REVERSE WIENER INDICES OF CONNECTED GRAPHS

Xiaochun Cai and Bo Zhou

Department of Mathematics, South China Normal University, Guangzhou 510631, P. R. China
e-mail: zhoubo@scnu.edu.cn

(Received September 20, 2007)

Abstract

We provide upper and lower bounds on the reverse Wiener index for connected graphs with given numbers of vertices, edges and diameter, and determine respectively the $n$-vertex trees of fixed number of pendent vertices, the $n$-vertex trees of fixed maximum degree, the $n$-vertex trees of fixed matching number, and the $n$-vertex trees of a given bipartition with greatest reverse Wiener indices. We also consider the Nordhaus–Gaddum–type result for the reverse Wiener index.

1. INTRODUCTION

We consider simple graphs. The Wiener index $W(G)$ of a connected graph $G$ is the sum of distances between all unordered pairs of vertices of $G$ [1]. It is one of the most thoroughly studied molecular–graph–based structure–descriptors, see, e.g., [2, 3, 4]. Its mathematical properties and its use in the structure–property–activity
modeling can be found in [5–11]. However, its degeneracy and low discriminating power have resulted in lack of unambiguousness and uniqueness in its properties [12].

With the aim to overcome the shortcomings of Wiener index, a number of mathematical chemists have come up with the modifications, extensions and variants of this index. Randić et al. [13] introduced a novel Wiener matrix, for its potential utilization in structure–property studies. Gutman et al. [14] introduced a multiplicative version of Wiener index, $\pi(G)$, which is equal to the product of distances between all pairs of vertices of $G$, and it has also been reported that in the case of alkanes, $\pi$ and $W$ are highly correlated. Ivanciuc et al. [15] introduced Wiener index extension by counting even/odd graph distances. Balaban et al. [16] proposed a novel structure–descriptor, the reverse Wiener index.

Let $G$ be a connected graph with $n$ vertices. Then the reverse Wiener index of $G$ is defined as [16]

$$\Lambda = \Lambda(G) = \frac{1}{2}n(n - 1)d - W(G)$$

where $d$ is the diameter of $G$. In [16], general formulae for $\Lambda$ were presented for several classes of graphs, including complete graph, star, path, cycle and linear polyacenes, relationships between $\Lambda$ and other structure–descriptors, especially Wiener index, were discussed, and QSPR investigations demonstrated the usefulness of this index. Ivanciuc et al. [17] have shown that $\Lambda$ is able to produce fair QSPR models for standard Gibbs energy of formation and refractive index for $C_6$–$C_{10}$ alkanes.

Let $P_n$ and $S_n$ be respectively the $n$-vertex path and $n$-vertex star. Let $T$ be a tree with $n > 4$ vertices, different from $S_n$ and $P_n$. Zhang and Zhou [18] reported that $\Lambda(S_n) < \Lambda(T) < \Lambda(P_n)$. Thus the reverse Wiener index can be used as a branching index.

In this paper, we establish some further properties of the reverse Wiener index of a connected graph. We provide upper and lower bounds on the reverse Wiener index for connected graphs with given numbers of vertices, edges and diameter. Moreover we characterize trees that have maximum reverse Wiener index within some classes of trees. Besides the number of vertices, these classes are specified by the number of pendent vertices, maximum degree, matching number and the numbers of vertices in the bipartition, respectively. We also consider the Nordhaus–Gaddum–type result [19] for the reverse Wiener index.

2. PROPERTIES OF $\Lambda$ FOR GENERAL GRAPHS

In this section, we present some properties of the reverse Wiener index of a connected graph. A pendent vertex is a vertex of degree one. Let $K_n$ denote the complete graph with $n$ vertices.
Theorem 1. Let $G$ be a connected graph with $n \geq 2$ vertices, $m$ edges and diameter $d$. Then
\[(d-1)m \leq \Lambda(G) \leq \frac{n(n-1)}{2}(d-2) + m\]
with either equality if and only if $d \leq 2$.

