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On the Nullity of Bicyclic Graphs

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Abstract

The nullity of a graph is the multiplicity of the eigenvalues zero in its spectrum. In this papers, we obtain the nullity set of $n-$vertex bicyclic graphs, and characterize the bicyclic graphs with maximal nullity.

1 Introduction

Let G be a simple undirected graph with vertex set V and edge set E. The number of vertices of G is denoted by $v(G)$. For any $v \in V$, the degree and neighborhood of v are denoted by $d(v)$ and $N(v)$, respectively. If W is a subset of V, the subgraph induced by W is the subgraph of G by tacking the vertices in W and joining those pairs of vertices in W which are joined in G. We write $G-\{v_1, v, \ldots, v_k\}$ for the graph obtained from G by removing the vertices $v_1, v_2, \ldots, v_k$ and all edges incident to any of them. The disjoint union of two graphs G_1 and G_2 is denoted by $G_1 \cup G_2$. The null graph of order n is the

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graph with n vertices and no edges. As usual, the complete graph, cycle, path and star of order n denoted by K_n, C_n, P_n and S_n, respectively. An isolated vertex is sometimes denoted by K_1 .

The adjacency matrix $A(G)$ of a graph G with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ is the $n \times n$ symmetric matrix $[a_{ij}]$, such that $a_{ij} = 1$ if v_i and v_j are adjacent and 0, otherwise. A graph is said to be singular(nonsingular) if its adjacency matrix A is a singular(nonsingular) matrix. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $A(G)$ are said to be the eigenvalues of the graph G, and to form the spectrum of this graph. The number of zero eigenvalues in the spectrum of the graph G is called its nullity and is denoted by $\eta(G)$. Let $\gamma(A(G))$ be the rank of $A(G)$. Clearly, $\eta(G) = n - \gamma(A(G))$. The following result is obvious.

Proposition 1.1. Let G be a graph of order n. Then $\eta(G) = n$ if and only if G is a null graph.

Proposition 1.2. Let $G = G_1 \cup G_2 \cup \cdots \cup G_t$, where $G_1, G_2, \ldots, G_t$ are connected components of G. Then $\eta(G) = \sum_{i=1}^{t} \eta(G_i)$.

Definition 1.1[13]. An elementary graph is a simple graph, each component of which is regular has degree 1 or 2. In other words, each component is a single edge(K_2) or a cycle(C_r). A spanning elementary subgraph of G is an elementary subgraph which contains all vertices of G.

Proposition 1.3[13]. Let A be the adjacency matrix of a graph G. Then

$$detA = \sum (-1)^{r(H)} 2^{s(H)},$$

where the summation is over all spanning elementary subgraphs H of G.

In [1], Collatz and Sinogowitz first posed the problem of characterizing all graphs which satisfy $\eta(G) > 0$. This question is of great interest in both chemistry and mathematics. For a bipartite graph G, which correspond to an alternant hydrocarbon in chemistry, if $\eta(G) > 0$, it is indicated in [2] that the corresponding molecule is unstable. The nullity of a graph is also meaningful in mathematics since it is related to the singularity of $A(G)$. The problem has not yet been solved completely. Some results on trees, bipartite graphs and unicyclic graphs are known (see[2,3,4]). More recent results can be fond in [5-12].

For trees the following theorem gives a concise formula.

Theorem 1.1[3]. If T is a tree of order n and m is the size of its maximum matchings, then $\eta(T) = n - 2m$.

If a tree contains a perfect matching, we call it a PM-tree for convenience. In fact, Theorem 1.1 implies the following corollary.

Corollary 1.1. Let T be a tree of order n. The nullity $\eta(T)$ of T is zero if and only if T is a PM-tree.

A unicyclic graph is a simple connected graph with equal number of vertices and edges. Denote by $\mathcal{U}_n$ the set of all unicyclic graphs of order n.

Theorem 1.2[4]. For any $U \in \mathcal{U}_n$ $(n \geq 5)$, $\eta(U) \leq n-4$.

Let $\mathcal{G}_n$ be the set of all graphs of order n, and let $[0, n] = \{0, 1, 2, \ldots, n\}$. A subset N of $[0, n]$ is said to be the nullity set of $\mathcal{G}_n$ provided that for any $k \in N$, there exists at least one graph $G \in \mathcal{G}_n$ such that $\eta(G) = k$.

Theorem 1.3[4]. The nullity set of $\mathcal{U}_n (n \geq 5)$ is $[0, n-4]$.

Theorem 1.4[4]. Let $U \in \mathcal{U}_n$ $(n \geq 5)$. Then $\eta(U) = n-4$ if and only if $U \cong U_1^*$ or $U \cong U_2^*$ or $U \cong U_3^*$, where U_1^*, U_2^* and U_3^* are shown in $Fig.$1.1.

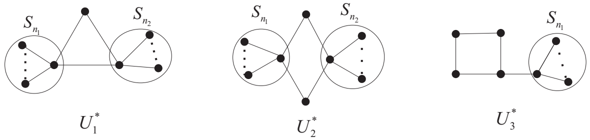

Fig.1.1

A bicyclic graph is a simple connected graph in which the number of edges equals the number of vertices plus one. Denoted by $\mathcal{B}_n$ the set of all bicyclic graphs of order n. Let C_k and C_l be two vertex-disjoint cycles. Suppose that v_1 is a vertex of C_k and v_q is a vertex of C_l. Joining v_1 and v_q by a path $v_1v_2\ldots v_q$ of length $q-1$, where $q \geq 1$ and $q = 1$ means identifying v_1 with v_q, the resulting graph, denoted by $B(k, q, l)$ shown in $Fig.$1.2, is called an $\infty-$graph; Let P_{l+1}, P_{p+1} and P_{q+1} be three vertex-disjoint paths, where $l, p, q \geq 1$, and at most one of them is 1. Identifying the three initial vertices and terminal vertices of them, respectively, the resulting graph, denoted by $P(l, p, q)$ shown in $Fig.$1.2, is called a $\theta-$graph. Obviously, $\mathcal{B}_n$ consists of three types of graphs: first type denoted by B_n^+ is the set of those graphs each of which is an $\infty-$graph with trees attached when $q > 1$; second type denoted by B_n^{++} is the set of those graphs each of which is an $\infty-$graph with trees attached when $q = 1$; third type denoted by θ_n is the set of those graphs each of which is an $\theta-$graph with trees attached. Then $\mathcal{B}_n = B_n^+ \bigcup B_n^{++} \bigcup \theta_n$.

