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A Note on Markaracter Tables of Finite Groups

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Abstract

The concept of markaracter tables of finite groups was introduced first by a Japanese chemist Shinsaku Fujita. He applied this notion in the context of stereochemistry and enumeration of molecules. In this paper, a simple computational method is described, by means of which it is possible to calculate the markaracter tables of finite groups. Using this method, the markaracter table of a dihedral group of order 2n and some abelian groups are computed. A GAP program is also included which is efficient for computing markaracter table of groups of order ≤ 10000. Using this program, the markaracter table of I_h point group symmetry is computed. This group appears as the point group symmetry of a Buckminster fullerene.

1. Introduction

The concept of the table of marks of a finite group was introduced by one of the pioneers of finite groups, William Burnside, in the second edition of his classical book [4]. This table describes a characterization of the permutation representations of a group G by certain numbers of fixed points and in some detail the partially ordered set of all conjugacy classes of subgroups of G. Hence it provides a very compact description of the subgroup lattice of G, see [21] for details.

Let the finite group G act on a finite set $X = \{x_1, x_2, \ldots, x_k\}$. The permutation representation PR (P_G) is a set of permutations (P_g) on X, each of which is associated with an element $g \in G$ so that P_G and G are homomorphic, $p_g p_{g'} = p_{gg'}$ for any $g,g' \in G$. Let H be a subgroup of G. It is well-known fact that the set of cosets of H in G provides a partition of G as $G = Hg_1 + Hg_2 + \ldots + Hg_m$, where g_1 = I, the identity element of G, and $g_i \in G$. The set of $\{g_1, g_2, \ldots, g_m\}$ is called a transversal. Consider the set of cosets $\{Hg_1, Hg_2, \ldots, Hg_m\}$. Following Shinsaku Fujita [6], for any $g \in G$, the set of permutations,

$$G(/H)_g = \begin{pmatrix} Hg_1 & Hg_2 & \ldots & Hg_m \\ Hg_1g & Hg_2g & \ldots & Hg_mg \end{pmatrix},$$

constructs a permutation representation of G, which is called a coset representation (CR) of G by H and notified as G(/H). The degree of G(/H) is m = |G|/|H|, where |G| is the number of elements in G. Obviously, the coset representation G(/H) is transitive, i.e. has one orbit.

The Burnside's theorem states that any permutation representation P_G of a finite group G acting on X can be reduced into transitive CRs in accord with equation $P_G = \sum_{i=1}^{s} \alpha_i G(/G_i)$, wherein the multiplicity α_i is a non- negative integer obtained by solving equations $\mu_j = \sum_{i=1}^{s} \alpha_i M_{ij}$, $(1 \le j \le s)$. Here μ_j is the number of fixed points of G_j in P_G named mark of G_j, and the symbol M_{ij} denotes the mark of G_j in $G(/G_i)$. Following Burnside [4], the matrix $M(G) = [M_{ij}]$ is called the table of marks or mark table of G. The matrix MC(G) obtained from M(G) in which we select rows and columns corresponding to cyclic subgroups of G is called the markaracter table of G. Shinsaku Fujita in some of his leading papers [6-16] introduced the term "markaracter" to discuss marks for permutation representations and characters for linear representations in a common basis.

For any two arbitrary matrices A and B, we have the direct product or Kronecker product $A \otimes B$ defined as

$$\begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}.$$

Note that if A is m-by-n and B is p-by-r then $A \otimes B$ is an mp-by-nr matrix. This multiplication is not usually commutative.

Throughout this paper our notation is standard and taken mainly from [17-19]. We encourage the reader to consult also papers by Balasubramanian [2,3], Kerber [20] and Pfeiffer [21], and references therein for background material as well as basic computational techniques.

2. Main Results and Discussion

If K is any subset in a group G, we designate by <K> the subgroup consisting of all finite products $x_1x_2\ldots x_n$, where each x_i is an element of K or the inverse of an element of K. We say that <K> is generated by K. It is easy to see that <K> is contained in any subgroup of G which contains K. The dihedral group D_n is the symmetry group of an n-sided regular polygon for n > 1. These groups are one of the most important classes of finite groups currently applicable in chemistry. For example D_3, D_4, D_5 and D_6 point groups are dihedral groups. One group presentation for D_n is $<x,y \mid x^n = y^2 = e, yxy = x^{-1}>$. This means that D_n is generated by a two elements set {x,y} with the condition $x^n = y^2 = 1$ and $yxy = x^{-1}$. The aim of this section is to calculate generally the markaracter tables of dihedral groups. We also prepare a GAP program for computing markaracters of finite groups. We apply our program to the I_h point group symmetry, which is the symmetry of a buckyball. This presents a new method for solving such problems.

Theorem 1. Let G and H be groups acting on sets X and Y, respectively. Then $|Fix_{X\times Y}(U \times V)| = |Fix_X(U)| \times |Fix_Y(V)|$, where $U \le G$, $V \le H$ and $Fix_X(U) = \{x \in X \mid xg = x; \forall g \in U\}$.

