DISTANCE EQUIENERGETIC GRAPHS

H. S. Ramane, a I. Gutman b and D. S. Revankar a

a Department of Mathematics, Gogte Institute of Technology, Udyambag, Belgaum - 590008, India
e-mail: hsramane@yahoo.com, revankards@rediffmail.com

b Faculty of Science, University of Kragujevac, P. O. Box 60, 34000 Kragujevac, Serbia
e-mail: gutman@kg.ac.yu

(Received December 2, 2007)

Abstract

The distance energy $E_D(G)$ of a graph $G$ is defined as the sum of the absolute values of the eigenvalues of the distance matrix of $G$. The graphs $G_1$ and $G_2$ are said to be distance equienergetic ($D$-equienergetic) if $E_D(G_1) = E_D(G_2)$. In this paper we obtain the eigenvalues of the distance matrix of the join of two graphs whose diameter is less than or equal to 2, and construct pairs of non $D$-cospectral, $D$-equienergetic graphs on $n$ vertices for all $n \geq 9$. 
INTRODUCTION

In this paper we are concerned with simple graphs, that is graphs without loops, multiple edges or directed edges. Let $G$ be such graph on $n$ vertices and $m$ edges. Let its vertices be labelled as $v_1, v_2, \ldots, v_n$. The distance between the vertices $v_i$ and $v_j$, denoted by $d_{ij}$, is the length of the shortest path between them. The diameter of a graph $G$, denoted by $\text{diam}(G)$, is the maximum distance between any pair of vertices of $G$ [4,12].

The distance matrix of a graph $G$ is an $n \times n$ matrix $D(G) = [d_{ij}]$. The characteristic polynomial of $D(G)$ is defined as $\psi(G : \mu) = \det(\mu I - D(G))$, where $I$ is the identity matrix of order $n$. The eigenvalues of the distance matrix $D(G)$, denoted by $\mu_1, \mu_2, \ldots, \mu_n$, are said to be the distance or $D$-eigenvalues of $G$ and their collection is called the distance or $D$-spectrum of $G$. Two non-isomorphic graphs are said to be $D$-cospectral if they have same $D$-spectra [4,5,6]. Since the distance matrix is symmetric, its eigenvalues are real and can be ordered as $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$.

The characteristic polynomial and the eigenvalues of the distance matrix of a graph were considered in [7–9,13,14,29].

The distance energy $E_D(G)$ of a graph $G$ is defined as

$$E_D(G) = \sum_{i=1}^{n} |\mu_i| .$$

Eq. (1) was recently introduced by Indulal et al. [15] and was conceived in full analogy to the ordinary graph energy $E(G)$ defined as [10,11]

$$E(G) = \sum_{i=1}^{n} |\lambda_i|$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the adjacency matrix of $G$ [6].

Several bounds for the distance energy of a graph are obtained in [15,20,21].

The graphs $G_1$ and $G_2$ are said to be equienergetic if $E(G_1) = E(G_2)$. Numerous results on (non-isomorphic) equienergetic graphs can be found in [1–3,16–19,23–28].

The connected graphs $G_1$ and $G_2$ are said to be $D$-equienergetic (distance equienergetic) if $E_D(G_1) = E_D(G_2)$. For obvious reason $D$-cospectral graphs are $D$-equienergetic. Therefore we are interested in non $D$-cospectral, $D$-equienergetic graphs having equal number of vertices. Indulal et al. [15] constructed pairs of $D$-equienergetic
graphs on \( n \) vertices for \( n \equiv 1 \pmod{3} \) and for \( n \equiv 0 \pmod{6} \). Ramane, Revankar, Gutman and Walikar [22] proved that if \( G_1 \) and \( G_2 \) are \( r \)-regular graphs on \( n \) vertices and \( diam(G_i) \leq 2, i = 1, 2 \), then \( E_D(L^k(G_1)) = E_D(L^k(G_2)) \) for \( k \geq 1 \), where \( L^k(G) \) is the \( k \)-th iterated line graph of \( G \).