In section 2, we determine the nullity set of $\mathcal{B}_n$. In section 3, we characterize the bicyclic graphs with maximal nullity.

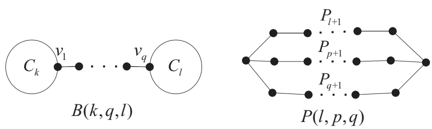

Fig.1.2

2 The nullity set of $\mathcal{B}_n$

First, we introduce some lemmas.

Lemma 2.1[3]. A path with four vertices of degree 2 in a bipartite graph G can be replaced by an edges without changing the value of $\eta(G)$.

Lemma 2.2[3]. For a graph G containing a vertex of degree 1, if the induced subgraph H of G is obtained by deleting this vertex together with the vertex adjacent to it, then the relation $\eta(H) = \eta(G)$ holds.

Lemma 2.3. Let G_1, G_2 and G_3 be the graphs of order n shown in $Fig.2.1$, respectively. Then $\eta(G_1) = 0$, $\eta(G_2) = 1$ and $\eta(G_3) = \begin{cases} 2 & n \equiv 0(mod2) \\ 3 & n \equiv 1(mod2) \end{cases}$.

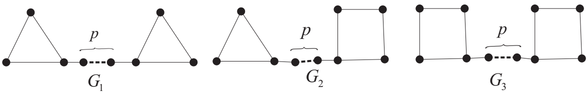

Fig.2.1

Proof. We can easy calculate that $\gamma(A(G_1)) = n$, $\gamma(A(G_2)) = n - 1$. Then $\eta(G_1) = 0$, $\eta(G_2) = 1$. For the graph G_3, if $p \in [0, 3]$, it must be one of the following graphs shown in $Fig.2.2$. If $p \geq 4$, since it is a bipartite graph, it can be transformed into one of the graphs shown in $Fig.2.2$ by Lemma 2.1 without changing its nullity. It is not difficult to get that $\eta(G_3^1) = \eta(G_3^3) = 2$ and $\eta(G_3^2) = \eta(G_3^4) = 3$. Therefore $\eta(G_3) = \begin{cases} 2 & n \equiv 0(mod2) \\ 3 & n \equiv 1(mod2) \end{cases}$. $\square$

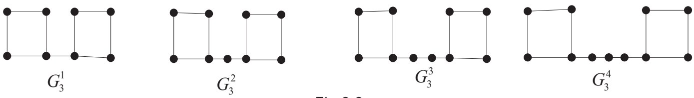

Fig.2.2

Lemma 2.4. $\eta(C_n) = \begin{cases} 2 & n \equiv 0(mod4) \\ 0 & otherwise \end{cases}$.

Proof. When $n \equiv 0(mod2)$, by Lemma 2.1, since C_n is a bipartite graph, we can get that

$$\eta(C_n) = \begin{cases} 2 & \text{if } n \equiv 0(mod4) \\ 0 & \text{if } n \equiv 2(mod4) \end{cases}$$

When $n \equiv 1(mod2)$, the spanning elementary subgraph of C_n is itself. Since $r(C_n) = 0$, $s(C_n) = 1$, $detA(C_n) = 2 \neq 0$ by Proposition 1.3. Thus $\eta(C_n) = 0$. $\square$

Lemma 2.4 is equivalent to the following

Lemma 2.5. $\gamma(A(C_n)) = \begin{cases} n-2 & n \equiv 0(mod4) \\ n & otherwise \end{cases}$.

Lemma 2.6. For any $G \in B_n^+$ $(n \geq 7)$, $\eta(G) \leq n-6$.

Proof. Let C_k, C_l are two vertex-disjoint cycles in G, we distinguish the following two cases:

Case 1. $k, l \in \{3, 4\}$. There must exists one of graphs shown in $Fig.2.1$ as a vertex-induced subgraph of G since $G \in B_n^+$. By Lemma 2.3, for each G_i $(i = 1, 2, 3)$, we have $\gamma(A(G_i)) \geq 6$. Hence $\gamma(A(G)) \geq \gamma(A(G_i)) \geq 6$. Thus $\eta(G) \leq n - \gamma(A(G_i)) \leq n-6$.

Case 2. $k \geq 5$ or $l \geq 5$. Without loss of generally, we assume that $k \geq 5$. There must exists H_1 shown in $Fig.2.3$ as a vertex-induced subgraph of G since $G \in B_n^+$. By Lemma 2.2, it is easy to get that

$$\eta((H_1)) = \begin{cases} 1 & \text{if } k \equiv 0(mod2) \\ 0 & \text{if } k \equiv 1(mod2) \end{cases}$$

Hence

$$\gamma(A(H_1)) = \begin{cases} k & \text{if } n \equiv 0(mod2) \\ k+1 & \text{if } n \equiv 1(mod2) \end{cases}$$

Since $k \geq 5$, we have $\gamma(A(H_1)) \geq 6$. Therefore $\gamma(A(G)) \geq \gamma(A(H_1)) \geq 6$. Thus $\eta(G) \leq n - \gamma(A(H_1)) \leq n-6$. □

Theorem 2.1. The nullity set of $B_n^+ (n \geq 7)$ is $[0, n-6]$.

Proof. By Lemma 2.6, it suffices to show that for each $k \in [0, n-6]$, there exist a graph $G \in B_n^+$ such that $\eta(G) = n-6$.

When $k = 0$, let $G = G_1$ shown in $Fig.2.1$, we have $\eta(G_1) = 0$; When $1 \leq k \leq n-7$, let $G = G_4$ shown in $Fig.2.3$. Using Lemma 2.2 repeatedly, if $n \not\equiv k(mod2)$, after $\frac{n-k-5}{2}$ steps, we get $P_2 \bigcup C_3 \bigcup kK_1$. Hence $\eta(G) = \eta(P_2 \bigcup C_3 \bigcup kK_1) = k$. If $n \equiv k(mod2)$, after $\frac{n-k-4}{2}$ steps, we get $P_2 \bigcup P_2 \bigcup kK_1$. Hence $\eta(G) = \eta(P_2 \bigcup P_2 \bigcup kK_1) = k$; When $k = n-6$, let $G = G_5$ shown in $Fig.2.3$. By Lemma 2.2, we get $P_2 \bigcup C_4 \bigcup (n-8)K_1$. Hence $\eta(G) = \eta(P_2 \bigcup C_4 \bigcup (n-8)K_1) = n-8+2 = n-6$. □

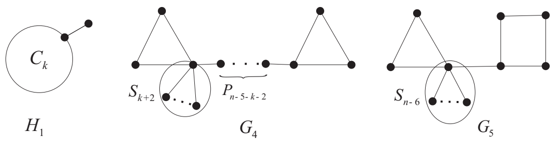

Fig.2.3

Lemma 2.7. For any $G \in B_n^{++}$ $(n \geq 8)$, $\eta(G) \leq n-6$.