Proof: Note that
\[m + 2 \left[ \frac{n(n-1)}{2} - m \right] \leq W(G) \leq m + d \left[ \frac{n(n-1)}{2} - m \right]\]
i.e.,
\[n(n-1) - m \leq W(G) \leq \frac{n(n-1)}{2}d - (d-1)m\]
with either equality if and only if $d \leq 2$. Now the result follows by the definition of $\Lambda(G)$. \ \square

Let $\mathcal{G}$ be a class of connected graphs with $n$ vertices, in which every graph possesses $f(n)$ edges, where $f(n)$ is a function of $n$ with $n - 1 \leq f(n) < \frac{n(n-1)}{2}$. By Theorem 1, for any $G \in \mathcal{G}$, $\Lambda(G) \geq f(n)$ with equality if and only if the diameter of $G$ is 2. In particular, if $f(n) = n - 1$, i.e., $G$ is a tree with $n$ vertices, then $\Lambda(G) \geq n - 1$ with equality if and only if $G = S_n$, while if $f(n) = n$, i.e., $G$ is a unicyclic graph with $n$ vertices, then for $n \geq 4$, $\Lambda(G) \geq n$ with equality if and only if $G$ is a quadrangle, a pentagon, or the the graph obtained by attaching $n - 3$ pendent vertices to a vertex of a triangle.

From [20, Theorem 2], we have

Theorem 2. Let $G$ be a graph with $n$ vertices and diameter $d$. Then
\[\Lambda(G) \leq \frac{n(n-1)d}{2} - \frac{d(d+1)(d+2)}{6} - \frac{n - d - 1}{2} \left( n + \left\lfloor \frac{d^2 + 1}{2} \right\rfloor \right)\]
with equality if and only if there is a vertex $v_0$ such that the distance layers $V_i$, where $V_i$ is a subset of the vertex set consisting of the vertices that are at distance $i$ from $v_0$ for $i = 0, 1, \ldots, d$, fulfill the condition that the subgraphs induced $V_{i-1} \cup V_i$ are complete whenever $1 \leq i \leq d$ and all noncentral layers are trivial.

Corollary 3. Let $G$ be a connected graph with $n \geq 2$ vertices. Then
\[0 \leq \Lambda(G) \leq \frac{n(n-1)(n-2)}{3}\]
with left equality if and only if $G = K_n$, and with right equality if and only if $G = P_n$.

Proof. Let $m$ and $d$ be respectively the number of edges and the diameter of $G$. Note that $d \geq 1$ with equality if and only if $G = K_n$. By Theorem 1, $\Lambda(G) \geq 0$ with equality if and only if $G = K_n$.\[\]
Let
\[ F(n, d) = \frac{n(n-1)d}{2} - \frac{d(d+1)(d+2)}{6} - \frac{n-d-1}{2} \left( n + \left\lfloor \frac{d^2 + 1}{2} \right\rfloor \right), \]
where \( 1 \leq d \leq n - 1 \). If \( d = n - 1 \), then \( G = P_n \) and so \( \Lambda(G) = \frac{n(n-1)(n-2)}{3} \). Suppose that \( 1 \leq d \leq n - 2 \). If \( d \) is even, then
\[
F(n, d + 1) - F(n, d) = \frac{n(n-d)}{2} + \frac{d^2}{4} - \frac{n}{2} > 0.
\]
If \( d \) is odd, then
\[
F(n, d + 1) - F(n, d) = \frac{n(n-d)}{2} - \frac{3}{4} + \frac{d^2}{4} - \frac{d}{2} > 0.
\]
In either case, we have \( F(n, d) < F(n, d + 1) \). Now by Theorem 2, we have
\[
\Lambda(G) \leq F(n, d) < F(n, d + 1) \leq \cdots \leq F(n, n - 1) = \Lambda(P_n).
\]
Thus \( \Lambda(G) \leq \Lambda(P_n) \) with equality if and only if \( G = P_n \). \( \square \)

**Corollary 4.** Let \( G \) be a connected bipartite graph with \( n \geq 3 \) vertices. Then
\[
n - 1 \leq \Lambda(G) \leq \frac{n(n-1)(n-2)}{3}
\]
with left equality if and only if \( G = S_n \), and with right equality if and only if \( G = P_n \).

