Applications of a theorem by Ky Fan in the theory of Laplacian energy of graphs

María Robbiano* and Raúl Jiménez†

Universidad Católica del Norte, Avenida Angamos 0610, Antofagasta, Chile

(Received December 19, 2008)

Abstract

The Laplacian energy of a graph \( G \) is equal to the sum of distances of the Laplacian eigenvalues to their average, which in turn is equal to the sum of singular values of a shift of Laplacian matrix of \( G \). Let \( X, Y, \) and \( Z \) be matrices, such that \( Z = X + Y \). Ky Fan has established an inequality between the sum of singular values of \( Z \) and the sum of the sum of singular values of \( X \) and \( Y \) respectively. We apply this inequality to obtain new results in the theory of Laplacian energy of a graph.

1 Preliminaries

Let \( G = (V, E) \) be a simple graph, with nonempty vertex set \( V = \{v_1, \ldots, v_n\} \) and edge set \( E = \{e_1, \ldots, e_m\} \). That is to say, \( G \) is a simple \((n, m)\)-graph. For any of these graphs \( d_1 \geq d_2 \geq \ldots \geq d_n \) corresponds to its vertex degree sequence. In particular \( \Delta (G) \) stands for the largest vertex degree of \( G \). The diagonal matrix of order \( n \) whose \((i, i)\)-entry is \( d_i \) is the diagonal

*E-mail address: mariarobbiano@gmail.com
†The authors were partially supported by Project Mecesup 2 UCN0605.
The vertex degree matrix of $G$ is denoted by $D(G)$. The $(0, 1)$-adjacency matrix $A(G) = (a_{ij})$ is defined by $a_{ij} = 1$ if, and only if, vertices $i$ and $j$ are connected. Its eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ form the spectrum of $G$. The matrix $L(G) = D(G) - A(G)$ is the Laplacian matrix of $G$. The Laplacian spectrum of $G$ corresponds to eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ of $L(G)$. It is well known that for bipartite graphs, Laplacian matrix and the signless Laplacian matrix $Q(G) = A(G) + D(G)$ have equal spectra [2].

The notion of the energy $E(G)$ of an $(n, m)$-graph $G$ was introduced by Gutman in connection with the $\pi$-molecular energy (cf. [8, 9, 11, 14]). It is defined by

$$E(G) = \sum_{j=1}^{n} |\lambda_j|$$

whereas the Laplacian energy $LE(G)$ of an $(n, m)$-graph $G$ (cf. [1, 4, 10, 12, 21]) is defined by

$$LE(G) = \sum_{j=1}^{n} |\mu_j - (2m/n)| . \quad (1)$$

The concept of matrix energy [16] was established by analogy with graph energy. For a matrix $C$, with singular values $s_1(C), s_2(C), \ldots$ its energy $E(C)$ is equal to $s_1(C) + s_2(C) + \cdots$. Consequently, if $C \in \mathbb{R}^{n \times n}$ is symmetric with eigenvalues $\lambda_1(C), \lambda_2(C), \ldots, \lambda_n(C)$ its energy is given by

$$E(C) = \sum_{i=1}^{n} |\lambda_i(C)| .$$

Let $s \in \mathbb{N}$. Denote by $I_s$ the corresponding identity matrix of order $s$. Evidently the energy of any graph $G$ is the energy of its adjacency matrix
and its Laplacian energy is provided by

\[ LE(G) = \mathcal{E}\left(L(G) - \frac{2m}{n} I_n\right). \]

(2)

The following results are already known.

**Theorem 1** Let \( A \) and \( B \) be two real square matrices of order \( n \) and let \( C = A + B \). Then

\[ \mathcal{E}(C) \leq \mathcal{E}(A) + \mathcal{E}(B). \]

(3)

Moreover equality holds if, and only if, there exists an orthogonal matrix \( P \) such that \( PA \) and \( PB \) are both positive semidefinite matrices.

**Lemma 2 ([3])** If \( A = (a_{ij})_{i,j=1}^n \) is a positive semidefinite matrix and \( a_{ii} = 0 \) for some \( i \), then \( a_{ij} = 0 = a_{ji}, j = 1, 2, \ldots, n \).


