj} v_j \end{pmatrix}.
\]
Recall that \((\gamma_{sj}, \bar{w}_{sj})\), \(s = 1, 2\) is an eigenpair of \(\tilde{C}_j = \begin{pmatrix} \alpha_j & \rho \\ \rho & \beta_j \end{pmatrix}\), where \(\bar{w}_{sj} = (w_{1sj}, w_{2sj})^T\). Then, for \(s = 1, 2\), we have,
\[
\begin{pmatrix} \alpha_j & \rho \\ \rho & \beta_j \end{pmatrix} \begin{pmatrix} w_{1sj} \\ w_{2sj} \end{pmatrix} = \gamma_{sj} \begin{pmatrix} w_{1sj} \\ w_{2sj} \end{pmatrix} .
\]
Therefore, for \(j = 1, 2, \ldots, k, s = 1, 2\), \((\gamma_{sj}, \begin{pmatrix} w_{1sj} u_j \\ w_{2sj} v_j \end{pmatrix})\) are \(2k\) eigenpairs for \(C\). For \(i = k + 1, \ldots, m\), we have
\[
\begin{pmatrix} A & \rho U V^T \\ \rho V U^T & B \end{pmatrix} \begin{pmatrix} u_i \\ 0 \end{pmatrix} = \begin{pmatrix} A u_i \\ 0 \end{pmatrix} = \alpha_i \begin{pmatrix} u_i \\ 0 \end{pmatrix}
\]
and for \(t = k + 1, \ldots, n\), we set
\[
\begin{pmatrix} A & \rho U V^T \\ \rho V U^T & B \end{pmatrix} \begin{pmatrix} 0 \\ v_t \end{pmatrix} = \begin{pmatrix} 0 \\ B v_t \end{pmatrix} = \beta_t \begin{pmatrix} 0 \\ v_t \end{pmatrix} .
\]
Therefore, \((\alpha_i, \begin{pmatrix} u_i \\ 0 \end{pmatrix})\) for \(i = k + 1, \ldots, m\), and \((\beta_t, \begin{pmatrix} 0 \\ v_t \end{pmatrix})\) for \(t = k + 1, \ldots, n\), are eigenpairs for \(C\). Thus the result is proved. \(\square\)

2 APPLICATIONS

A simple graph \(G\) is a pair of sets \((V, E)\), such that \(V\) is a nonempty finite set of \(n\) vertices and \(E\) is the set of \(m\) edges. We say that \(G\) is a simple \((n, m)\)-graph. Let \(A(G)\) be the adjacency matrix of the graph \(G\). Its eigenvalues \(\lambda_1, \ldots, \lambda_n\) form the spectrum of \(G\) (cf. [2]).

The notion of energy of an \((n, m)\)-graph \(G\) (written \(E(G)\)) is a nowadays much studied spectral invariant, see the reviews [3, 4] and the recent works [5–9]. This concept is of great interest in a vast range of fields, especially in chemistry since it can be used to approximate the total \(\pi\)-electron energy of a molecule. It is defined as [10]
\[
E(G) = \sum_{j=1}^{n} |\lambda_j| .
\]

Given a complex \(m \times n\) matrix \(C\), we index its singular values by \(s_1(C), s_2(C), \ldots\). The value
\[
E(C) = \sum_{j} s_j(C)
\]
is the energy of $C$ (cf. [11]), thereby extending the concept of graph energy. Consequently, if the matrix $C \in \mathbb{R}^{n \times n}$ is symmetric with eigenvalues $\beta_1(C), \ldots, \beta_n(C)$, then its energy is given by

$$E(C) = \sum_{i=1}^{n} |\beta_i(C)| .$$

In this section, using Lemma 1.2, we present an application of the concept of energy to some special kinds of graphs.

We first formulate an auxiliary result.

Let $Q = (q_{ij})$ be an $n_1 \times n_2$ matrix with real-valued elements. Without loss of generality we may assume that $n_1 \leq n_2$. Let the singular values of $Q$ be $s_1, s_2, \ldots, s_{n_1}$.