Proof. Let C_k, C_l are two cycles in G, we distinguish the following two cases:
Case 1. $k, l \in \{3,4\}$. There must exists one of the graphs shown in $Fig.2.4$ as a vertex-induced subgraph of G since $n \geq 8$. It is easy to calculate that $\eta(G_6) = \eta(G_7) = \eta(G_{10}) = 0$, $\eta(G_8) = \eta(G_9) = \eta(G_{11}) = 1$ and $\eta(G_{12}) = \eta(G_{13}) = \eta(G_{14}) = 2$. For each G_i $(i = 6, 7, \ldots, 14)$, we have $\gamma(A(G_i)) \geq 6$. Hence $\gamma(A(G)) \geq \gamma(A(G_i)) \geq 6$. Thus $\eta(G) \leq n - \gamma(A(G_i)) \leq n - 6$.
Case 2. $k \geq 5$ or $l \geq 5$. Without loss of generally, we assume that $k \geq 5$, There must exists H_1 shown in $Fig.2.3$ as a vertex-induced subgraph of G since $G \in B_n^{++}$. Similar to the proof of Case 2 in Lemma 2.6, we have $\gamma(A(H_1)) \geq 6$. Hence $\gamma(A(G)) \geq \gamma(A(H_1)) \geq 6$. Thus $\eta(G) \leq n - \gamma(A(H_1)) \leq n - 6$. □

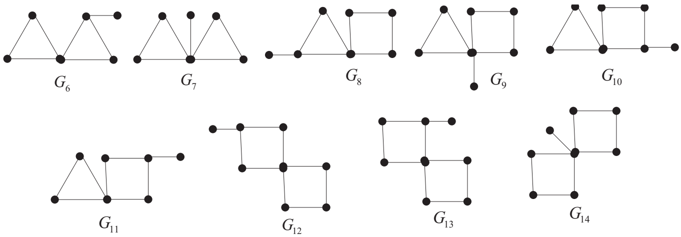

Fig.2.4

Theorem 2.2. The nullity set of $B_n^{++} (n \geq 8)$ is $[0, n - 6]$.

Proof. By Lemma 2.7, it suffices to show that for each $k \in [0, n - 6]$, there exists a graph $G \in B_n^{++}$ such that $\eta(G) = n - 6$.

When $k = 0$, if $n \equiv 1(mod 2)$, let $G = G_{15}$ shown in $Fig.2.5$, Using Lemma 2.2 repeatedly, after $\frac{n-5}{2}$ steps, we get H_3 shown in $Fig.2.5$. Hence $\eta(G) = \eta(H_3) = 0$. If $n \equiv 0(mod 2)$, let $G = G_{16}$ shown in $Fig.2.5$, Using Lemma 2.2 repeatedly, after $\frac{n-4}{2}$ steps, we get $P_2 \bigcup P_2$. Hence $\eta(G) = \eta(P_2 \bigcup P_2) = 0$; When $1 \leq k \leq n - 6$, if $n \not\equiv k(mod 2)$, let $G = G_{17}$ shown in $Fig.2.5$. Using Lemma 2.2 repeatedly, after $\frac{n-k-5}{2}$ steps, we get $kK_1 \bigcup H_3$. Hence $\eta(G) = \eta(kK_1 \bigcup H_3) = k$. If $n \equiv k(mod 2)$, let $G = G_{18}$ shown in $Fig.2.5$. Using Lemma 2.2 repeatedly, after $\frac{n-k-4}{2}$ steps, we get $kK_1 \bigcup P_2 \bigcup P_2$. Hence $\eta(G) = \eta(kK_1 \bigcup P_2 \bigcup P_2) = k$. □

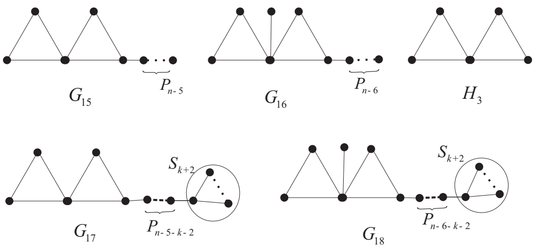

Fig.2.5

Lemma 2.8. For any $G \in \theta_n$ $(n \geq 6)$, $\eta(G) \leq n-4$.

Proof. Let C_k, C_l are two elementary cycles in G, we distinguish the following two cases: Case 1. $k, l \in \{3,4\}$, There must exists one of the graphs shown in $Fig.2.6$ as a vertex-induced subgraph of G since $n \geq 6$. It is easy to calculate that $\eta(G_{19}) = \eta(G_{25}) = 0$, $\eta(G_{20}) = \eta(G_{21}) = 1$ and $\eta(G_{22}) = \eta(G_{23}) = \eta(G_{24}) = 2$. For each G_i $(i = 19, 20, \ldots, 25)$, we have $\gamma(A(G_i)) \geq 4$. Hence $\gamma(A(G)) \geq \gamma(A(G_i)) \geq 4$. Thus $\eta(G) \leq n - \gamma(A(G_i)) \leq n-4$. Case 2. $k \geq 5$ or $l \geq 5$. Without loss of generally, we assume that $k \geq 5$. Since C_k is a vertex-induced subgraph of G, by Lemma 2.5, $\gamma(A(G)) \geq \gamma(A(C_k)) \geq 4$. Hence $\eta(G) \leq n-4$. $\square$

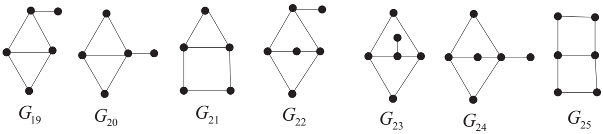

Fig.2.6

Theorem 2.3. The nullity set of $\theta_n (n \geq 6)$ is $[0, n-4]$.