For a matrix \( A \), define \( |A| \triangleq (A^T A)^{1/2} \). Here we present the following version of the polar decomposition theorem.

**Theorem 3 ([15])** Let \( A \in \mathbb{R}^{n \times n} \). Then there exist positive semidefinite matrices \( X, Y \in \mathbb{R}^{n \times n} \) and orthogonal matrices \( P, F \in \mathbb{R}^{n \times n} \) such that \( A = PX = YF \). Moreover, the matrices \( X, Y \) are unique, \( X = |A|, Y = (AA^T)^{1/2} \). The matrices \( P \) and \( F \) are uniquely determined if and only if \( A \) is nonsingular.
The aim of this paper is to study cases of equality.

2 Graphs $G$ for which $LE(G) = E(G) + \mathcal{E}(D(G) - (2m/n)I_n)$

Theorem 4 Let $G$ be a connected $(n, m)$-graph. Then

$$E(G) + \sum_{i=1}^{n} \left| d_i - \frac{2m}{n} \right| \geq LE(G).$$

(4)

Moreover equality in (4) holds if, and only if, $G$ is a regular graph.

Proof. The inequality in (4) is proved in [17]. If $G$ is a regular graph then the equality in (4) holds (see [12]). Conversely, suppose the equality in (4) holds. In order to obtain a contradiction, we suppose that $G$ is not regular. Therefore

$$\Delta(G) = d_1 > \frac{2m}{n}.$$  

(5)

For $i = 1, \ldots, n$, let $a_i \triangleq d_i - (2m/n)$. We have $a_1 > 0$, via (5). Bearing in mind that $L(G) - (2m/n)I_n = D(G) - (2m/n)I_n - A(G)$ and the equality in (4), we see that Theorem 1 asserts that there exists an orthogonal matrix $P$ such that $X = P(D - (2m/n)I_n)$ and $Y = P(-A(G))$ are both positive semidefinite. Hence $P^TX$ and $P^TY$ are polar decompositions of the matrices $D - \frac{2m}{n}I_n$ and $-A(G)$, respectively. Here, using Theorem 3 we obtain $X = |D - \frac{2m}{n}I_n|$ and $Y = |A(G)|$. Therefore $X = \ldots$
$\text{diag}(|a_1|, |a_2|, \ldots, |a_n|)$. Setting

$$P^T = \begin{pmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & \ddots & \vdots \\ q_{n1} & \cdots & q_{nn} \end{pmatrix} \quad \text{and} \quad A(G) = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ a_{12} & 0 & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & 0 \end{pmatrix},$$

$P^T X = D - (2m/n)I_n$, implies

$$\begin{pmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & \ddots & \vdots \\ q_{n1} & \cdots & q_{nn} \end{pmatrix} \begin{pmatrix} |a_1| \\ \vdots \\ |a_n| \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}.$$ Then,

$$\begin{pmatrix} |a_1| q_{11} & |a_2| q_{12} & \cdots & |a_n| q_{1n} \\ |a_1| q_{21} & |a_2| q_{22} & \cdots & |a_n| q_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ |a_1| q_{n1} & |a_2| q_{n2} & \cdots & |a_n| q_{nn} \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$ Equality at first column imposes $q_{11} = 1$ and, $q_{i1} = 0$, $i = 2, \ldots, n$. It follows that

$$P = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ q_{12} & \cdots & q_{n2} \\ \vdots & \ddots & \vdots \\ q_{1n} & \cdots & q_{nn} \end{pmatrix}.$$ We must then have

$$Y = -\begin{pmatrix} 1 & 0 & \cdots & 0 \\ q_{12} & \cdots & q_{n2} \\ \vdots & \ddots & \vdots \\ q_{1n} & \cdots & q_{nn} \end{pmatrix} \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ a_{12} & 0 & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & 0 \end{pmatrix}$$

$$= -\begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{pmatrix}.$$
The previous matrix is positive semidefinite and by Lemma 2, we obtain $a_{1j} = 0, \ j = 2, \ldots, n$. This contradicts our assumption that $G$ is a connected graph and the result follows.