Then $E(Q) = \sum_{i=1}^{n_1} s_i$.

**Lemma 2.1.**

$$E(Q) \leq \sqrt{n_1} \|Q\|_F$$

where $\|Q\|_F$ is the Frobenius norm of $Q$, defined as [12]

$$\|Q\|_F = \sqrt{\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} q_{ij}^2} .$$

**Proof.** Let $\xi_1, \xi_2, \ldots, \xi_p$ be real numbers. Their variance

$$\frac{1}{p} \sum_{i=1}^{p} \xi_i^2 - \left( \frac{1}{p} \sum_{i=1}^{p} \xi_i \right)^2$$

is known to be non-negative. Therefore,

$$\sum_{i=1}^{p} \xi_i \leq \sqrt{p} \sum_{i=1}^{p} \xi_i^2 .$$

Setting in the above inequality $p = n_1$ and $\xi_i = s_i$, we obtain

$$E(Q) = \sum_{i=1}^{n_1} s_i \leq \sqrt{n_1} \sqrt{\sum_{i=1}^{n_1} s_i^2} .$$

Now, the singularities of the matrix $Q$ are just the square roots of the eigenvalues of $QQ^T$. Therefore, $\sum_{i=1}^{n_1} s_i^2$ is equal to the sum of the eigenvalues of $QQ^T$, which in turn is the trace of $QQ^T$. Lemma 2.1 follows from

$$\text{Tr}(QQ^T) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} q_{ij} q_{ji}^T = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} q_{ij}^2 = \|Q\|_F^2 .$$

$\square$
Let 
\[ A(G) = \begin{pmatrix} B & X \\ X^T & C \end{pmatrix} \]
be a partition of the adjacency matrix of a graph \(G\), where \(B\) and \(C\) have orders \(n_1 \times n_1\) and \(n_2 \times n_2\), respectively. Thus \(X\) has order \(n_1 \times n_2\).

**Theorem 2.2.** Let \(G\) be a graph of order \(n = n_1 + n_2\), such that its adjacency matrix is given by 
\[ A(G) = \begin{pmatrix} B & X \\ X^T & C \end{pmatrix}, \]
where \(B\) and \(C\) represent adjacency matrices of graphs. Moreover, let \(k \leq \min\{n_1, n_2\}\) and consider the eigenpairs \(\{(\alpha_i, u_i) : 1 \leq i \leq k\}\) of \(B\) and \(\{(\beta_i, v_i) : 1 \leq i \leq k\}\) of \(C\), such that the sets \(\{u_1, \ldots, u_k\}\) and \(\{v_1, \ldots, v_k\}\) are orthonormal vectors. Let \(U = (u_1 | \ldots | u_k)\) and \(V = (v_1 | \ldots | v_k)\). Then

\[
E(G) \leq \frac{1}{2} \sum_{i=1}^{k} \left| \alpha_i + \beta_i + \sqrt{(\alpha_i - \beta_i)^2 + 4} \right| + \frac{1}{2} \sum_{i=1}^{k} \left| \alpha_i + \beta_i - \sqrt{(\alpha_i - \beta_i)^2 + 4} \right|
\]

\[
+ \sqrt{n_1 - k} \sqrt{\|B\|_F^2 - \sum_{i=1}^{k} |\alpha_i|^2} + \sqrt{n_2 - k} \sqrt{\|C\|_F^2 - \sum_{i=1}^{k} |\beta_i|^2}
\]

\[
+ 2 \sqrt{\min\{n_1, n_2\}} \|X - UV^T\|_F .
\]

**Proof.**

\[
A(G) = \begin{pmatrix} B & X \\ X^T & C \end{pmatrix} = \begin{pmatrix} B & UV^T \\ VU^T & C \end{pmatrix} + \begin{pmatrix} 0 & X - UV^T \\ X^T - VU^T & 0 \end{pmatrix} = M + N .
\]

Thus, according by the Ky Fan theorem [13], 
\(E(G) \leq E(M) + E(N)\).