**Proof.** Let \( m \) and \( d \) be respectively the number of edges and the diameter of \( G \). Note that \( d \geq 2 \) and \( m \geq n - 1 \) with both equalities if and only if \( G = S_n \). Now the result follows from Theorem 1 and Corollary 3. \( \square \)

### 3. PROPERTIES of \( \Lambda \) FOR TREES

Recall that a caterpillar is a tree in which removal of all pendent vertices gives a path. Let \( P_{n,d,i} \) be the caterpillar obtained from the path \( P_{d+1} \) labelled as \( v_0, v_1, \ldots, v_d \) by attaching \( n - d - 1 \) pendent vertices labelled consecutively as \( v_{d+1}, \ldots, v_{n-1} \) to the vertex \( v_i \) of the path, where \( 3 \leq d \leq n - 2 \) (see Fig. 1). Clearly, \( P_{n,d,i} \) has diameter \( d \) for any \( 1 \leq i \leq d - 1 \).

Fig. 1. The graph \( P_{n,d,i} \).
Lemma 5. [18] Let $T$ be a tree with $n$ vertices and diameter $d$, where $3 \leq d \leq n-2$. If $T \neq P_{n,d,\left\lfloor \frac{d}{2} \right\rfloor}$, then $\Lambda(T) < \Lambda \left( P_{n,d,\left\lfloor \frac{d}{2} \right\rfloor} \right)$.

Lemma 6. [18] For $3 \leq d \leq n-3$, $\Lambda \left( P_{n,d,\left\lfloor \frac{d}{2} \right\rfloor} \right) < \Lambda \left( P_{n,d+1,\left\lfloor \frac{d+1}{2} \right\rfloor} \right)$.

Let $T$ be a tree with $n$ vertices, $p$ of which are pendent vertices, where $2 \leq p \leq n-1$. Obviously, if $p = 2$ then $T = P_n$, and if $p = n-1$ then $T = S_n$. So we can assume that $3 \leq p \leq n-2$.

Theorem 7. Let $T$ be a tree with $n$ vertices, $p$ of which are pendent vertices, where $3 \leq p \leq n-2$. Then $\Lambda(T) \leq \Lambda \left( P_{n,n-p+1,\left\lfloor \frac{n-p+1}{2} \right\rfloor} \right)$ with equality if and only if $T = P_{n,n-p+1,\left\lfloor \frac{n-p+1}{2} \right\rfloor}$.

Proof. Let $d$ be the diameter of $T$. It is easy to see that $d \leq n-p+1$. If $d \leq n-p$, then by Lemmas 5 and 6, $\Lambda(T) \leq \Lambda \left( P_{n,d,\left\lfloor \frac{d}{2} \right\rfloor} \right) \leq \Lambda \left( P_{n,n-p,\left\lfloor \frac{n-p}{2} \right\rfloor} \right) < \Lambda \left( P_{n,n-p+1,\left\lfloor \frac{n-p+1}{2} \right\rfloor} \right)$. If $d = n-p+1$ and $T \neq P_{n,n-p+1,\left\lfloor \frac{n-p+1}{2} \right\rfloor}$, then by Lemma 5, $\Lambda(T) < \Lambda \left( P_{n,n-p+1,\left\lfloor \frac{n-p+1}{2} \right\rfloor} \right)$. Thus the result follows.

Let $T$ be a tree with maximum degree $\Delta$, where $2 \leq \Delta \leq n-1$. Obviously, if $\Delta = 2$ then $T = P_n$, and if $\Delta = n-1$ then $T = S_n$. So we can assume that $3 \leq \Delta \leq n-2$. Note that the diameter is at most $n-\Delta + 1$. Similar to the proof of Theorem 7, we have

Theorem 8. Let $T$ be a tree with maximum degree $\Delta$, where $3 \leq \Delta \leq n-2$. Then $\Lambda(T) \leq \Lambda \left( P_{n,n-\Delta+1,\left\lfloor \frac{n-\Delta+1}{2} \right\rfloor} \right)$ with equality if and only if $T = P_{n,n-\Delta+1,\left\lfloor \frac{n-\Delta+1}{2} \right\rfloor}$.