3 Graphs $G$ for which $LE(G) = E(G)$

Gutman and Zhou [12] showed that if $G$ is a regular graph then

$$LE(G) = E(G).$$  \hspace{1cm} (6)

In particular, if $G$ is bipartite and regular, then the equality (6) holds. In this section we give conditions for the converse:

**Theorem 5** Let $G$ be a bipartite graph. Then the equality (6) holds if, and only if, $G$ is a regular graph.

**Proof.** Let $G$ be a regular graph. We must then have (6) [12]. Conversely, suppose the equality (6) holds. From definition of Laplacian and signless Laplacian matrices it is clear that

$$\left( Q(G) - \frac{2m}{n} I_n \right) - \left( L(G) - \frac{2m}{n} I_n \right) = 2A(G).$$ \hspace{1cm} (7)

Therefore,

$$E \left( Q(G) - \frac{2m}{n} I_n - \left( L(G) - \frac{2m}{n} I_n \right) \right) = 2E(A(G)) = E(G) + E(G) = LE(G) + LE(G).$$
Bearing in mind that $G$ is bipartite we obtain
\[
\mathcal{E} \left( Q(G) - \frac{2m}{n} I_n - \left( L(G) - \frac{2m}{n} I_n \right) \right) = \mathcal{E} \left( Q(G) - \frac{2m}{n} I_n \right) + \mathcal{E} \left( - \left( L(G) - \frac{2m}{n} I_n \right) \right). \tag{8}
\]
Therefore, Theorem 1 asserts that there exists an orthogonal matrix $P$, such that
\[
X = P \left( Q(G) - \frac{2m}{n} I_n \right) \quad \text{and} \quad Y = P \left( - \left( L(G) - \frac{2m}{n} I_n \right) \right) \tag{9}
\]
are both positive semidefinite matrices. Hence $P^T X$ and $P^T Y$ are polar decompositions of
\[
Q(G) - \frac{2m}{n} I_n \quad \text{and} \quad - \left( L(G) - \frac{2m}{n} I_n \right)
\]
respectively. By Theorem 3 we obtain
\[
X = \left| Q(G) - \frac{2m}{n} I_n \right| \quad \text{and} \quad Y = \left| L(G) - \frac{2m}{n} I_n \right|.
\]
In view of the fact that $G$ is bipartite, we conclude that $X = Y$. By using Eq. (9) we arrive at
\[
Q(G) + L(G) = \frac{4m}{n} I_n
\]
which implies the result. \[\blacksquare\]

4 A new upper bound on $\text{LE}(G)$

We shall be considering $G$ with nonempty edge set $\overline{E}$. Let $u, v$ be two vertices of $G$. The Laplacian matrix of the graph $G(u, v)$ with $n$ vertices and just one
edge between vertices $u$ and $v$, is determined via

$$L(G(u, v))_{i,j} = \begin{cases} 
1 & \text{if } (i, j) = (u, u) \text{ or } (i, j) = (v, v) \\
-1 & \text{if } (i, j) = (u, v) \text{ or } (i, j) = (v, u) \\
0 & \text{otherwise.}
\end{cases}$$

Spielman [18] expresses the Laplacian matrix of $G$ in terms of $L(G(u, v))$ by

$$L(G) = \sum_{(u,v) \in E} L(G(u, v)). \quad (10)$$

We consider $\alpha \in \mathbb{R}$. The energy $\mathcal{E}(L(G(u, v)) - \alpha I_n)$ can be computed directly as:

$$\mathcal{E}(L(G(u, v)) - \alpha I_n) = (n - 1)|\alpha| + |2 - \alpha|. \quad (11)$$

On the other hand, for $0 < a < 1$, let $A$, $Q$, $P$, and $D$ be the following matrices:

$$A = \begin{bmatrix} a & -1 \\ -1 & a \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} a - 1 & 0 \\ 0 & a + 1 \end{bmatrix}$$

The proof of the next result is a matter of straightforward computation, and depends on the spectrum of $A$.

Lemma 6 Let $A$, $Q$, $P$ and $D$ be as above. Then $A = QDQ^{-1}$. Moreover $A = P|A|$.

As an immediate consequence we have $|A| = Q|D|Q^{-1}$.