In this paper we obtain the characteristic polynomial of the distance matrix of the join of two regular graphs whose diameter is less than or equal to 2 and thereby construct pairs of non \( D \)-cospectral, \( D \)-equienergetic graphs on \( n \) vertices for all \( n \geq 9 \).

**ON THE JOIN OF GRAPHS**

**Definition** [6, 12]. The join of two graphs \( G_1 \) and \( G_2 \), denoted by \( G_1 \square G_2 \), is a graph obtained from \( G_1 \) and \( G_2 \) by joining each vertex of \( G_1 \) to all vertices of \( G_2 \).

![Fig. 1](https://example.com/f1.png)

**Theorem 1.** Let \( G_i \) be an \( r_i \)-regular graph on \( n_i \) vertices and \( diam(G_i) \leq 2, i = 1, 2 \). Then the characteristic polynomial of the distance matrix of \( G_1 \square G_2 \) is

\[
\psi(G_1 \square G_2 : \mu) = \frac{[(\mu - 2n_1 + 2 + r_1)(\mu - 2n_2 + 2 + r_2) - n_1 n_2]}{(\mu - 2n_1 + 2 + r_1)(\mu - 2n_2 + 2 + r_2)} \psi(G_1 : \mu) \psi(G_2 : \mu). \tag{3}
\]

**Proof.**

\[
\psi(G_1 \square G_2 : \mu) = \det(\mu I - D(G_1 \square G_2))
= \begin{vmatrix}
\mu I_{n_1} - D(G_1) & -J_{n_1 \times n_2} \\
-J_{n_2 \times n_1} & \mu I_{n_2} - D(G_2)
\end{vmatrix} \tag{4}
\]

where \( J \) is a matrix whose all entries are equal to unity.
The determinant (4) can be written as

\[
\begin{vmatrix}
\mu & -d_{12} & \ldots & -d_{1n_1} & -1 & -1 & \ldots & -1 \\
-d_{21} & \mu & \ldots & -d_{2n_1} & -1 & -1 & \ldots & -1 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\
-d_{n_{11}} & -d_{n_{12}} & \ldots & \mu & -1 & -1 & \ldots & -1 \\
-1 & -1 & \ldots & -1 & \mu & -d'_{12} & \ldots & -d'_{1n_2} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & \ldots & -1 & -d'_{n_{21}} & -d'_{n_{22}} & \ldots & \mu \\
\end{vmatrix}
\]

(5)

where \(d_{ij}\) is the distance between the vertices \(v_i\) and \(v_j\) in \(G_1\) and \(d'_{ij}\) is the distance between the vertices \(u_i\) and \(u_j\) in \(G_2\). In \(G_i\), every vertex is at distance one from \(r_i\) vertices and at distance two from remaining \(n_i - 1 - r_i\) vertices. Therefore

\[
\sum_{j=1}^{n_1} d_{ij} = 2n_1 - r_1 - 2 \quad \text{for } i = 1, 2, \ldots, n_1
\]

(6)

and

\[
\sum_{j=1}^{n_2} d'_{ij} = 2n_2 - r_2 - 2 \quad \text{for } i = 1, 2, \ldots, n_2.
\]

(7)

We now perform the number of transformations that leave the value of the determinant (5) unchanged.