Note that the spectrum of the matrix \(\widehat{M}_i = \begin{pmatrix} \alpha_i & 1 \\ 1 & \beta_i \end{pmatrix}\) is

\[
\sigma(\widehat{M}_i) = \left\{ \frac{1}{2} \left( \alpha_i + \beta_i \pm \sqrt{(\alpha_i - \beta_i)^2 + 4} \right) \right\}.
\]

Then, by applying Lemma 1.2,

\[
E(M) = \frac{1}{2} \sum_{i=1}^{k} \left| \alpha_i + \beta_i + \sqrt{(\alpha_i - \beta_i)^2 + 4} \right| + \frac{1}{2} \sum_{i=1}^{k} \left| \alpha_i + \beta_i - \sqrt{(\alpha_i - \beta_i)^2 + 4} \right|
\]

\[
+ \sum_{i=k+1}^{n_1} |\alpha_i| + \sum_{i=k+1}^{n_2} |\beta_i| .
\]
By Lemma 2.1,
\[
\sum_{i=k+1}^{n_1} |\alpha_i| + \sum_{i=k+1}^{n_2} |\beta_i| \leq \sqrt{n_1 - k} \sqrt{\sum_{i=k+1}^{n_1} |\alpha_i|^2} + \sqrt{n_2 - k} \sqrt{\sum_{i=k+1}^{n_2} |\beta_i|^2}.
\]

Moreover, also by Lemma 2.1,
\[
E(N) = 2E(X - UV^T) \leq 2\sqrt{\min\{n_1, n_2\}} \|X - UV^T\|_F.
\]

Consider the following special case of Theorem 2.2. Let both submatrices \( B \) and \( C \) be equal, say equal to \( A \), the adjacency matrix of an \((n, m)\)-graph \( G \). Further, let \( k = n_1 = n_2 = n \). By setting in Lemma 1.2, \( A = B = A(G) \), and considering \((\lambda_i, u_i), \ i = 1, \ldots, n\), such that \( U = (|u_1|, \ldots, |u_n|) \) is an orthonormal matrix and \( \rho = 1 \), the matrix
\[
C = \begin{pmatrix} A & \rho UU^T \\ \rho UU^T & A \end{pmatrix} = \begin{pmatrix} A & I_n \\ I_n & A \end{pmatrix}
\]

has eigenvalues \( \beta_{s_1}, \beta_{s_2}, \ldots, \beta_{sn}, \ s = 1, 2 \), where for \( j = 1, 2, \ldots, n, \ s = 1, 2, \ \beta_{sj} \) is an eigenvalue of
\[
\hat{C}_j = \begin{pmatrix} \lambda_j & 1 \\ 1 & \lambda_j \end{pmatrix}.
\]

Thus \( \beta_{1j} = \lambda_j - 1, \ \beta_{2j} = \lambda_j + 1 \), and for the symmetric matrix \( C \), we have
\[
E(C) = \sum_{i=1}^{n} |\lambda_i - 1| + \sum_{i=1}^{n} |\lambda_i + 1| \geq \sum_{i=1}^{n} (\lambda_i - 1) + \sum_{i=1}^{n} (\lambda_i + 1) = n + n = 2n.
\]

Hence \( E(C) > 2n \) holds, except if for all \( i = 1, 2, \ldots, n \), the value of the terms \( |\lambda_i - 1| \) and \( |\lambda_i + 1| \) is either zero or equal to some constant \( \gamma > 0 \). This can happen only if either for all \( i = 1, 2, \ldots, n, \ \lambda_i = 0 \) or \( \lambda_i \in \{-1, +1\} \), i.e., if either (a) the underlying graph \( G \) is without edges, or (b) all components of \( G \) are isomorphic to \( K_2 \). In these two cases, \( E(C) \) is equal to the number of vertices of the graph whose adjacency matrix is \( C \).

