On Incidence Energy of Graphs

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Abstract


1. INTRODUCTION

Two of the present authors, together M. R. Jooyandeh [1], conceived recently a graph–energy like quantity, called incidence energy, $IE(G)$, calculated from the incidence matrix $I(G)$ of a graph $G$.

The energy $E(G)$ of a graph $G$ is equal to the sum of the absolute values of the graph eigenvalues, i. e., of the eigenvalues of the adjacency matrix $A(G)$ of $G$. (For
details on graph energy see the reviews [2, 3].) Nikiforov [4] recently extended the concept of energy to all (not necessarily square) matrices, defining the energy of a matrix \( M \) as the sum of the singular values of \( M \). Recall that the singular values of a matrix \( M \) are equal to the square roots of the eigenvalues of the (square) matrix \( MM^t \).

Let \( G = (V, E) \) be a simple graph, where \( V = \{v_1, v_2, \ldots, v_n\} \) is its vertex set, and \( E = \{e_1, e_2, \ldots, e_m\} \) its edge set. Thus \( n \) and \( m \) denote, respectively, the number of vertices and edges of \( G \). In [1] as well as in the present paper, \( I(G) \) is the vertex-edge incidence matrix, an \((n \times m)\)-matrix whose \((i, j)\)-element is equal to 1 if the vertex \( v_i \) is incident to the edge \( e_j \), and is equal to 0 otherwise (see, for instance [5], p. 16).

In line with Nikiforov’s idea, the incidence energy of a graph \( G \) was defined [1] as the sum of the singular values of the incidence matrix \( I(G) \), that, in turn, are equal to the square roots of the eigenvalues of \( I(G)I(G)^t \). Recall that \( I(G)I(G)^t \) is a square matrix of order \( n \).

Let \( D(G) \) be the diagonal matrix of order \( n \) whose \((i, i)\)-entry is the degree (= number of first neighbors) of the vertex \( v_i \) of the graph \( G \). Then the matrix \( L(G) = D(G) - A(G) \) is the Laplacian matrix of the graph \( G \), for details see [6, 7]. The matrix \( L^+(G) = D(G) + A(G) \) is the signless Laplacian matrix, for details see [8].

Denote by \( \mu_1, \mu_2, \ldots, \mu_n \) the eigenvalues of the Laplacian matrix \( L(G) \) and by \( \mu_1^+, \mu_2^+, \ldots, \mu_n^+ \) the eigenvalues of the signless Laplacian matrix \( L^+(G) \). All eigenvalues of both \( L(G) \) and \( L^+(G) \) are real and non-negative. If the graph \( G \) is connected, then \( n - 1 \) eigenvalues of \( L(G) \) are positive, and one is equal to zero [6, 7]. If \( G \) is a connected non-bipartite graph, then all eigenvalues of \( L^+(G) \) are positive; if \( G \) is connected and bipartite, then exactly one eigenvalue of \( L^+(G) \) is equal to zero [8].

The following result is well known [6, 7, 8]:

**Lemma 1.** The spectra of \( L(G) \) and \( L^+(G) \) coincide if and only if the graph \( G \) is bipartite.

Short time ago, J. Liu and B. Liu [9] introduced the so-called Laplacian–energy like invariant, \( \text{LEL}(G) \) of a graph \( G \), as the sum of the square roots of the eigenvalues...
of the Laplacian matrix of \( G \), i. e.,

\[
LEL(G) := \sum_{i=1}^{n} \sqrt{\mu_i} .
\]  

(1)

In [9] and in the subsequent papers [10, 11, 12] a number of properties of \( LEL \) were established.

2. RELATION BETWEEN INCIDENCE ENERGY AND LAPLACIAN–ENERGY LIKE INVARIANT

A well known identity for the incidence matrix of any graph \( G \) is (see, for instance, [5], p. 16):

\[
I(G) I(G)^t = A(G) + D(G)
\]

i. e.,

\[
I(G) I(G)^t = L^+(G) .
\]  

(2)

Then an immediate consequence of Eq. (2) and the definition of the incidence energy is:

**Theorem 2.** If \( IE(G) \) is the incidence energy of an \( n \)-vertex graph \( G \), and if \( \mu_1^+, \mu_2^+, \ldots, \mu_n^+ \) are the eigenvalues of the signless Laplacian matrix of \( G \), then

\[
IE(G) = \sum_{i=1}^{n} \sqrt{\mu_i^+} .
\]

In view of Lemma 1 and Eq. (1), we arrive at the following noteworthy:

**Corollary 3.** If \( IE(G) \) is the incidence energy of a bipartite graph \( G \), and \( \mu_1, \mu_2, \ldots, \mu_n \) are the eigenvalues of the Laplacian matrix of \( G \), then

\[
IE(G) = \sum_{i=1}^{n} \sqrt{\mu_i} \equiv LEL(G) .
\]

In other words, for bipartite graphs the incidence energy \( IE \) and the Laplacian–energy like invariant \( LEL \) coincide.

In analogy with Eq. (1), we define

\[
LEL^+(G) := \sum_{i=1}^{n} \sqrt{\mu_i^+}
\]
which immediately implies:

**Corollary 4.** If $G$ is any graph, then $IE(G) \equiv LEL^+(G)$.

### 3. ON A RELATION BETWEEN INCIDENCE ENERGY AND GRAPH ENERGY

In [1] it was demonstrated that if $G$ is any graph, it is possible to find a bipartite graph $\hat{G}$, such that

$$IE(G) = \frac{1}{2} E(\hat{G})$$

where $E(\hat{G})$ is the ordinary energy of the graph $\hat{G}$. In [1] the graph $\hat{G}$ is constructed so that its adjacency matrix is of the form:

$$A(\hat{G}) = \begin{bmatrix} 0 & I(G) \\ I(G)^t & 0 \end{bmatrix}.$$  \hfill (3)

In graph theory it is well known that the graph defined via Eq. (3) is the subdivision graph $S(G)$ of the graph $G$, obtained by inserting an additional vertex into each edge of $G$ (see, for instance, [5], p. 16). Thus $\hat{G}$ is just the subdivision graph of $G$.