Subtract the row \((n_1 + 1)\) from the rows \((n_1 + 2), (n_1 + 3), \ldots, (n_1 + n_2)\) of (5) to obtain (8):

\[
\begin{vmatrix}
\mu & -d_{12} & \ldots & -d_{1n_1} & -1 & -1 & \ldots & -1 \\
-d_{21} & \mu & \ldots & -d_{2n_1} & -1 & -1 & \ldots & -1 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\
-d_{n_{11}} & -d_{n_{12}} & \ldots & \mu & -1 & -1 & \ldots & -1 \\
-1 & -1 & \ldots & -1 & \mu & -d'_{12} & \ldots & -d'_{1n_2} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & -d'_{21} - \mu & \mu + d'_{12} & \ldots & -d'_{2n_2} + d'_{1n_2} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & -d'_{n_{21}} - \mu & -d'_{n_{22}} + d'_{12} & \ldots & \mu + d'_{1n_2} \\
\end{vmatrix}
\]

(8)

Adding the columns \((n_1 + 2), (n_1 + 3), \ldots, (n_1 + n_2)\) to the column \((n_1 + 1)\) of (8),
using Eq. (7), and noting that \( d'_{ij} = d''_{ji} \) we arrive at the determinant (9):

\[
\begin{vmatrix}
\mu & -d_{12} & \ldots & -d_{1n} & -n_2 & -1 & \ldots & -1 \\
-d_{21} & \mu & \ldots & -d_{2n} & -n_2 & -1 & \ldots & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-d_{n1} & -d_{n2} & \ldots & \mu & -n_2 & -1 & \ldots & -1 \\
-1 & -1 & \ldots & -1 & \mu - 2n_2 + 2 + r_2 & -d'_{12} & \ldots & -d'_{1n} \\
0 & 0 & \ldots & 0 & 0 & \mu + d'_{12} & \ldots & -d''_{2n} + d''_{1n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 0 & -d''_{n2} + d'_{12} & \ldots & \mu + d'_{1n}
\end{vmatrix}
\]

which evidently is equal to (10):

\[
\begin{vmatrix}
\mu & -d_{12} & \ldots & -d_{1n} & -n_2 \\
-d_{21} & \mu & \ldots & -d_{2n} & -n_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-d_{n1} & -d_{n2} & \ldots & \mu & -n_2 \\
-1 & -1 & \ldots & -1 & \mu - 2n_2 + 2 + r_2
\end{vmatrix}
\]

where

\[
|B| = \begin{vmatrix}
\mu + d'_{12} & -d_3 + d'_{13} & \ldots & -d''_{2n} + d''_{1n} \\
-d''_{32} + d''_{12} & \mu + d'_{13} & \ldots & -d''_{3n} + d''_{1n} \\
\vdots & \vdots & \ddots & \vdots \\
-d''_{n2} + d'_{12} & -d''_{n3} + d''_{13} & \ldots & \mu + d''_{1n}
\end{vmatrix}
\]

In (10) the determinant is of order \((n_1 + 1)\). Subtract the first row from the rows 2, 3, \ldots, \(n_1\), to obtain (12):

\[
\begin{vmatrix}
\mu & -d_{12} & \ldots & -d_{1n} & -n_2 \\
-d_{21} - \mu & \mu + d_{12} & \ldots & -d_{2n} + d_{1n} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-d_{n1} - \mu & -d_{n2} + d_{12} & \ldots & \mu + d_{1n} & 0 \\
-1 & -1 & \ldots & -1 & \mu - 2n_2 + 2 + r_2
\end{vmatrix}
\]

Adding columns 2, 3, \ldots, \(n_1\) to the first column of (12) and using Eq. (6) we get (13):

\[
\begin{vmatrix}
\mu - 2n_1 + 2 + r_1 & -d_{12} & \ldots & -d_{1n} & -n_2 \\
0 & \mu + d_{12} & \ldots & -d_{2n} + d_{1n} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & -d_{n2} + d_{12} & \ldots & \mu + d_{1n} & 0 \\
-n_1 & -1 & \ldots & -1 & \mu - 2n_2 + 2 + r_2
\end{vmatrix}
\]