Proof. We have:

$$\begin{aligned}|Fix_{X\times Y}(U \times V)| &= |\{(x,y) \mid (x,y)(g,h) = (x,y);\ \forall\ (g,h) \in U \times V\}|\\ &= |\{(x,y) \mid (xg,yh) = (x,y);\ \forall\ (g,h) \in U \times V\}|\\ &= |\{(x,y) \mid xg = x\ \&\ yh = y;\ \forall g \in U\ \&\ \forall h \in V\}|\\ &= |\{(x,y) \mid x \in Fix_X(U)\ \&\ y \in Fix_Y(V)\}|\\ &= |Fix_X(U)| \times |Fix_Y(V)|.\end{aligned}$$ □

Corollary. Let G and H be groups of co-prime orders acting on sets X and Y, respectively. If $M(G) = [a_{ij}]$ and $M(H) = [b_{ij}]$ then $M(G \times H) = [c_{rs}]$, where $c_{rs} = a_{i_ri_s} b_{j_rj_s}$ and $G_{i_r} \times H_{j_r}, G(/G_{i_s} \times H_{j_s})$ are the r^{th} column and s^{th} row of $M(G \times H)$, respectively.

Proof. The proof is straightforward. □

It is an easy task to show that in the last Corollary, $M(G \times H)$ is the Kronecker product $M(G) \otimes M(H)$. In [6] Fujita obtained the form of the mark table of a cyclic group. We now apply Theorem 1 to find another method for computing this table. To simplify our argument, in the following example we only compute the mark table of a cyclic group of order $p^n q^m$.

Example 1. Let G be a cyclic group of order $p^n q^m$. It is a well-known fact that G is isomorphic to $H \times K$ in which H and K are subgroups of G of order p^n and q^m, respectively. Suppose $H_1, H_2, \ldots, H_{n+1}$ and $K_1, K_2, \ldots, K_{m+1}$ are all subgroups of H and K, respectively. One can see that $M(H) = [a_{ij}]$ and $M(K) = [b_{ij}]$, where

$$a_{ij} = \begin{cases} p^{n-j+1} & j \le i \\ 0 & \text{otherwise} \end{cases} \text{ and } b_{ij} = \begin{cases} q^{m-j+1} & j \le i \\ 0 & \text{otherwise} \end{cases}.$$

Then $M(H \times K) = [c_{rs}]$, in which

$$c_{rs} = \begin{cases} p^{n-j_r+1} q^{m-j_s+1} & j_r \le i_r, j_s \le i_s \\ 0 & \text{otherwise} \end{cases}.$$

In the following theorem, we calculate the markaracter tables of dihedral groups.

Theorem 2. Suppose $G = D_n$ is the dihedral group of order 2n. Then $1 = G_1$, $\langle b \rangle = G_2$, $\langle ab \rangle = G_3$, $\langle a^{n/2} \rangle = G_4$, $\langle a^{v_5} \rangle = G_5$, $\langle a^{v_6} \rangle = G_6$, …, and $\langle a^{v_{t+1}} \rangle = \langle a \rangle = G_{t+2}$ are all cyclic non-conjugate subgroups of D_n such that v_i divides n (i = 5, 6, ..., t+1), where t is the number of divisors of n. Moreover the markaracter table of G is as follows

The Markaracter Table of D_n, When n is Even.					
Cyclic Subgroups	G_1	G_2	G_3	G_4	$G_i = \langle a^{v_i} \rangle (5 \le i \le t+2)$
G/<>	2n	0	0	0	0
G/G_2	n	2	0	0	0
G/G_3	n	0	2	0	0
G/G_4	n	0	0	n	0
$G/G_j(5 \le j \le t+2)$	2j	0	0	α	γ

The Markaracter Table of D_n, When n is Odd.			
Cyclic Subgroups	**G_1**	**G_2**	**$G_i = <x^i>(3 \le j \le t+2)$**
G/<>	2n	0	0
G/G_2	n	1	0
$G/G_j(3 \le j \le t+2)$	2j	0	α

where, $\alpha = \begin{cases} 2j & v_j \mid \frac{n}{2} \\ 0 & \text{Otherwise} \end{cases}$ and $\gamma = \begin{cases} 2j & v_j \mid v_i \\ 0 & \text{Otherwise} \end{cases}$.