If $T$ is a tree with $n \geq 2$ vertices, then $[1, 2] W(T) = \sum_{e \in E(T)} W(e,T)$, where $E(T)$ is the edge set of $T$, $W(e,T) = n_{T,1}(e) \cdot n_{T,2}(e)$, $n_{T,1}(e)$ and $n_{T,2}(e)$ are the respectively numbers of vertices of $T$ lying on the two sides of the edge $e$. We will use this fact in the proof of Lemmas 9 and 11.

A matching is a set of pairwise non-adjacent edges. A maximum matching is a matching that contains the largest possible number of edges. The matching number of a graph $G$ is the size of a maximum matching, denoted by $\beta(G)$.

Let $T$ be a tree with $n$ vertices and matching number $k$, where $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$. If $k = 1$, then $T = S_n$. So we can assume that $k \geq 2$.

Lemma 9. For even $k$ with $2 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$, $\Lambda(P_{n,2k-1,k-1}) < \Lambda(P_{n,2k,k-1})$.

Proof. Note that

$$\Lambda(P_{n,2k,k-1}) - \Lambda(P_{n,2k-1,k-1})$$
\[= \frac{n(n - 1)}{2} - W(v_{k-1}v_k, P_{n,2k,k-1}) + W(v_{k-1}v_{n-1}, P_{n,2k-1,k-1}) \]
\[\geq \frac{n(n - 1)}{2} - \frac{n^2}{4} + W(v_{k-1}v_{n-1}, P_{n,2k-1,k-1}) > 0.\]

The result follows. \(\square\)

**Lemma 10.** For \(2 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1\), \(\beta(P_{n,2k,k}) = k\) if \(k\) is odd, and \(\beta(P_{n,2k,k-1}) = k\) if \(k\) is even.

For a tree \(T\) with at least two vertices and \(u \in V(T)\), \(d_u\) denotes the degree of \(u\) in \(T\). There are \(d_u\) components in \(T - u\), each containing a neighbor of \(u\) in \(T\). These components are called the branches of \(T\) at \(u\).

**Lemma 11.** Let \(T\) be a tree with \(n\) vertices and matching number \(k\), where \(k\) is even, \(2 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1\), and \(T \neq P_{n,2k,k-1}\). If the diameter of \(T\) is \(2k\), then \(\Lambda(T) < \Lambda(P_{n,2k,k-1})\).

**Proof.** Let \(P(T) = v_0v_1 \cdots v_d\) be a diametrical path of \(T\), where \(d = 2k\). There is a matching of size \(k\) in \(P(T)\). So \(T\) is a caterpillar.

If there exists an even \(i_0\), \(2 \leq i_0 \leq d - 2\), such that \(d_{v_{i_0}} \geq 3\), then \(v_{i_0}w \in E(T)\) for some \(w\) outside \(P(T)\), and so there is a matching of \(T\) containing the edge \(v_{i_0}w\) of size \(1 + \frac{i_0}{2} + \frac{d-i_0}{2} = k + 1\), a contradiction. Hence \(d_{v_i} = 2\) for all even \(i\) with \(2 \leq i \leq d - 2\).

Suppose that there are two vertices \(v_i, v_j\) of degree at least three such that the distance between \(v_i\) and \(v_j\) is as small as possible, where \(i\) and \(j\) are odd with \(1 \leq i < j \leq d - 1\). Then the vertices \(v_{i+1}, \ldots, v_{j-1}\) have equal degree two. Let \(n_1\) (resp. \(n_2\)) be the number of vertices of the branch at \(v_{i+1}\) (resp. \(v_{j-1}\)) containing \(v_i\) (resp. \(v_j\)). Then \(n_1 + n_2 + (j - i - 1) = n\). Assume that \(n_1 \geq n_2\). Let \(w\) be a pendant vertex adjacent to \(v_j\). Let \(T'\) denote the tree formed from \(T\) by deleting edge \(v_jw\) and adding edge \(v_1w\). It is easy to see that

\[\Lambda(T') - \Lambda(T) = W(T) - W(T')\]
\[= W(v_0v_{i+1}, T) - W(v_{j-1}v_j, T')\]
\[= n_1(n - n_1) - (n_2 - 1)(n - n_2 + 1).\]

Since \(n_2 - 1 < \min\{n_1, n - n_1\}\), we have \(\Lambda(T') > \Lambda(T)\). By iterating the transformation from \(T\) to \(T'\), we have \(\Lambda(T) < \Lambda(P_{n,2k,i})\) for some odd \(i\) with \(1 \leq i \leq 2k - 1\).