Expand it along the first column to obtain (14):

\[
\{(\mu - 2n_1 + 2 + r_1) \Delta_1 - (-1)^{n_1} n_1 \Delta_2\} |B|
\]
where

\[ \Delta_1 := \begin{vmatrix} \mu + d_{12} & -d_{23} + d_{13} & \ldots & -d_{2n_1} + d_{1n_1} & 0 \\ -d_{32} + d_{12} & \mu + d_{13} & \ldots & -d_{3n_1} + d_{1n_1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -d_{n_12} + d_{12} & -d_{n_13} + d_{13} & \ldots & \mu + d_{1n_1} & 0 \\ -1 & -1 & \ldots & -1 & \mu - 2n_2 + 2 + r_2 \end{vmatrix} \]

and

\[ \Delta_2 = \begin{vmatrix} -d_{12} & -d_{13} & \ldots & -d_{1n_1} & -n_2 \\ \mu + d_{12} & -d_{23} + d_{13} & \ldots & -d_{2n_1} + d_{1n_1} & 0 \\ -d_{32} + d_{12} & \mu + d_{13} & \ldots & -d_{3n_1} + d_{1n_1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -d_{n_12} + d_{12} & -d_{n_13} + d_{13} & \ldots & \mu + d_{1n_1} & 0 \end{vmatrix} . \]

The expression (14) can be rewritten as

\[ \left\{ (\mu - 2n_1 + 2 + r_1)(\mu - 2n_2 + 2 + r_2) \right\} \cdot A - n_1 n_2 \cdot \left\| A \right\| \cdot B \]

\[ = \left\| A \right\| \cdot \left\{ (\mu - 2n_1 + 2 + r_1)(\mu - 2n_2 + 2 + r_2) - n_1 n_2 \right\} \]  \hspace{1cm} (15)

where

\[ |A| = \begin{vmatrix} \mu + d_{12} & -d_{23} + d_{13} & \ldots & -d_{2n_1} + d_{1n_1} \\ -d_{32} + d_{12} & \mu + d_{13} & \ldots & -d_{3n_1} + d_{1n_1} \\ \vdots & \vdots & \ddots & \vdots \\ -d_{n_12} + d_{12} & -d_{n_13} + d_{13} & \ldots & \mu + d_{1n_1} \end{vmatrix} . \]  \hspace{1cm} (16)

The determinant (16) can be written as

\[ |A| = \frac{1}{(\mu - 2n_1 + 2 + r_1)} \times \begin{vmatrix} \mu - 2n_1 + 2 + r_1 & -d_{12} & -d_{13} & \ldots & -d_{1n_1} \\ 0 & \mu + d_{12} & -d_{23} + d_{13} & \ldots & -d_{2n_1} + d_{1n_1} \\ 0 & -d_{32} + d_{12} & \mu + d_{13} & \ldots & -d_{3n_1} + d_{1n_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -d_{n_12} + d_{12} & -d_{n_13} + d_{13} & \ldots & \mu + d_{1n_1} \end{vmatrix} . \]  \hspace{1cm} (17)

From Eq. (6) the sum of the \( i \)-th row in (17) is \( \mu + d_{i1} \) for \( i = 2, 3, \ldots, n_1 \). Therefore, by subtracting the columns 2, 3, \ldots, \( n_1 \) of (17) from the first column, we
obtain (18):

$$|A| = \frac{1}{(\mu - 2n_1 + 2 + r_1)} \times$$

$$\begin{vmatrix}
\mu & -d_{12} & -d_{13} & \ldots & -d_{1n_1} \\
-\mu - d_{21} & \mu + d_{12} & -d_{23} + d_{13} & \ldots & -d_{2n_1} + d_{1n_1} \\
-\mu - d_{31} & -d_{32} + d_{12} & \mu + d_{13} & \ldots & -d_{3n_1} + d_{1n_1} \\
\vdots \\
-\mu - d_{n_11} & -d_{n_12} + d_{12} & -d_{n_13} + d_{13} & \ldots & \mu + d_{1n_1}
\end{vmatrix} \quad (18)$$

Add the first row of (18) to the rows 2, 3, \ldots, \ n_1 to obtain (19):

$$|A| = \frac{1}{(\mu - 2n_1 + 2 + r_1)} \psi(G_1 : \mu). \quad (19)$$

In a similar manner we can show that from (11) follows

$$|B| = \frac{1}{(\mu - 2n_2 + 2 + r_2)} \psi(G_2 : \mu). \quad (20)$$

Substituting (19) and (20) back into (15) yields Eq. (3).