Proof. By Lemma 2.8, it suffices to show that for each $k \in [0, n-4]$, there exists a graph $G \in \theta_n$ such that $\eta(G) = n-4$.

When $k = 0$, if $n \equiv 1(mod 2)$, let $G = G_{26}$ shown in $Fig.2.7$. Using Lemma 2.2 repeatedly, after $\frac{n-3}{2}$ steps, we get C_3. Hence $\eta(G) = \eta(C_3) = 0$. If $n \equiv 0(mod 2)$, let $G = G_{27}$ shown in $Fig.2.7$. Using Lemma 2.2 repeatedly, after $\frac{n-2}{2}$ steps, we get P_2. Hence $\eta(G) = \eta(P_2) = 0$; When $1 \leq k \leq n-6$, let $G = G_{28}$ shown in $Fig.2.7$. Using Lemma 2.2 repeatedly, if $n \neq k(mod 2)$, after $\frac{n-k-3}{2}$ steps, we get $kK_1 \bigcup C_3$. Hence $\eta(G) = \eta(kK_1 \bigcup C_3) = k$. If $n \equiv k(mod 2)$, after $\frac{n-k-2}{2}$ steps, we get $kK_1 \bigcup P_2$. Hence $\eta(G) = \eta(kK_1 \bigcup P_2) = k$; When $k = n-5$, let $G = G_{29}$ shown in $Fig.2.7$. By Lemma 2.2, we get $(n-6)K_1 \bigcup H_4$, where H_4 is shown in $Fig.2.7$. Hence $\eta(G) = \eta((n-6)K_1 \bigcup H_4) = n-6+1 = n-5$; When $k = n-6$, let $G = G_{30}$ shown in $Fig.2.7$. By Lemma 2.2, we get $(n-5)K_1 \bigcup P_3$. Hence $\eta(G) = \eta((n-6)K_1 \bigcup P_3) = n-5+1 = n-4$. $\square$

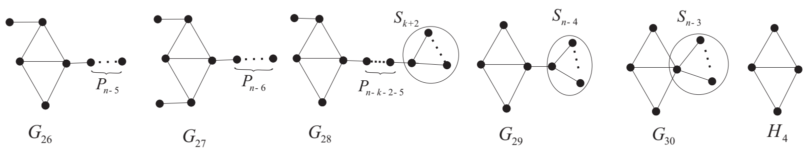

Fig.2.7

3 The bicyclic graph with maximal nullity

Theorem 3.1. Let $G \in B_n^+$ $(n \geq 10)$. Then $\eta(G) = n - 6$ if and only if $G \cong G_1^*$ or $G \cong G_2^*$ or $G \cong G_3^*$, where G_1^*, G_2^* and G_3^* are shown in $Fig.3.1$.

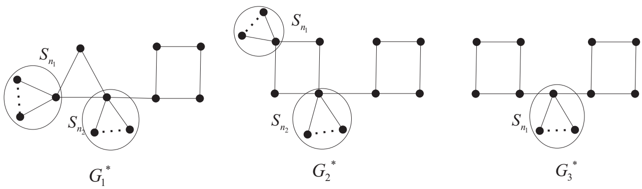

Fig.3.1

Proof. If $G \cong G_i^*$ $(i = 1, 2, 3)$, it is easy to check that $\eta(G) = n - 6$. So it is suffices to prove the converse side of the theorem.

Let C_k, C_l be two vertex-disjoint cycles in G, we first prove the following two claims.

Claim 1. If $G \in B_n^+$ $(n \geq 10)$, $\eta(G) = n - 6$, then $k, l \in \{3, 4\}$.

Otherwise, without loss of generality, we assume that $k \geq 5$. We can find H_2 shown in Fig.3.2 as a vertex-induced subgraph of G. By Lemma 2.2 and 2.4, we easy get

$$\eta((H_2)) = \begin{cases} 0 & k \neq 0(mod4) \\ 2 & k \equiv 0(mod4) \end{cases}$$

Then

$$\gamma(A((H_2))) = \begin{cases} k+2 & k \neq 0(mod4) \\ k & k \equiv 0(mod4) \end{cases}$$

Since $k \geq 5$, we have $\gamma(A((H_2))) \geq 7$. Hence $\eta(G) \leq n - 7 < n - 6$, a contradiction. So Claim 1 holds.

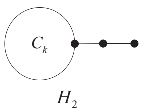

Fig.3.2

Claim 2. If $\eta(G) = n - 6$ $(n > 9)$ and $k, l \in \{3, 4\}$, then there exists at least one pendent vertex in G.

Otherwise, G must be the one of graphs shown in Fig.2.1 since $G \in B_n^+$ and $n > 9$. By Lemma 2.3, for each G_i $(i = 1, 2, 3)$, we have $\eta(G_i) < n - 6$, a contradiction. So Claim 2 holds.

Let x be a pendant vertex in G and y the adjacent vertex of x. Let $G_1 = G_{11} \bigcup G_{12} \bigcup \ldots \bigcup G_{1t}$ be the graph obtained by deleting x, y from G, where $G_{11}, G_{12}, \ldots, G_{1t}$ are connected components of G_1. At least one of G_{1i} $(i = 1, 2, \ldots, t)$ is nontrivial. Otherwise, G would be a star.