Now suppose that \(T = P_{n,2k,i}\) for some odd \(i\) with \(1 \leq i \leq 2k - 1\). Since \(T \neq P_{n,2k,k-1}\), we may assume that \(1 \leq i \leq k - 3\). It is easy to see that

\[\Lambda(P_{n,2k,i+2}) - \Lambda(P_{n,2k,i})\]
\[ W(v_i v_{i+1}, P_{n,2k,i}) - W(v_i v_{i+1}, P_{n,2k,i+2}) \]
+ \[ W(v_{i+1} v_{i+2}, P_{n,2k,i}) - W(v_{i+1} v_{i+2}, P_{n,2k,i+2}) \]
\[ \equiv [(n - 2k + i)(2k - i) - (i + 1)(n - i - 1)] \]
+ \[ [(n - 2k + i + 1)(2k - i - 1) - (i + 2)(n - i - 2)] . \]

Since \( i + 1 < \min \{n - 2k + i, 2k - i\} \) and \( i + 2 < \min \{n - 2k + i + 1, 2k - i - 1\} \), we have \( \Lambda(P_{n,2k,i+2}) > \Lambda(P_{n,k,i}) \). Iterating the procedure, we prove the lemma. \( \square \)

**Theorem 12.** Let \( T \) be a tree with \( n \) vertices and matching number \( k \), where \( 2 \leq k \leq \lfloor \frac{n}{2} \rfloor \).

(i) If \( k = \lfloor \frac{n}{2} \rfloor \), then \( \Lambda(T) \leq \Lambda(P_n) \) with equality if and only if \( T = P_n \).

(ii) If \( k \leq \lfloor \frac{n}{2} \rfloor - 1 \) and \( k \) is odd, then \( \Lambda(T) \leq \Lambda(P_{n,2k,k}) \) with equality if and only if \( T = P_{n,2k,k} \).

(iii) If \( k \leq \lfloor \frac{n}{2} \rfloor - 1 \) and \( k \) is even, then \( \Lambda(T) \leq \Lambda(P_{n,2k,k-1}) \) with equality if and only if \( T = P_{n,2k,k-1} \).

**Proof.** Note that \( \Lambda(T) < \Lambda(P_n) \) for any \( n \)-vertex tree different from \( P_n \) and that \( \beta(P_n) = \lfloor \frac{n}{2} \rfloor \). Hence (i) follows easily.

Suppose that \( k \leq \lfloor \frac{n}{2} \rfloor - 1 \). Let \( d \) be the diameter of \( T \). Obviously, \( k \geq \lfloor \frac{d+1}{2} \rfloor \), i.e., \( d \leq 2k \).

Suppose that \( k \) is odd. If \( d \leq 2k - 1 \), then by Lemmas 5 and 6, we have \( \Lambda(T) \leq \Lambda(P_{n,d,\lfloor \frac{d}{2} \rfloor}) \leq \Lambda(P_{n,2k-1,k-1}) < \Lambda(P_{n,2k,k}) \). If \( d = 2k \) and \( T \neq P_{n,2k,k} \), then by Lemma 5, \( \Lambda(T) < \Lambda(P_{n,2k,k}) \). Furthermore, by Lemma 10, we have \( \beta(P_{n,2k,k}) = k \). Hence (ii) follows.

Now suppose that \( k \) is even. If \( d \leq 2k - 1 \), then by Lemmas 5, 6 and 9, we have \( \Lambda(T) \leq \Lambda(P_{n,d,\lfloor \frac{d}{2} \rfloor}) \leq \Lambda(P_{n,2k-1,k-1}) < \Lambda(P_{n,2k,k-1}) \). If \( d = 2k \) and \( T \neq P_{n,2k,k-1} \), then by Lemma 11, \( \Lambda(T) < \Lambda(P_{n,2k,k-1}) \). Furthermore, by Lemma 10, we have \( \beta(P_{n,2k,k-1}) = k \). Hence (iii) follows. \( \square \)

The vertex set of a connected bipartite graph \( G \) with at least two vertices can be uniquely partitioned into two disjoint sets \( V_1 \) and \( V_2 \) such that all edges join a vertex in \( V_1 \) to a vertex in \( V_2 \). In this case we say that \( G \) has a \((|V_1|, |V_2|)\)-bipartition.