**Theorem 2.** Let $G_i$ be an $r_i$-regular graph on $n_i$ vertices and $\text{diam}(G_i) \leq 2$, $i = 1, 2$. Then

$$E_D(G_1 \nabla G_2) = E_D(G_1) + E_D(G_2), \quad \text{if} \quad XY \geq n_1 n_2$$

$$= E_D(G_1) + E_D(G_2) - (X + Y) + \sqrt{(X + Y)^2 - 4(XY - n_1 n_2)}, \quad \text{if} \quad XY < n_1 n_2$$

where $X = 2n_1 - 2 - r_1$ and $Y = 2n_2 - 2 - r_2$.

**Proof.** If $G_i$ is an $r_i$-regular graph on $n_i$ vertices and $\text{diam}(G_i) \leq 2$, $i = 1, 2$, then from Theorem 1,

$$\psi(G_1 \nabla G_2 : \mu) = \frac{[(\mu - 2n_1 + 2 + r_1)(\mu - 2n_2 + 2 + r_2) - n_1 n_2]}{(\mu - 2n_1 + 2 + r_1)(\mu - 2n_2 + 2 + r_2)} \psi(G_1 : \mu) \psi(G_2 : \mu)$$

$$= \frac{[(\mu - X)(\mu - Y) - n_1 n_2]}{(\mu - X)(\mu - Y)} \psi(G_1 : \mu) \psi(G_2 : \mu)$$
which gives that
\[(\mu - X)(\mu - Y)\psi(G_1 \nabla G_2 : \mu) = (\mu - X)(\mu - Y)\psi(G_1 : \mu)\psi(G_2 : \mu) .\]

Let
\[P_1(\mu) = (\mu - X)(\mu - Y)\psi(G_1 \nabla G_2 : \mu)\]
and
\[P_2(\mu) = (\mu - X)(\mu - Y)\psi(G_1 : \mu)\psi(G_2 : \mu) .\]
The roots of the equation \(P_1(\mu) = 0\) are \(X, Y\) and the \(D\)-eigenvalues of \(G_1 \nabla G_2\).
Therefore the sum of the absolute values of the roots of \(P_1(\mu) = 0\) is
\[X + Y + E_D(G_1 \nabla G_2) .\]

The roots of \(P_2(\mu) = 0\) are the \(D\)-eigenvalues of \(G_1\) and \(G_2\) and
\[\frac{1}{2} \left( X + Y \pm \sqrt{(X + Y)^2 - 4(XY - n_1 n_2)} \right) .\]
Therefore the sum of the absolute values of the roots of \(P_2(\mu) = 0\) is
\[E_D(G_1) + E_D(G_2) + \frac{1}{2} \left[ X + Y + \sqrt{(X + Y)^2 - 4(XY - n_1 n_2)} \right] + \frac{1}{2} \left[ X + Y - \sqrt{(X + Y)^2 - 4(XY - n_1 n_2)} \right] .\]

Since \(P_1(\mu) = P_2(\mu)\), equating Eqs. (21) and (22) we get
\[E_D(G_1 \nabla G_2) = E_D(G_1) + E_D(G_2) - (X + Y) + \frac{1}{2} \left[ X + Y + \sqrt{(X + Y)^2 - 4(XY - n_1 n_2)} \right] + \frac{1}{2} \left[ X + Y - \sqrt{(X + Y)^2 - 4(XY - n_1 n_2)} \right] .\]

Case 1: If \(XY \geq n_1 n_2\), then Eq. (23) reduces to
\[E_D(G_1 \nabla G_2) = E_D(G_1) + E_D(G_2) .\]