In fact, there are at most two nontrivial components in G_1. Otherwise, we assume that G_{11}, G_{12}, G_{13} are three nontrivial components in G_1. Let $v(G_{11}) = n_1, v(G_{12}) = n_2$ and $v(G_{13}) = n_3$. At least one of G_{11}, G_{12}, G_{13} contains pendant vertices since $G \in B_n^+$. On the other hand, at least two of G_{11}, G_{12}, G_{13} contains pendant vertices, without loss of generality, we assume that G_{11} and G_{12} contain pendent vertices. Let v be a pendent vertex of G_{11} and u the adjacent vertex of v. Let G_{21} be the graph obtained by deleting u, v from G_{11}. Let w be a pendent vertex of G_{12} and p the adjacent vertex of w. Let G_{31} be the graph obtained by deleting w, p from G_{12}. Denote the graph $G_{21} \bigcup G_{31} \bigcup G_{13} \bigcup \ldots \bigcup G_{1t}$ by G_2, and obviously, $v(G_2) = n - 6$. By Lemma 2.2, we have $\eta(G) = n - 6 = \eta(G_1) = \eta(G_2)$. By proposition 1.1, G_2 is the null graph. Therefore, G_{13} is trivial, a contradiction. So only one of G_{11}, G_{12}, G_{13} contains pendant vertices. We assume that G_{11} contains a pendent vertex v and u is the adjacent vertex of v. Let G'_{21} be the graph obtained by deleting u, v from G_{11}. Denote the graph $G'_{21} \bigcup G_{12} \bigcup G_{13} \bigcup \ldots \bigcup G_{1t}$ by G'_2, and $v(G'_2) = n - 4$. Since G_{12}, G_{13} no contain pendent vertices, and $G \in B_n^+$, G_{12}, G_{13} must be C_{n_2}, C_{n_3}, respectively, where $n_2, n_3 \in \{3, 4\}$. By Lemma 2.4, we have

$$\eta((C_{n_2})) = \begin{cases} 2 & n_2 = 4 \\ 0 & n_2 = 3 \end{cases}$$

$$\eta((C_{n_3})) = \begin{cases} 2 & n_3 = 4 \\ 0 & n_3 = 3 \end{cases}$$

Hence $\eta(G) = \eta(G_1) = \eta(G'_2) \leq n_1 - 2 + \eta(C_2) + \eta(C_3) + (n - 2 - n_1 - n_2 - n_3) \leq n - 8 < n - 6$, a contradiction.

We distinguish the following two cases:

Case 1. There is a unique nontrivial component in G_1. Without loss of generality, we assume that G_{11} is nontrivial. Let $v(G_{11}) = n_1$. Then $G_1 = G_{11} \bigcup (n - 2 - n_1) K_1$. It is easy to see that deleting x, y destroy at most one cycle since $G \in B_n^+$. Hence G_{11} contains cycles. By Lemma 2.2, we have $\eta(G) = n - 6 = \eta(G_1) = \eta(G_{11}) + (n - 2 - n_1)$. Thus $\eta(G_{11}) = n_1 - 4$. If $G_{11} \in B_{n_1}^+$, by Lemma 2.6, we have $\eta(G_{11}) \leq n_1 - 6 < n_1 - 4$, a contradiction. Thus $G_{11} \in \mathcal{U}_{n_1}$. By Theorem 1.4, $\eta(G_{11}) = n_1 - 4$ if and only if $G_{11} \cong U_1^*$ or

$G_{11}{\cong}U_2^*$ or $G_{11}{\cong}U_3^*$, where U_1^*, U_2^* and U_3^* are shown in Fig.1.1. If $G_{11}{\cong}U_1^*$ or $G_{11}{\cong}U_2^*$, we can't get two vertex-disjoint cycles by recovering x, y to G_1. This is impossible. Therefore, $G_{11}{\cong}U_3^*$, and $G_1 = U_3^*\bigcup(n-2-n_1)K_1$. Now recover x, y to G_1, we need to insert edges from y to each $n-2-n_1$ isolated vertices of G_1. This gives a star S_{n-n_1}. In order to produce two vertex-disjoint cycles in G, two edges must be added from the center of S_{n-n_1} to G_1. If we select the center and a pendant vertex in U_3^* as two ends of these two edges, then $G{\cong}G_1^*$; If both ends chosen in U_3^* are pendant vertices, then $G{\cong}G_2^*$.

Case 2. There are two nontrivial components in G_1. Without loss of generality, we assume that G_{11} and G_{12} are nontrivial. Let $v(G_{11}) = n_1$ and $v(G_{12}) = n_2$. Then $G_1 = G_{11}\bigcup G_{12}\bigcup(n-2-n_1-n_2)K_1$. Now we consider the following three subcases:

Subcase 2.1. Both G_{11} and G_{12} contain pendent vertex. Let v be a pendent vertex of G_{11} and u the adjacent vertex of v. Let G_{21} be the graph obtained by deleting u, v from G_{11}. Let w be a pendent vertex of G_{12} and p the adjacent vertex of w. Let G_{31} be the graph obtained by deleting w, p from G_{12}. Denote the graph $G_{21}\bigcup G_{31}\bigcup(n-2-n_1-n_2)K_1$ by G_2 and obviously $v(G_2) = n-6$. By Lemma 2.2, we have $\eta(G) = n-6 = \eta(G_1) = \eta(G_2)$. By proposition 1.1, G_2 is the null graph, Hence $G_{21} = (n_1-2)K_1$, $G_{31} = (n_2-2)K_1$. In order to recover G_{11}, G_{12}, respectively, return u, v to G_{21}, G_{11} must be a star S_{n_1}, return w, p to G_{31}, G_{12} must be a star S_{n_2}, and $G_1 = S_{n_1}\bigcup S_{n_2}\bigcup(n-2-n_1-n_2)K_1$. We can't get two vertex-disjoint cycles by adding x, y to G_1, a contradiction.

Subcase 2.2. Only one of G_{11}, G_{12} contains pendent vertices. Without loss of generality, we assume that G_{11} contains pendent vertices. Let v be a pendent vertex of G_{11} and u the adjacent vertex of v. Let G_{21} be the graph obtained by deleting u, v from G_{11}. Denote the graph $G_{21}\bigcup G_{12}\bigcup(n-2-n_1-n_2)K_1$ by G_2 and obviously $v(G_2) = n-4$. By Lemma 2.2, we have $\eta(G) = n-6 = \eta(G_1) = \eta(G_2){\leq}n_1-2+\eta(G_{12})+(n-2-n_1-n_2)$. Then $\eta(G_{12}) \geq n_2-2$. Since G_{12} no contain pendent vertices, if $G_{12} \in B_{n_2}^+$, By Lemma 2.6, we have $\eta(G_{12}) \leq n_2-6 < n_2-2$, a contradiction; If $G_{12} \in \mathcal{U}_{n_2}$, G_{12} must be C_{n_2}, where $n_2 \in \{3, 4\}$. It is easy to check that $\eta(G_{12}) \geq n_2-2$ holds only if $n_2 = 4$. Then $G_{12} = C_4$. Since $n-6 = \eta(G_2){\leq}n_1-2+2+(n-2-n_1-4) = n-6$, then $G_{21} = (n_1-2)K_1$, return u, v to G_{21}, G_{11} must be a star S_{n_1}, and $G_1 = S_{n_1}\bigcup C_4\bigcup(n-2-n_1-4)K_1$. Now recover x, y to G_1, we get $G{\cong}G_1^*$ or $G{\cong}G_2^*$.