Theorem 7 Let $G$ be an $(n, m)$-graph. Then

$$LE(G) \leq 4m \left( 1 - \frac{1}{n} \right). \quad (12)$$

Equality holds if, and only if, $\mathcal{E} = \emptyset$ or $G$ is the union of one edge and $n - 2$ isolated vertices.
Proof. The description of $L(G)$ in (10) makes the next equality evident, that is:

$$L(G) - \frac{2m}{n} I_n = \sum_{(u,v) \in \mathcal{E}(G)} \left( L(G(u,v)) - \frac{2}{n} I_n \right).$$  \hspace{1cm} (13)

The inequality in (12) is a consequence of Eqs. (2), (13), Theorem 1 and Eq. (11), by changing $\alpha$ to $2/n$. On the equality case in (12), it is easily checked that (12) is an equality in the cases considered in the statement. Conversely, suppose that we have equality in (12), $\mathcal{E} \neq \emptyset$ and $m \geq 2$. Consider $\mathcal{E} = \{e_1, e_2, \ldots, e_m\}$ where $e_i = \{u_i, v_i\}$, $i = 1, \ldots, m$. Thus, the next equality is implied by equality in (12), (13) and (11).

$$\mathcal{E} \left( L(G) - \frac{2m}{n} I_n \right) = \sum_{i=1}^{m} \mathcal{E} \left( L(G(e_i)) - \frac{2}{n} I_n \right).$$  \hspace{1cm} (14)

Using (13) and Theorem 1 we obtain

$$\mathcal{E} \left( L(G) - \frac{2m}{n} I_n \right) \leq \mathcal{E} \left( L(G(e_1)) - \frac{2}{n} I_n \right)$$

$$+ \mathcal{E} \left( \sum_{i=2}^{m} \left( L(G(e_i)) - \frac{2}{n} I_n \right) \right).$$  \hspace{1cm} (15)

Replacing (14) into (15) and apply Theorem 1 to obtain

$$\mathcal{E} \left( \sum_{i=2}^{m} \left( L(G(e_i)) - \frac{2}{n} I_n \right) \right) = \sum_{i=2}^{m} \mathcal{E} \left( L(G(e_i)) - \frac{2}{n} I_n \right).$$  \hspace{1cm} (16)

By the same kind of reasoning, but this time considering (16) rather than (14), we obtain

$$\mathcal{E} \left( \sum_{i=3}^{m} \left( L(G(e_i)) - \frac{2}{n} I_n \right) \right) = \sum_{i=3}^{m} \mathcal{E} \left( L(G(e_i)) - \frac{2}{n} I_n \right).$$
Using a reasoning analogous to that above, we arrive at

\[
\begin{align*}
\mathcal{E} \left( \left( L(G(e_m)) - \frac{2}{n} I_n \right) + \left( L(G(e_{m-1})) - \frac{2}{n} I_n \right) \right) \\
= \mathcal{E} \left( L(G(e_m)) - \frac{2}{n} I_n \right) + \mathcal{E} \left( L(G(e_{m-1})) - \frac{2}{n} I_n \right). 
\end{align*}
\] (17)

Invoking again Theorem 1, there exists an orthogonal matrix \( P \) such that

\[
X = P \left( L(G(e_m)) - \frac{2}{n} I_n \right) \quad \text{and} \quad Y = P \left( L(G(e_{m-1})) - \frac{2}{n} I_n \right) \quad (18)
\]

are positive semidefinite matrices. Hence \( P^T X \) and \( P^T Y \) are polar decompositions of nonsingular matrices \( L(G(e_m)) - \frac{2}{n} I_n \) and \( L(G(e_{m-1})) - \frac{2}{n} I_n \) respectively. By Theorem 3 we conclude that

\[
X = \begin{vmatrix} L(G(e_m)) - \frac{2}{n} I_n \end{vmatrix} \quad \text{and} \quad Y = \begin{vmatrix} L(G(e_{m-1})) - \frac{2}{n} I_n \end{vmatrix}. 
\]