Case 2: If \(XY < n_1 n_2\), then Eq. (23) reduces to
\[E_D(G_1 \nabla G_2) = E_D(G_1) + E_D(G_2) - (X + Y) + \sqrt{(X + Y)^2 - 4(XY - n_1 n_2)} .\]
This completes the proof. ■

**Corollary 2.1.** If $H_1$ and $H_2$ are non $D$-cospectral, $D$-equienergetic regular graphs on $n$ vertices and of same degree and $diam(H_i) \leq 2$, $i = 1, 2$, then for any regular graph $G$ with $diam(G) \leq 2$, $E_D(H_1 \nabla G) = E_D(H_2 \nabla G)$. ■

**CONSTRUCTION OF DISTANCE EQUIENERGETIC GRAPHS**

**Theorem 3.** There exist pairs of non $D$-cospectral, $D$-equienergetic graphs on $n$ vertices for all $n \geq 9$.

**Proof.** Consider the graphs $H_a$ and $H_b$ as shown in Fig. 2.

![Fig. 2](image)

By direct computation,

$$
\psi(H_a: \mu) = (\mu - 12)(\mu + 3)^4 \mu^4
$$

(24)

and

$$
\psi(H_b: \mu) = (\mu - 12)(\mu + 4)(\mu + 3)^2 (\mu + 1)^2 \mu^3.
$$

(25)

Both $H_a$ and $H_b$ are regular graphs on 9 vertices and of degree 4. Also $diam(H_i) \leq 2$, $i = a, b$, and $E_D(H_a) = 24 = E_D(H_b)$.

Let $H$ be any $r$-regular graph on $p \geq 1$ vertices and $diam(H) \leq 2$. Then by Theorem 2

$$
E_D(H_a \nabla H) = E_D(H_b \nabla H) = 24 + E_D(H) \quad \text{if} \quad 5p \geq 8 + 4r
$$
and

\[ E_D(H_a \nabla H) = E_D(H_b \nabla H) = E_D(H) + 14 - 2p + r \]
\[ + \sqrt{(10 + 2p - r)^2 - 12(5p - 8 - 4r)} \quad \text{if } 5p < 8 + 4r. \]

Thus from both cases, \( H_a \nabla H \) and \( H_b \nabla H \) are \( D \)-equienergetic. By Eqs. (24) and (25), \( H_a \) and \( H_b \) are non \( D \)-cospectral, so from Theorem 1, \( H_a \nabla H \) and \( H_b \nabla H \) are also non \( D \)-cospectral. Further \( H_a \nabla H \) and \( H_b \nabla H \) possesses equal number of vertices \( n = 9 + p, p = 1, 2, \ldots \).

That the theorem holds also for \( n = 9 \) is directly verified from Eqs. (24) and (25).

Let \( K_p \) be the complete graph on \( p \) vertices. It is regular of degree \( p - 1 \) and \( diam(K_p) = 1 \). The adjacency matrix and the distance matrix of \( K_p \) are same. Therefore \( E_D(K_p) = E(K_p) = 2(p - 1) \) [6,10]. Using this in Theorem 2 we have following result.

**Theorem 4.** If \( H_a \) and \( H_b \) are the graphs as shown in Fig. 2, then

\[ E_D(H_a \nabla K_p) = E_D(H_b \nabla K_p) = 2(p + 11) \quad \text{if } p \geq 4 \]

and

\[ E_D(H_a \nabla K_p) = E_D(H_b \nabla K_p) = p + 11 + \sqrt{(p + 11)^2 - 12(p - 4)} \quad \text{if } p < 4. \]

**Conclusion.** From Corollary 2.1 it is easy to construct a pair of non \( D \)-cospectral, \( D \)-equienergetic graphs. In particular from Theorem 3 and Theorem 4, it is easy to construct a pair of non \( D \)-cospectral, \( D \)-equienergetic \( n \)-vertex graphs for all \( n \geq 9 \).

**References**