subcase 2.3. G_{11} and G_{12} no contain pendent vertices. Since $G \in B_n^+$, G_{11}, G_{12} must be C_{n_1}, C_{n_2}, respectively, where $n_1, n_2 \in \{3, 4\}$. $G_1 = C_{n_1}\bigcup C_{n_2}\bigcup(n-2-n_1-n_2)K_1$, By Lemma 2.2, we have $\eta(G) = n-6 = \eta(G_1) = \eta(C_{n_1})+\eta(C_{n_2})+(n-2-n_1-n_2)$. Then $n_1-\eta(C_{n_1})+n_2-\eta(C_{n_2}) = 4$. It is easy to check that $n_1-\eta(C_{n_1})+n_2-\eta(C_{n_2}) = 4$ holds only if $n_1 = n_2 = 4$. Then $G_1 = C_4\bigcup C_4\bigcup(n-2-8)K_1$. Now recover x, y to G_1, we get

$G \cong G_3^*$. □

Theorem 3.2. Let $G \in B_n^{++}$ $(n \geq 8)$. Then $\eta(G) = n - 6$ if and only if $G \cong G_1^*$ or $G \cong G_2^*$ or $G \cong G_3^*$ or $G \cong G_4^*$ or $G \cong G_5^*$ or $G \cong G_6^*$ or $G \cong G_7^*$ or $G \cong G_8^*$, where G_1^*, G_2^*, G_3^*, G_4^*, G_5^*, G_6^*, G_7^* and G_8^* are shown in $Fig.3.3$.

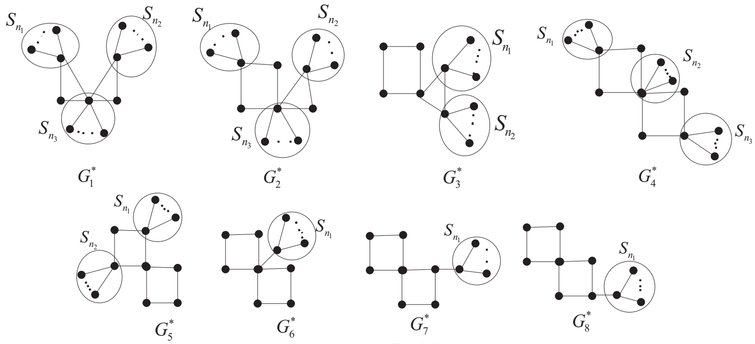

Fig.3.3

Proof. If $G \cong G_i^*$ $(i = 1, 2, \ldots, 8)$, It is easy to check that $\eta(G) = n - 6$. So it is suffices to prove the converse side of the theorem.

Let C_k, C_l be two cycles in G, we first prove the following two claims.

Claim 1. If $G \in B_n^{++}$ $(n \geq 8)$, $\eta(G) = n - 6$, then $k, l \in \{3, 4\}$.

Otherwise, without loss of generality, we assume that $k \geq 5$. We distinguish the following two cases:

Case 1. $l \geq 4$. We can find H_2 shown in $Fig.3.2$ as a vertex-induced subgraph of G. Similar to the proof of Claim 1 in Theorem 3.1, we have $\gamma(A(H_2)) \geq 7$, Hence $\eta(G) \leq n - 7 < n - 6$, a contradiction.

Case 2. $l = 3$. If $k \geq 7$, we can find H_1 shown in $Fig.2.3$ as a vertex-induced subgraph of G, Similar to the proof of Case 2 in Lemma 2.6, we have $\gamma(A(H_1)) \geq 8$. Hence $\eta(G) \leq n - 8 < n - 6$, a contradiction; If $k = 5, 6$, these must exists one of the following graphs shown in $Fig.3.4$ as a vertex-induced subgraph of G since $n \geq 8$. It is easy to calculate that $\eta(G_{31}) = \eta(G_{32}) = \eta(G_{33}) = \eta(G_{34}) = \eta(G_{35}) = 0$. For each G_i $(i = 31, \ldots, 35)$, we have $\gamma(A(G_i)) = 8 > 6$. Hence $\eta(G) \leq n - \gamma(A(G_i)) \leq n - 8 < n - 6$, a contradiction.

Thus in each case we get contradiction, so Claim 1 holds.

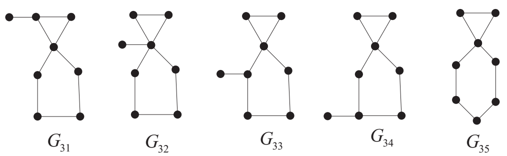

Fig.3.4

Claim 2. If $G \in B_n^{++}$, $\eta(G) = n - 6$ $(n \geq 8)$, and $k, l \in \{3, 4\}$, then there exists at least one pendent vertex in G.

Otherwise, since $n \geq 8$, $G \in B_n^{++}$, and $k, l \in \{3, 4\}$, this is impossible. So Claim 2 holds.

Let x be a pendant vertex in G and y the adjacent vertex of x. Let $G_1 = G_{11} \bigcup G_{12} \bigcup \ldots \bigcup G_{1t}$ be the graph obtained by deleting x, y from G, where $G_{11}, G_{12}, \ldots, G_{1t}$ are connected components of G_1. At least one of G_{1i} $(i = 1, 2, \ldots, t)$ is nontrivial. Otherwise, G would be a star.

In fact, there are at most two nontrivial components in G_1. Otherwise, we assume that G_{11}, G_{12}, G_{13} are three nontrivial components in G_1. Let $v(G_{11}) = n_1, v(G_{12}) = n_2$ and $v(G_{13}) = n_3$. At most one of G_{11}, G_{12}, G_{13} contains cycle since $G \in B_n^{++}$. Then at least two of G_{11}, G_{12}, G_{13} contain pendent vertices. Without loss of generality, we assume that G_{11} and G_{12} contain pendent vertices. Let v be a pendent vertex of G_{11} and u the adjacent vertex of v. Let G_{21} be the graph obtained by deleting u, v from G_{11}. Let w be a pendent vertex of G_{12} and p the adjacent vertex of w. Let G_{31} be the graph obtained by deleting w, p from G_{12}. Denote the graph $G_{21} \bigcup G_{31} \bigcup G_{13} \bigcup \ldots \bigcup G_{1t}$ by G_2, and obviously $v(G_2) = n - 6$. By Lemma 2.2, we have $\eta(G) = n - 6 = \eta(G_1) = \eta(G_2)$. By proposition 1.1, G_2 is the null graph. Therefore, G_{13} is trivial, a contradiction.