As

\[
L(G(e_m)) - \frac{2}{n} I_n \quad \text{and} \quad L(G(e_{m-1})) - \frac{2}{n} I_n
\]

are invertible matrices, \( P \) is the unique orthogonal matrix for which (18) is true. Let \( b = 1 - 2/n \). Then the matrix \( L(G(e_m)) - \frac{2}{n} I_n \) can be expressed as

\[
\begin{pmatrix}
-\frac{2}{n} & 0 & \ldots & 0 \\
0 & b & \ldots & 0 & -1 \\
\vdots & 0 & -\frac{2}{n} & \vdots \\
\vdots & \vdots & \ddots & 0 \\
0 & -1 & 0 & \ldots & b \\
0 & \ldots & 0 & \vdots \\
0 & \ldots & 0 & -\frac{2}{n}
\end{pmatrix}
\]
By relatively straightforward means one can show that

\[ L(G(e_m)) - \frac{2}{n} I_n = F_m B F_m^T \]  \hspace{1cm} (19)

with \( F_m \) denoting a particular permutation matrix and \( B \) is the nonsingular matrix

\[
\begin{pmatrix}
  b & -1 & \ldots & 0 \\
  -1 & b & \ldots & \vdots \\
  \vdots & 0 & -\frac{2}{n} & \vdots \\
  \vdots & \ddots & \ddots & 0 \\
  0 & \ldots & 0 & -\frac{2}{n}
\end{pmatrix}.
\]

Now let \( Q \) be the matrix

\[
\begin{pmatrix}
  0 & -1 \\
  -1 & 0 \\
  \vdots & \ddots \\
  \vdots & \ddots \\
  0 & \ldots & 0 & -1
\end{pmatrix}.
\]

By Lemma 6 and Theorem 3 we conclude that \( Q \) is the unique orthogonal matrix for which

\[ B = Q |B| . \] \hspace{1cm} (20)

Therefore, Eq. (19) implies

\[ \left| L(G(e_m)) - \frac{2}{n} I_n \right| = F_m |B| F_m^T = F_m Q^{-1} B F_m^T . \] \hspace{1cm} (21)

Now we can replace (19) and (21) in

\[ L(G(e_m)) - \frac{2}{n} I_n = P^T \left| L(G(e_m)) - \frac{2}{n} I_n \right| \]

to obtain

\[ F_m Q F_m^T = P^T . \]
Then \( P = F_m Q F_m^T \) is the unique orthogonal matrix such that

\[
P \left( L(G(e_m)) - \frac{2}{n} I_n \right)
\]

is positive definite and it is depending of edge \( e_m \). We see that this contradicts to the requirement that

\[
Y = P \left( L(G(e_{m-1})) - \frac{2}{n} I_n \right)
\]

is a positive definite matrix. This proves the assertion. \( \blacksquare \)

5 An upper bound on the Laplacian energy for the union of graphs

Here and throughout this section, \( \bigoplus \) denotes the block matrix direct sum [13].

**Theorem 8** Let \( k \in \mathbb{N} \). Let \( \{G_i\}_{i=1}^{k} \) be a collection of \( k, (n_i, m_i) \)-graphs, \( i = 1, \ldots, k \). Consider \( G = G_1 \cup G_2 \cup \ldots \cup G_k \) so that \( n = \sum_{i=1}^{k} n_i \) is the order of \( G \) and \( m = \sum_{i=1}^{k} m_i \) is the size of \( G \). Then

\[
LE(G) \leq \sum_{i=1}^{k} LE(G_i) + \sum_{i=1}^{k} \left| \frac{2m_i}{n_i} - \frac{2m}{n} \right| n_i.
\]

Equality holds if, and only if, \( 2m_i/n_i = 2m/n \) for all \( i = 1, \ldots, k \).

**Proof.** The following equality follows immediately from the statement,

\[
\frac{2m}{n} = \left( \frac{1}{\sum_{j=1}^{k} n_j} \right) \left( \sum_{i=1}^{k} 2m_i \right) = \sum_{i=1}^{k} \frac{2m_i}{n_i} \left( \frac{n_i}{\sum_{j=1}^{k} n_j} \right).
\]

In other words \( 2m/n \) is a convex combination of \( 2m_i/n_i, i = 1, \ldots, k \).
In order to simplify the writing and omit some subscripts, we take $I_{n_i} \equiv I_i$ and $2m_i/n_i - 2m/n \equiv b_i$. It is clear that

$$L(G) - \frac{2m}{n} I_n = \bigoplus_{i=1}^{k} \left( L(G_i) - \frac{2m_i}{n_i} I_i \right)$$

$$= \bigoplus_{i=1}^{k} \left( L(G_i) - \frac{2m_i}{n_i} I_i \right) + \bigoplus_{i=1}^{k} b_i I_i.$$  