Proof. Suppose $MC(G) = [a_{ij}]$. We first assume n is even. Then the conjugacy classes of D_n are $\{1\}$, $\{a^{n/2}\}$, $\{a^r,a^{-r}\}$ $(1 \le r \le n/2)$, $\{a^sb \mid 0 \le s \le n-1 \ \&\ s \text{ is even}\}$ and $\{a^sb \mid 0 \le s \le n-1 \ \&\ s \text{ is odd}\}$. Hence up to conjugacy there are three subgroups of order 2, $G_2 = <b>$, $G_3 = <ab>$, $G_4 = <a^{n/2}>$ and there are $t = d(n)$ cyclic subgroups whose orders divide n, say $G_5, \ldots, G_{t+2} = <a>$. By a result of Pfeiffer [21], $a_{ij} = |\{G_ig \mid G_j \subseteq g^{-1}G_ig\}|$ and so $a_{ii} = \frac{|N_G(G_i)|}{|G_i|}$. Clearly, $N_G(<b>) = \{1, b, a^{n/2}, a^{n/2}b\}$, $N_G(<a^{n/2}>) = G$ and $N_G(<ab>) = \{1, ab, a^{n/2}, a^{1+n/2}b\}$. So $a_{22} = a_{33} = 2$ and $a_{44} = n$. Suppose $j \mid n$. By an elementary fact in finite groups $o(a^j) = n/j$. Since every subgroup of $<a>$ is normal in G, $a_{ij} = 2n/(n/j) = 2j$. If $v_j \mid v_i$ then $G_j \subseteq G_i$ and so $a_{ij} = 2j$, as desired. We now assume that n is odd. Then the conjugacy classes of D_n are $\{1\}$, $\{a^r,a^{-r}\}$ $(1 \le r \le (n-1)/2)$, $\{a^sb \mid 0 \le s \le n-1\}$ and up to conjugacy there is one only subgroup of order 2 and d(n) cyclic subgroups whose orders divide n. Now a similar argument as above, complete the proof. □

In the end of this paper, we compute the markaracter table of the I_h point group. This is the symmetry group of the Buckminster fullerene, Figure 1. To do this, we notice that this group is isomorphic to the direct product of a cyclic group of order 2 and an alternating group on five symbols. In permutation group language [23], $I_h = <a,b>$, where permutations a and b are defined as follows:

a = (1,10,11)(2,14,19)(4,15,17)(5,50,28)(6,21,27)(8,51,26)(13,20,35)(16,18,33)(22,34,55)(23,36,54)(29,38,49)(30,59,45)(31,58,48)(32,39,52)(7,24,25) (37,53,46)(40,56,47)(41,43,60)(42,57,44)(3,12,9),

b = (1,25,9,33,17)(2,26,10,34,18)(3,27,11,35,19)(4,28,12,36,20)(5,29,13,37,21)(6,30,14,38,22)(7,31,15,39,23)(8,32,16,40,24)(41,49,57,45,53)(42,50,58,46,54)(43,51,59,47,55)(44,52,60,48,56).

In the end of this paper, a GAP program is prepared by which it is possible to compute markaracter tables of finite groups. We encourage the reader to consult [1,22] for computational techniques, as well as basic functions of GAP. We apply our GAP function as follows:

gap> f(DirectProduct(CyclicGroup(2),AlternatingGroup(5)))=$Z_2 \times A_5 = I_h$

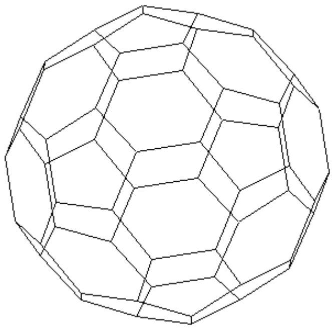

Figure 1. The Buckminster Fullerene C_{60}.

and output will be the markaracter table of I_h point group symmetry, i.e.

$$MC(I_h) = \begin{bmatrix} 120 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 60 & 60 & 0 & 0 & 0 & 0 & 0 & 0 \\ 60 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 60 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 40 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 24 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 20 & 20 & 0 & 0 & 2 & 0 & 2 & 0 \\ 12 & 12 & 0 & 0 & 0 & 2 & 0 & 2 \end{bmatrix}.$$

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A GAP Program For Computing Markaracter Table of Finite Groups

```
f:=function(G)
local u,r,s,k,kk,a,l,ll,lll,dd,ddd,dddd,z,h,v,vv,ss,i,j;;
    s:=[];
    k:=[];
    kk:=[];l:=[];ll:=[];lll:=[];dd:=[];ddd:=[];dddd:=[];
    a:=List(ConjugacyClassesSubgroups(G),x->Elements(x));
    z:=Length(a);
        for i in [1..z] do
            if IsCyclic((a[i][1])) then
                Add(s,i);
            fi;
        od;
h:=TableOfMarks(G);
v:=MatTom(h);
vv:=TransposedMat(v);
ss:=Difference([1..z],s);
    for i in s do
        Add(k,v[i]);
    od;
    for j in [1..Length(k)] do
        for i in [1..z] do
            if i in s then
                Add(ll,k[j][i]);
            fi;
        od;
        Add(lll,ll);
        ll:=[];
    od;
Print("MarkaracterTable is: ","\n");
PrintArray(lll);
return;
end;
```

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