We distinguish the following two cases:

Case 1. There is a unique nontrivial component in G_1. Without loss of generality, we assume that G_{11} is nontrivial. Let $v(G_{11}) = n_1$. Then $G_1 = G_{11} \bigcup (n - 2 - n_1)K_1$. G_{11} must contains cycles since $G \in B_n^{++}$. By Lemma 2.2, we have $\eta(G) = n - 6 = \eta(G_1) = \eta(G_{11}) + (n - 2 - n_1)$. Hence $\eta(G_{11}) = n_1 - 4$. Now, we consider the following two subcases:

Subcase 1.1. $G_{11} \in B_{n_1}^{++}$. If G_{11} contains pendent vertices, there must exists one of graphs shown in Fig.2.4 as a vertex-induced subgraph of G_{11}. Similar to the proof of Lemma 2.7, we have $\gamma(A(G_i)) \geq 6$ $(i = 6, \ldots, 14)$. Hence $\gamma(A(G_{11})) \geq \gamma(A(G_i)) \geq 6$. Thus $\eta(G_{11}) \leq n_1 - \gamma(A(G_i)) \leq n_1 - 6 < n_1 - 4$, a contradiction; If G_{11} no contain pendent vertex, there must one of graphs shown in Fig.3.5. It is easy to check that only graph which satisfies $\eta(G_{11}) = n_1 - 4$ is G_{38}. Then $G_1 = G_{38} \bigcup (n - 2 - n_1)K_1$. Now recover x, y to G_1 , we get $G \cong G_6^*$ or $G \cong G_7^*$ or $G \cong G_8^*$.

Subcase 1.2. $G_{11} \in \mathcal{U}_{n_1}$. By lemma 1.4, $\eta(G_{11}) = n_1 - 4$ if and only if $G_{11} \cong U_1^*$ or $G_{11} \cong U_2^*$ or $G_{11} \cong U_3^*$, where U_1^*, U_2^* and U_3^* are shown in Fig.1.1. If $G_{11} \cong U_1^*$, recover x, y to G_1, we get $G \cong G_1^*$ or $G \cong G_2^*$; if $G_{11} \cong U_2^*$, recover x, y to G_1, we get $G \cong G_2^*$ or $G \cong G_4^*$; if $G_{11} \cong U_3^*$, recover x, y to G_1, we get $G \cong G_3^*$ or $G \cong G_5^*$.

Case 2. There are two nontrivial components in G_1. Without loss of generality, we assume that G_{11} and G_{12} are nontrivial. Let $v(G_{11}) = n_1$ and $v(G_{12}) = n_2$. Then $G_1 = G_{11}\bigcup G_{12}\bigcup(n-2-n_1-n_2)K_1$. At least one of G_{11}, G_{12} contains pendent vertices since $G \in B_n^{++}$. Otherwise G_{11}, G_{12} must be C_{n_1},C_{n_2}, respectively. We get two vertex-disjoint cycles in G by recovering x, y to G_1, a contradiction. Now, we consider the following two subcases:

Subcase 2.1. Only one of G_{11}, G_{12} contains pendent vertices. Without loss of generality, we assume that G_{11} contains pendent vertices. Let v be a pendent vertex of G_{11} and u the adjacent vertex of v. Let G_{21} be the graph obtained by deleting u, v from G_{11}. Denote the graph $G_{21}\bigcup G_{12}\bigcup(n-2-n_1-n_2)K_1$ by G_2 and obviously $v(G_2) = n-4$. By Lemma 2.2, we have $\eta(G) = n-6 = \eta(G_1) = \eta(G_2) \leq n_1 - 2 + \eta(G_{12}) + (n-2-n_1-n_2)$. Thus $\eta(G_{12}) \geq n_2 - 2$. Since G_{12} no contain pendent vertices, when $G_{12} \in B_{n_2}^{++}$, G_{12} must be one of graphs shown in $Fig.$3.5. It is easy to check that each $G_i (i = 36, 37, 38)$ does not satisfy $\eta(G_{12}) \geq n_2 - 2$, a contradiction. When $G_{12} \in \mathcal{U}_{n_2}$, similar to the proof of Subcase 2.2 in Theorem 3.1, $G_1 = S_{n_1}\bigcup C_4\bigcup(n-2-n_1-4)K_1$. Since there is no edges jointing a vertex in C_4 to that of S_{n_1}, recover x, y to G_1, only one edge be inserted from y to C_4, we can't get two only one common vertex cycles in G, a contradiction.

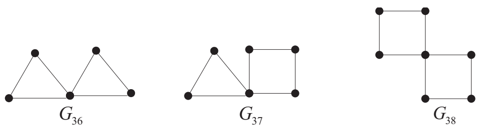

Fig.3.5

Subcase 2.2. Both G_{11} and G_{12} contain pendent vertices. Similar to the proof of Subcase 2.1 in Theorem 3.1. $G_1 = S_{n_1}\bigcup S_{n_2}\bigcup(n-2-n_1-n_2)K_1$. Recover x, y to G_1, we get $G\cong G_1^*$ or $G\cong G_2^*$ or $G\cong G_4^*$. □

Theorem 3.3. Let $G \in \theta_n$ $(n \geq 6)$. Then $\eta(G) = n-4$ if and only if $G\cong G_1^*$ or $G\cong G_2^*$ or $G\cong G_3^*$ or $G\cong G_4^*$ or $G\cong G_5^*$ or $G\cong G_6^*$, where G_1^*, G_2^*, G_3^*, G_4^*, G_5^* and G_6^* are shown in $Fig.$3.6.