Therefore, as a consequence of Eq. (2) and Theorem 1, the inequality in (22) follows. On the equality case, the condition is sufficient [12]. Conversely we suppose the equality in (22) is true and suppose that the equalities $2m_i/n_i = 2m/n$ for all $i = 1, \ldots, k$, fail. Therefore, by (23) there exists $\ell$ such that $2m_\ell/n_\ell > 2m/n$. We can assume that $\ell = 1$. As a consequence of Theorem 1 and equality in (22), there exists an orthogonal matrix $P$ such that

$$X = P \bigoplus_{i=1}^{k} \left( L(G_i) - \frac{2m_i}{n_i} I_i \right) \quad \text{and} \quad Y = P \bigoplus_{i=1}^{k} b_i I_i$$

are both positive semidefinite. Hence $P^T X$ and $P^T Y$ are polar decompositions of the matrices

$$\bigoplus_{i=1}^{k} \left( L(G_i) - \frac{2m_i}{n_i} I_i \right) \quad \text{and} \quad \bigoplus_{i=1}^{k} b_i I_i$$

respectively. By Theorem 3, we arrive at $Y = \bigoplus_{i=1}^{k} |b_i| I_i$. Thus

$$\bigoplus_{i=1}^{k} |b_i| I_i \equiv P \bigoplus_{i=1}^{k} b_i I_i. \quad (24)$$

We can write the orthogonal matrix $P$ as

$$P = \begin{pmatrix}
    P_{11} & P_{12} & \cdots & P_{1k} \\
    P_{21} & P_{22} & \cdots & P_{2k} \\
    \vdots & \ddots & \ddots & \vdots \\
    P_{k1} & \cdots & \cdots & P_{kk}
\end{pmatrix}, \quad (25)$$
with the diagonal matrices $P_{jj}$, $j = 1, \ldots, k$, of order $n_j$ respectively. From (24) we have
\[
\begin{pmatrix}
|b_1| I_1 & 0 & \cdots & 0 \\
0 & |b_2| I_2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & |b_k| I_k
\end{pmatrix}
= \begin{pmatrix}
P_{11} & P_{12} & \cdots & P_{1k} \\
P_{21} & P_{22} & \cdots & P_{2k} \\
\vdots & \ddots & \ddots & \vdots \\
P_{k1} & \cdots & P_{kk}
\end{pmatrix}
\begin{pmatrix}
|b_1| I_1 & 0 & \cdots & 0 \\
0 & |b_2| I_2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & |b_k| I_k
\end{pmatrix}
\]
and then
\[
\begin{pmatrix}
|b_1| I_1 & 0 & \cdots & 0 \\
0 & |b_2| I_2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & |b_k| I_k
\end{pmatrix}
= \begin{pmatrix}
b_1P_{11} & P_{12} & \cdots & P_{1k} \\
b_1P_{21} & P_{22} & \cdots & P_{2k} \\
\vdots & \ddots & \ddots & \vdots \\
b_1P_{k1} & \cdots & P_{kk}
\end{pmatrix}
\begin{pmatrix}
b_1P_{11} & P_{12} & \cdots & P_{1k} \\
b_1P_{21} & P_{22} & \cdots & P_{2k} \\
\vdots & \ddots & \ddots & \vdots \\
b_1P_{k1} & \cdots & P_{kk}
\end{pmatrix}
\]
As $b_1 = 2m_1/n_1 - 2m/n > 0$, via (26) we obtain $P_{11} = I_1$ and $P_{j1} = 0$, $j = 2, \ldots, k$. Substituting these $P_{j1}$ into (25) and then replacing the matrix $P$ in the equality $X = P \bigoplus_{i=1}^k (L(G_i) - (2m_i/n_i)I_i)$, we conclude that $L(G_1) - (2m_1/n_1)I_1$ is a positive semidefinite matrix. Now we have the required contradiction since
\[-\frac{2m_1}{n_1} \in \sigma \left( L(G_1) - \frac{2m_1}{n_1} I_1 \right).\]
Hence the assertion follows. ■

References