In conclusion, except in the two “pathological” cases (a) and (b), the energy of the graph whose adjacency matrix is \( C \) is greater than its number of vertices.
At this point it is worth noting that the graph whose adjacency matrix is \( C \) is just the sum of the graphs \( G \) and \( K_2 \), denoted by \( G + K_2 \), where \( K_2 \) is the complete graph on two vertices.

The graph operation marked by + is described in detail elsewhere [2, 14]. Let \( G_1 \) and \( G_2 \) be two graphs, with (disjoint) vertex sets \( V(G_1) \) and \( V(G_2) \). Then the vertex set of \( G_1 + G_2 \) is \( V(G_1) \times V(G_2) \). The vertices \( (x_1, x_2) \) and \( (y_1, y_2) \) of \( G_1 + G_2 \) are adjacent if and only if \( x_1 = y_1 \) and \( (x_2, y_2) \) is an edge of \( G_2 \) or if \( x_2 = y_2 \) and \( (x_1, y_1) \) is an edge of \( G_1 \).

For the present consideration it is important that the spectrum of \( G_1 + G_2 \) consists of the sums of eigenvalues of \( G_1 \) and \( G_2 \). In view of this, the finding that \( \beta_{ij} = \lambda_i - 1 \), \( \beta_{2j} = \lambda_j + 1 \) is an immediate consequence of the fact that the spectrum of \( K_2 \) consists of the numbers +1 and −1.

The result \( E(C) \geq 2n \) can now be generalized as follows.

Let \( \alpha_1, \ldots, \alpha_{n_1} \) and \( \beta_1, \ldots, \beta_{n_2} \) be, respectively, the eigenvalues of \( G_1 \) and \( G_2 \). Then the eigenvalues of \( G_1 + G_2 \) are of the form \( \alpha_i + \beta_j \), \( i = 1, \ldots, n_1 \), \( j = 1, \ldots, n_2 \), and

\[
E(G_1 + G_2) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} |\alpha_i + \beta_j| .
\]

We thus have,

\[
E(G_1 + G_2) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} (\alpha_i + \beta_j) = \sum_{i=1}^{n_1} n_2 |\alpha_i| = n_2 \sum_{i=1}^{n_1} |\alpha_i| = n_2 E(G_1)
\]

because of

\[
\sum_{j=1}^{n_2} \alpha_i = n_2 \alpha_i \quad \text{and} \quad \sum_{j=1}^{n_2} \beta_j = 0 .
\]

In an analogous manner it can be shown that \( E(G_1 + G_2) \geq n_1 E(G_2) \).

Same as in the above example, equality \( E(G_1 + G_2) = n_2 E(G_1) \) occurs if \((a')\) \( G_2 \) is without edges, whereas equality \( E(G_1 + G_2) = n_1 E(G_2) \) occurs if \((a'')\) \( G_1 \) is without edges. Both equalities \( E(G_1 + G_2) = n_2 E(G_1) \) and \( E(G_1 + G_2) = n_1 E(G_2) \) occur if \((b')\) all components of both \( G_1 \) and \( G_2 \) are isomorphic to \( K_2 \).

The graph \( G_1 + G_2 \) has \( n_1 n_2 \) vertices. Bearing this in mind we arrive at:
Theorem 2.3. With the exception of the (above specified) “pathological” cases \((a')\), \((a'')\), and \((b')\), if either the energy of \(G_1\) exceeds or is equal to the number of vertices of \(G_1\), or the energy of \(G_2\) exceeds or is equal to the number of vertices of \(G_2\), then the energy of \(G_1 + G_2\) exceeds the number of vertices of \(G_1 + G_2\).