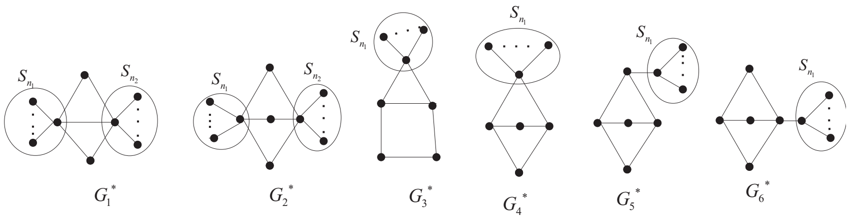

Fig.3.6

Proof. If $G \cong G_i^*$ $(i = 1, 2, \ldots, 6)$, it is easy to check that $\eta(G) = n - 4$. So it is suffices to prove the converse side of the theorem.

Let C_k, C_l be two elementary cycles in G, we first prove the following two claims.

Claim 1. If $G \in \theta_n$ $(n \geq 6)$ and $\eta(G) = n - 4$, then $k, l \in \{3, 4\}$.

Otherwise, without loss of generality, we assume that $k \geq 5$. Then C_k is a vertex-induced subgraph of G. By Lemma 2.5, we have $\gamma(A(C_k)) \geq 5$. Then $\gamma(A(G)) \geq \gamma(A(C_k)) \geq 5$. Hence $\eta(G) \leq n - 5 < n - 4$, a contradiction. So Claim 1 holds.

Claim 2. If $G \in \theta_n$, $\eta(G) = n - 4$ $(n \geq 6)$, and $k, l \in \{3, 4\}$, then there must exists a pendent vertex in G.

Otherwise, since $n \geq 6$, $k, l \in \{3, 4\}$, G must be G_{25} shown in $Fig.$2.6. Hence $\eta(G_{25}) = 0 \neq 6 - 4$, a contradiction. So Claim 2 holds.

Let x be a pendant vertex in G and y the adjacent vertex of x. Let $G_1 = G_{11} \bigcup G_{12} \bigcup \ldots \bigcup G_{1t}$ be the graph obtained by deleting x, y from G, where $G_{11}, G_{12}, \ldots, G_{1t}$ are connected components of G_1. At least one of G_{1i} $(i = 1, 2, \ldots, t)$ is nontrivial. Otherwise, G would be a star.

In fact, there is a unique nontrivial components in G_1. Otherwise , we assume that G_{11}, G_{12} are two nontrivial components in G_1. Let $v(G_{11}) = n_1$ and $v(G_{12}) = n_2$. At least one of G_{11}, G_{12} contains pendent vertices since $G \in \theta_n$. Without loss of generality, Let v be a pendent vertex of G_{11} and u the adjacent vertex of v. Let G_{21} be the graph obtained by deleting u, v from G_{11}. Denoted $G_{21} \bigcup G_{12} \bigcup \ldots \bigcup G_{1t}$ by G_2 and obviously $v(G_2) = n - 4$. By Lemma 2.2, we have $\eta(G) = \eta(G_1) = \eta(G_2) = n - 4$. By proposition 1.1, G_2 is the null graph. Then G_{12} is trivial, a contradiction.

We assume that G_{11} is nontrivial. Let $v(G_{11}) = n_1$. Then $G_1 = G_{11} \bigcup (n - 2 - n_1) K_1$. We distinguish the following two cases.

Case 1. The minimum degree of G_{11} is 1.

Let v be a pendent vertex of G_{11} and u the adjacent vertex of v. Let G_{21} be the graph obtained by deleting u, v from G_{11}. Denoted $G_{21} \bigcup (n - 2 - n_1) K_1$ by G_2 and obviously $v(G_2) = n - 4$. By Lemma 2.2, we have $\eta(G) = n - 4 = \eta(G_1) = \eta(G_2)$. By proposition 1.1, G_2 is the null graph. Hence $G_{21} = (n_1 - 2) K_1$, In order to recover G_{11}, return u, v to G_{21}, G_{11} must be a S_{n_1}, and $G_1 = S_{n_1} \bigcup (n - 2 - n_1) K_1$. Now we add x, y to G_1, we need to insert edges from y to each of $n - 2 - n_1$ isolated vertices of G_1. This gives another star S_{n-n_1}. In order to assure that there are two edge-joint cycles in G, three edges must be added from the center of S_{n-n_1} to S_{n_1}. If we select the center and two pendant vertices in S_{n_1} as three ends of these three edges, then $G \cong G_1^*$; If three ends chosen in S_{n_1} are pendent vertices, then $G \cong G_2^*$.

Case 2. The minimum degree of G_{11} is equal or greater than 2.

Since the minimum degree of G_{11} is equal or greater than 2, G_{11} contains cycles. If $G_{11} \in \mathcal{U}_{n_1}$, G_{11} must be C_{n_1}, where $n_1 \in \{3, 4\}$. Hence $G_1 = C_{n_1} \bigcup (n-2-n_1)K_1$. By Lemma 2.2, we have $\eta(G) = n-4 = \eta(G_1) = \eta(C_{n_1}) + (n-2-n_1)$. Then $\eta(C_{n_1}) = n_1 - 2$. It is easy to check that $\eta(C_{n_1}) = n_1 - 2$ holds only if $n_1 = 4$. Thus $G_1 = C_4 \bigcup (n-6)K_1$. To add x, y to G_1, we need to insert edges from y to each of $n-6$ isolated vertices of G_1. This gives a star S_{n-4}. In order to assure that there are two edge-joint cycles in G, two edges must be added from the center of S_{n-4} to C_4. If we select two adjacent vertices in C_4 as two ends of these two edges, then $G \cong G_3^*$. If both ends chosen in C_4 are nonadjacent vertices, then $G \cong G_4^*$; If $G_{11} \in \theta_{n_1}$, G_{11} must be one of the following graphs shown in $Fig.3.7$ since $G \in \theta_n$ and $k, l \in \{3, 4\}$. By Lemma 2.2, we have $\eta(G) = n-4 = \eta(G_1) = \eta(G_{11}) + (n-2-n_1)$. Then $\eta(G_{11}) = n_1 - 2$. It is easy to check that $\eta(G_{11}) = n_1 - 2$ holds only if $G_{11} = G_{41}$. Thus $G_1 = G_{41} \bigcup (n-7)K_1$. Now recover x, y to G_1, we get $G \cong G_5^*$ or $G \cong G_6^*$. □

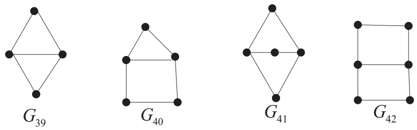

Fig.3.7

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