Let \( T \) be a tree with an \((s, n - s)\)-bipartition, where \( 1 \leq s \leq \lfloor \frac{n}{2} \rfloor \). If \( s = 1 \), then \( T = S_n \). So we can assume that \( s \geq 2 \). Note that \( P_n \) has the bipartition \((\lfloor \frac{n}{2} \rfloor, n - \lfloor \frac{n}{2} \rfloor)\) and that the diameter of \( T \) is at most \( 2s \) if \( s \leq \lfloor \frac{n}{2} \rfloor - 1 \). Similar to the proof of Theorem 12, we have
Theorem 13. Let $T$ be a tree with an $(s, n-s)$-bipartition, where $2 \leq s \leq \lfloor \frac{n}{2} \rfloor$.

(i) If $s = \lfloor \frac{n}{2} \rfloor$, then $\Lambda(T) \leq \Lambda(P_n)$ with equality if and only if $T = P_n$.

(ii) If $s \leq \lfloor \frac{n}{2} \rfloor - 1$ and $s$ is odd, then $\Lambda(T) \leq \Lambda(P_n, 2s, s)$ with equality if and only if $T = P_n, 2s, s$.

(iii) If $s \leq \lfloor \frac{n}{2} \rfloor - 1$ and $s$ is even, then $\Lambda(T) \leq \Lambda(P_n, 2s, s-1)$ with equality if and only if $T = P_n, 2s, s-1$.

4. THE NORDHAUS–GADDUM–TYPE RESULT FOR $\Lambda$

Nordhaus and Gaddum [19] reported bounds for the chromatic numbers of a graph and its complement. Eventually, Norhaus–Gaddum–type relations were established for many other graph invariants, see, e.g., [21]. Now we are ready to give bounds of this kind for the reverse Wiener index.

For simplicity, let $m(G)$ and $d(G)$ be respectively the number of edges and the diameter of the graph $G$.

Lemma 14. Let $G$ be a graph with $n \geq 6$ vertices. If $d(G) = d(\overline{G}) = 3$, then $\Lambda(G) + \Lambda(\overline{G}) \leq \frac{(n-1)(n-2)(2n+3)}{6}$.

Proof. Since $d(G) = d(\overline{G}) = 3$, we have $W(G) + W(\overline{G}) > m(G) + 2m(\overline{G}) + m(\overline{G}) + 2m(G) = \frac{3}{2}n(n-1)$. Thus

$$\Lambda(G) + \Lambda(\overline{G}) = \frac{1}{2}n(n-1) \cdot 6 - [W(G) + W(\overline{G})]$$

$$< 3n(n-1) - \frac{3}{2}n(n-1)$$

$$= \frac{3}{2}n(n-1) < \frac{(n-1)(n-2)(2n+3)}{6}.$$

The last inequality holds because $n \geq 6$. □

Lemma 15. Let $G$ be a graph of order $n \geq 5$. If $d(\overline{G}) = 2$, then $\Lambda(G) + \Lambda(\overline{G}) \leq \frac{(n-1)(n-2)(2n+3)}{6}$ with equality if and only if $G \cong P_n$.

Proof. Let $d = d(G)$. By Corollary 2, $\Lambda(G) \leq \Lambda(P_n)$ with equality if and only if $G = P_n$.

Since $n \geq 5$, we have $d(\overline{G}) = d(P_n) = 2$, and so $\Lambda(G) = m(\overline{G}) \leq \frac{n(n-1)}{2} - (n-1) = m(P_n) = \Lambda(P_n)$ with equality if and only if $G$ is a tree whose complement has diameter
2. Note that
\[\Lambda(P_n) + \Lambda(P_n) = \frac{n(n-1)(n-2)}{3} + \frac{n(n-1)}{2} - (n-1)\]
\[= \frac{(n-1)(n-2)(2n+3)}{6}.
\]
The result follows easily. □

**Remark.**
(i) There is exactly one pair of connected graphs \( G \) and \( \overline{G} \) with 4 vertices: \( P_4 \) and \( \overline{P}_4 = P_4 \). Obviously, \( d(P_4) = 3 \) and \( \Lambda(P_4) + \Lambda(P_4) = 16 \).