If $G$ is a graph with $n$ vertices and $m$ edges, then its subdivision graph $S(G)$ has $n + m$ vertices and $2m$ edges.

A connection between the spectrum of $S(G)$ and the Laplacian spectrum of $G$ was recently communicated [13]. The main result in [13] is the following:

**Theorem 5.** Let $G$ be a bipartite graph with $n$ vertices and $m$ edges, and let $S(G)$ be its subdivision graph. If $\mu_i$, $i = 1, \ldots, h$, are the non-zero eigenvalues of the Laplacian matrix of $G$, then the ordinary spectrum of $S(G)$ consists of the numbers $\pm \sqrt{\mu_i}$, $i = 1, \ldots, h$, and of $n + m - 2h$ zeros.

Although not stated in [13], the following extension of Theorem 5 to all graphs is evident (and is proven in the same way as Theorem 5 itself):
**Theorem 6.** Let \( G \) be a graph with \( n \) vertices and \( m \) edges, and let \( S(G) \) be its subdivision graph. If \( \mu_i^+, i = 1, \ldots, h, \) are the non-zero eigenvalues of the signless Laplacian matrix of \( G \), then the ordinary spectrum of \( S(G) \) consists of the numbers \( \pm \sqrt{\mu_i^+}, i = 1, \ldots, h \), and of \( n + m - 2h \) zeros.

From Theorem 5 it immediately follows (but is not stated in [13]) that for a bipartite graph \( G \), \( E(S(G)) = 2 \text{LEL}(G) \). Analogously, Theorem 6 implies that for any graph \( G \), \( E(S(G)) = 2 \text{LEL}^+(G) \). In view of Corollaries 3 and 4, we then get

**Corollary 7.** For any graph \( G \) whose subdivision graph is \( S(G) \),

\[
IE(G) = \frac{1}{2} E(S(G)).
\]

Corollary 7 is, of course, precisely identical to Theorem 1 in [1].

**4. TREES WITH MINIMAL AND MAXIMAL INCIDENCE ENERGY**

Let \( T_n \) be the set of all \( n \)-vertex trees. Let \( S_n \) and \( P_n \) be the \( n \)-vertex star and \( n \)-vertex path, respectively. In [11] it was shown that for any tree \( T \in T_n \),

\[
\text{LEL}(S_n) \leq \text{LEL}(T) \leq \text{LEL}(P_n)
\]

with equality if and only if \( T \cong S_n \) and \( T \cong P_n \), respectively. Because trees are bipartite graphs, by means of Corollary 3 the above result can be re-stated as:

**Theorem 8.** For any tree \( T \in T_n \),

\[
IE(S_n) \leq IE(T) \leq IE(P_n)
\]

with equality if and only if \( T \cong S_n \) and \( T \cong P_n \), respectively.

In view of the result reported in [11], no proof of Theorem 8 would be needed. We nevertheless offer an alternative proof.

**Proof.** Denote by \( \psi(G, \lambda) \) the characteristic polynomial of the Laplacian matrix of the graph \( G \). Its zeros are \( \mu_1, \mu_2, \ldots, \mu_n \). It is known [5, 13] that this polynomial is
of the form
\[ \psi(G, \lambda) = \sum_{k \geq 0} (-1)^k c_k(G) \lambda^{n-k} \]
where \(c_k(G) \geq 0\).\(^1\)

Let \(\Psi(G, \lambda)\) be an auxiliary polynomial defined as
\[ \Psi(G, \lambda) = \psi(G, \lambda^2) = \sum_{k \geq 0} (-1)^k c_k(G) \lambda^{2n-2k} . \]
Then the zeros of \(\Psi(G, \lambda)\) are \(\pm \sqrt{\mu_1}, \pm \sqrt{\mu_2}, \ldots, \pm \sqrt{\mu_n}\). Therefore, the sum of the positive zeros of \(\Psi(G, \lambda)\) is just \(\text{LEL}(G)\).

The Coulson integral formula (see [2, 14, 15] and the references cited therein) makes it possible to compute the sum of the positive zeros of a polynomial without knowing the actual values of these zeros. Applying to \(\Psi(G, \lambda)\) a variant of the Coulson integral formula from the work [16], we get
\[ \text{LEL}(G) = \frac{1}{\pi} \int_0^{+\infty} \ln \left[ \sum_{k \geq 0} c_k(G) x^{2k} \right] \frac{dx}{x^2} . \quad (5) \]
From formula (5) we see that \(\text{LEL}(G)\) is a monotonically increasing function of each of the coefficients \(c_k(G)\).

For \(n\)-vertex trees it has been shown [13] that for all \(k \geq 0\) and for all \(T \in \mathcal{T}_n\),
\[ c_k(S_n) \leq c_k(T) \leq c_k(P_n) \quad (6) \]
and that equality for all values of \(k\) occurs if and only if \(T \cong S_n\) and \(T \cong P_n\), respectively. Combining (5) and (6) we immediately obtain the relations (4), from which Theorem 8 follows. \(\square\)

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\(^1\)The dependence of the coefficients \(c_k(G)\) on the structure of the graph \(G\) is also fully understood [5], but is irrelevant for the present considerations.
References