In connection with Theorem 2.3 it is worth noting that the problem of constructing and characterizing graphs whose energy exceeds the number of vertices was first considered in [15] and thereafter in [16–19]. A closely related problem is the construction and characterization of hypoenergetic graphs, namely (connected) graphs whose energy is less than the number of vertices [13,20–25]. Also graphs whose energy is equal to the number of vertices were recently studied [26]. From this point of view, Theorem 2.3 provides an additional possibility to obtain (infinitely many) non-hypoenergetic graphs.

Let \(A, B, C, X, U\) and \(V\) be the same matrices as in the formulation and proof of Theorem 2.2. Let, as before

\[
M = \begin{pmatrix} B & UV^T \\ VU^T & C \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & UV^T \\ VU^T & 0 \end{pmatrix}.
\]

Theorem 2.4. \(E(A) \geq E(M) - E(Q)\).

Proof. Let

\[
\hat{A} = \begin{pmatrix} B & -X \\ -X^T & C \end{pmatrix}.
\]

Then, as well known, \(E(A) = E(\hat{A})\). We have \(A = M + P\) and \(\hat{A} = M + \hat{P}\), where

\[
P = \begin{pmatrix} 0 & X - UV^T \\ X^T - VU^T & 0 \end{pmatrix} \quad \text{and} \quad \hat{P} = \begin{pmatrix} 0 & -X - UV^T \\ -X^T - VU^T & 0 \end{pmatrix}.
\]

Therefore

\[
A + \hat{A} = 2M - 2Q.
\]

By Ky Fan theorem [13],

\[
2E(M) \leq E(A) + E(\hat{A}) + 2E(Q)
\]

implying the theorem. \(\square\)
As a special case of Theorem 2.4, for $k = 1$, 

$$M = \begin{pmatrix} B & u_1 v_1^T \\ v_1 u_1^T & C \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & u_1 v_1^T \\ v_1 u_1^T & 0 \end{pmatrix}. $$

Hence, $E(Q) = 2|v_1^T u_1|$ and $E(A) \geq E(M) - 2|v_1^T u_1|$. 

By Fiedler’s Lemma 1.1,

$$E(M) = E(B) + E(C) + \frac{1}{2} \left| \alpha_1 + \beta_1 + \sqrt{(\alpha_1 - \beta_1)^2 + 4} \right|$$

$$+ \left| \frac{1}{2} \alpha_1 - \beta_1 - \sqrt{(\alpha_1 - \beta_1)^2 + 4} \right| - (|\alpha_1| + |\beta_1|).$$

Therefore, $E(A) \geq E(B) + E(C) + \varepsilon$, where

$$\varepsilon = \left| \frac{1}{2} \alpha_1 + \beta_1 + \sqrt{(\alpha_1 - \beta_1)^2 + 4} \right| + \left| \frac{1}{2} \alpha_1 - \beta_1 - \sqrt{(\alpha_1 - \beta_1)^2 + 4} \right| - (|\alpha_1| + |\beta_1|).$$

The inequality $E(A) \geq E(B) + E(C)$ has been reported earlier [27]. Therefore our result will be an improvement of that inequality only if $\varepsilon > 0$. If $\alpha_1 > 0$, $\beta_1 > 0$, and $\alpha_1 \beta_1 \geq 1$, then

$$\frac{1}{2} \left| \alpha_1 + \beta_1 + \sqrt{(\alpha_1 - \beta_1)^2 + 4} \right| + \frac{1}{2} \left| \alpha_1 - \beta_1 - \sqrt{(\alpha_1 - \beta_1)^2 + 4} \right| - (|\alpha_1| + |\beta_1|) = 0$$

and therefore $\varepsilon < 0$. Thus, in order to get an improvement, we must choose $\alpha_1$ and $\beta_1$ such that $\alpha_1 \beta_1 < 1$. For instance, for $\alpha_1 = \beta_1 = 0$ we get $\varepsilon = 2 - 2|v_1^T u_1|$, which is positive because of $|v_1^T u_1| \leq \|v_1\| \|u_1\| = 1 \cdot 1 = 1$.

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References