(ii) There are exactly five pair of connected graphs \( G \) and \( \overline{G} \) with 5 vertices, in which three pairs satisfy \( d(G) = d(\overline{G}) = 3 \): \( T \) and \( T \), \( U_1 \) and \( U_1 \), \( U_2 \) and \( U_2 = U_2 \), where \( T \) be the unique tree with 5 vertices and diameter 3, \( U_1 \) is the graph formed from \( T \) by adding an edge between its two pendent vertices with a common end vertex, and \( U_2 \) is formed from the path \( P_5 \) by adding a edge between the two neighbors of its center. The values of \( \Lambda(G) + \Lambda(\overline{G}) \) for them are respectively 27, 27 and 28. The two other pairs are \( P_5 \) and \( \overline{P}_5 \), \( C_5 \) and \( \overline{C}_5 = C_5 \). Note that \( \Lambda(P_5) + \Lambda(\overline{P}_5) = 26 \).

**Theorem 16.** Let \( G \) be a graph on \( n \geq 6 \) vertices with a connected \( \overline{G} \). Then
\[\frac{1}{2} n(n-1) \leq \Lambda(G) + \Lambda(\overline{G}) \leq \frac{(n-1)(n-2)(2n+3)}{6}\]
with left equality if and only if \( G \) and \( \overline{G} \) have equal diameter 2 and with right equality if and only if \( G = P_n \) or \( \overline{P}_n \).

**Proof.** By Theorem 1,
\[\Lambda(G) + \Lambda(\overline{G}) \geq m(G) + m(\overline{G}) = \frac{1}{2} n(n-1)\]
with equality if and only if \( G \) and \( \overline{G} \) have equal diameter 2.

If \( d(G) = d(\overline{G}) = 3 \), and by Lemma 14, we have \( \Lambda(G) + \Lambda(\overline{G}) < \frac{(n-1)(n-2)(2n+3)}{6} \).

If \( d(\overline{G}) = 2 \), then by Lemma 15, we have \( \Lambda(G) + \Lambda(\overline{G}) \leq \frac{(n-1)(n-2)(2n+3)}{6} \) with equality if and only if \( G = P_n \). Similarly, if \( d(G) = 2 \), then \( \Lambda(G) + \Lambda(\overline{G}) \leq \frac{(n-1)(n-2)(2n+3)}{6} \) with equality if and only if \( G = \overline{P}_n \).

Note that if \( d(\overline{G}) \geq 3 \) then \( d(G) \leq 3 \). The result follows. □

5. CONCLUSIONS

The Wiener index is a well-known measure of graph or network structures with similarly useful variant of the reverse Wiener index. In this paper, we establish some properties of the reverse Wiener index of a connected graph. In particular, we
provide (in Theorems 1 and 2) upper and lower bounds on the reverse Wiener index for connected graphs with given numbers of vertices, edges and diameter, we show (in Theorems 7, 8, 12 and 13) that $P_{n,n-p+1,\lceil \frac{n-p+1}{2} \rceil}$ is the unique tree with the greatest reverse Wiener index in the class of $n$-vertex trees with $p$ pendent vertices or with maximum degree $p$, where $3 \leq p \leq n - 2$, and $P_{n,k}$ for $k = \lfloor \frac{n}{2} \rfloor$, $P_{n,2k,k}$ for odd $k$ with $2 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$ and $P_{n,2k,k-1}$ for even $k$ with $2 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$ is the unique tree with the greatest reverse Wiener index in the class of $n$-vertex trees with matching number $k$ or with a $(k,n-k)$-bipartition, where $2 \leq k \leq \lfloor \frac{n}{2} \rfloor$ and we also give (in Theorem 16) the Nordhaus–Gaddum–type result for the reverse Wiener index.

Acknowledgement. This work was supported by the National Natural Science Foundation of China (No. 10671076).

References


